ON PRINCIPAL EIGENVALUES OF BIHARMONIC SYSTEMS

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Abstract. We prove the existence, positivity, simplicity, uniqueness up to nonnegative eigenfunctions, and isolation of the principle eigenvalue of a biharmonic system. We also provide the extension of our results to a related system.

1. Introduction

We study the properties of the principal eigenvalue of the (p, q)-biharmonic system

$$\begin{cases} \Delta\left(|\Delta u|^{p-2}\Delta u\right) = \lambda a(x)|u|^{p-2}u + \lambda c(x)|u|^{\alpha-1}|v|^{\beta+1}u & \text{in } \Omega, \\ \Delta\left(|\Delta v|^{q-2}\Delta v\right) = \lambda b(x)|v|^{q-2}v + \lambda c(x)|u|^{\alpha+1}|v|^{\beta-1}v & \text{in } \Omega, \\ u = \Delta u = 0, \ v = \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, with $N \geq 1$, is a bounded and connected set, $\lambda > 0$ is a parameter, $p > 1$, $q > 1$, $\max\{p, q\} < N/2$, $\alpha \geq 0$, $\beta \geq 0$, and a, b, c are nonnegative functions defined in Ω and $c \not\equiv 0$ in Ω.

The eigenvalues of biharmonic problems have been studied by many authors in the literature, for example, in [1, 4, 5, 10, 14, 20, 21, 23, 24, 26]. In particular, in 2001, Drábek and Ótanik [10] proved, among many results, that the p-biharmonic boundary value problem

$$\begin{cases} \Delta\left(|\Delta u|^{p-2}\Delta u\right) = \lambda|u|^{p-2}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

has a principal eigenvalue λ_1 which is simple and isolated. In 2012, Benedikt and Drábek [4] obtained estimates from below and from above for λ_1 and applied their estimates to study the asymptotic behavior of λ_1 as $p \to \infty$. When Ω is a ball centered at the origin, in 2014, they [5] further studied the asymptotics of λ_1 as $p \to 1^+$. In 2015, Ge et al. [14] considered the eigenvalues of the $p(x)$-biharmonic problem with an indefinite weight

$$\begin{cases} \Delta\left(|\Delta u|^{p(x)-2}\Delta u\right) = \lambda V(x)|u|^{q(x)-2}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p, q : \overline{\Omega} \to (1, \infty)$ are continuous functions and V is an indefinite weight function. Under appropriate conditions on p and q, they showed the existence of a continuous family of

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eigenvalues of the problem. Recently, Sang and Ren [26] studied the (p,p)-biharmonic system

$$\begin{cases} \Delta\left(|\Delta u|^{p-2}\Delta u\right) = \frac{\lambda\alpha_1}{p^*}u^{\alpha_1-1}v^{\beta_1} + \frac{\alpha_2}{\alpha_2+\beta_2}h(x)u^{\alpha_2-1}v^{\beta_2} & \text{in } \Omega, \\ \Delta\left(|\Delta v|^{p-2}\Delta v\right) = \frac{\lambda\beta_1}{p^*}u^{\alpha_1}v^{\beta_1-1} + \frac{\beta_2}{\alpha_2+\beta_2}h(x)u^{\alpha_2}v^{\beta_2-1} & \text{in } \Omega, \\ u>0, \quad v>0 & \text{in } \Omega, \\ u=\Delta u=0, \ v=\Delta v=0 & \text{on } \partial\Omega, \end{cases}$$

where $\alpha_1 > 1$ and $\beta > 1$ satisfy $\alpha_1 + \beta_1 = p^*$ with $p^* = \frac{Np}{N-2p}$, $\alpha_2, \beta_2 \in (0,1)$, $h \in L^\infty(\Omega)$, and $h(x) > 0$ in Ω. They proved that there exists $\lambda_* > 0$ such that the above system has at least two solutions for all $\lambda \in (0, \lambda_*)$.

Motivated partly by the above papers, here we study system (1.1) and prove that it has a principal eigenvalue which is simple, unique up to nonnegative eigenfunctions, and isolated. See Theorems 2.1 and 2.2 in Section 2 for details. The proofs of this paper extend and develop the ideas in [10,29]. In [29], Zographopoulos studied the principal eigenvalue of the degenerate quasilinear elliptic system

$$\begin{cases} \nabla\left(\nu_1(x)|\nabla u|^{p-2}\nabla u\right) = \lambda a(x)|u|^{p-2}u + \lambda c(x)|u|^{\alpha-1}|v|^{\beta+1}u & \text{in } \Omega, \\ \nabla\left(\nu_2(x)|\nabla v|^{q-2}\nabla v\right) = \lambda b(x)|v|^{q-2}v + \lambda c(x)|u|^{\alpha+1}|v|^{\beta-1}v & \text{in } \Omega, \\ u=0, \ v=0 & \text{on } \partial\Omega, \end{cases}$$

where ν_1 and ν_2 are two weight functions satisfying some suitable conditions. In extending the ideas in [29] to system (1.1), a key point is to extend a "weak" form of Picone's identity ([29, Lemma 2.3]) to the p-biharmonic operator. See Lemma 3.1 in Section 3 for the extension. We comment that the technique in [29] to show the principal eigenvalue is isolated cannot be extended to system (1.1). This is because the property that

$$u \in W^{1,p}(\Omega) \Longrightarrow u^- := \min\{0,u\} \in W^{1,p}(\Omega)$$

was used in the proof of the isolation of the principal eigenvalue. To be able to extend the ideas in [29] to our problem, we will need that $u^- \in W^{2,p}(\Omega)$ if $u \in W^{2,p}(\Omega)$. But $u \in W^{2,p}(\Omega)$ does not imply the same regularity for u^-. The reader may refer to [27] for more discussion on this subject. To prove the isolation of the principal eigenvalue of system (1.1), in this paper, we extend and develop the ideas in [10] for problem (1.2).

For completeness, we note that simplicity of the principal eigenvalue and nonnegativity (up to constant multiples) of the corresponding eigenfunction are familiar phenomena in many linear second-order problems, even on minimally smooth domains and with very general non-local boundary conditions, see [15, Corollary 4.5]. These problems naturally arise in quantum mechanics in connection with nondegeneracy of the ground state [12]. Moreover, fourth order partial differential equations have applications in the fields of image and signal processing, nuclear physics, and engineering. For instance, they arise in the study of traveling waves in suspension bridges ([7,22,25]). Elliptic systems may be used to describe the multiplicative chemical reactions catalyzed by a catalyst ([8]). In recent years, various biharmonic elliptic systems have been studied by many authors, see, for example, [2,3,8,9,19,24,26] and the references therein. It is also well known that the eigenvalue problems of biharmonic operators are central in understanding the mechanical vibrations of plates ([18,28]).

In this paper, we also extend our theorems for system (1.1) to the following related (p,q)-biharmonic system

$$\begin{cases} \Delta\left(|\Delta u|^{p-2}\Delta u\right) = \lambda a(x)|u|^{p-2}u + \lambda c(x)|u|^{\alpha}|v|^{\beta}v & \text{in } \Omega, \\ \Delta\left(|\Delta v|^{q-2}\Delta v\right) = \lambda b(x)|v|^{q-2}v + \lambda c(x)|u|^{\alpha}|v|^{\beta}u & \text{in } \Omega, \\ u = \Delta u = 0,\ v = \Delta v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Throughout this paper, for any $r \in (1,\infty]$, we denote the norm in $L^r(\Omega)$ by $\|u\|_r = \left(\int_\Omega |u|^r\right)^{1/r}$, $u \in L^r(\Omega)$, and let $\|\cdot\|_X$ stand for the norm in a Banach space X. We use the notation $X \hookrightarrow Y$ for the continuous embedding $X \subset Y$, and the notation $X \hookrightarrow\hookrightarrow Y$ for the compact embedding $X \subset Y$.

In the sequel, let the space E be defined by

$$E = \left(W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)\right) \times \left(W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)\right).$$

Then, E is a separable and reflexive Banach space equipped with the standard norm

$$\|(u,v)\|_E = \|\Delta u\|_p + \|\Delta v\|_q, \quad (u,v) \in E.$$

Define

$$p^* = \frac{Np}{N-2p} \quad \text{and} \quad q^* = \frac{Nq}{N-2q}.$$

Lemma 1.1 below follows from the well-known Sobolev embedding theorem.

Lemma 1.1. *The following embeddings hold:*

(a) $E \hookrightarrow L^{r_1}(\Omega) \times L^{r_2}(\Omega)$ *for all* $1 \le r_1 \le p^*$ *and* $1 \le r_2 \le q^*$*;*

(b) $E \hookrightarrow\hookrightarrow L^{r_1}(\Omega) \times L^{r_2}(\Omega)$ *for all* $1 \le r_1 < p^*$ *and* $1 \le r_2 < q^*$*.*

We make the following assumptions.

(H1) $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$;

(H2) $a \in L^{\frac{p^*}{p^*-p}}(\Omega)$, $b \in L^{\frac{q^*}{q^*-q}}(\Omega)$, and $c \in L^{\rho}(\Omega)$, where $\rho = \left(1 - \frac{\alpha+1}{p^*} - \frac{\beta+1}{q^*}\right)^{-1}$.

The rest of this paper is organized as follows. Section 2 contains the main theorems for system (1.1) whose proofs are given in Sections 3 and 4. Section 5 discusses the extensions of the main theorems to system (1.3).

2. The principal eigenvalue of system (1.1)

Let the functionals $I, J : E \to \mathbb{R}$ be defined by

$$I(u,v) = \frac{\alpha+1}{p}\int_\Omega |\Delta u|^p dx + \frac{\beta+1}{q}\int_\Omega |\Delta v|^q dx \tag{2.1}$$

and

$$J(u,v) = \frac{\alpha+1}{p}\int_\Omega a(x)|u|^p dx + \frac{\beta+1}{q}\int_\Omega b(x)|v|^q dx + \int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1} dx, \tag{2.2}$$

where $(u,v) \in E$.

Using a standard argument, we can show the following lemma.

Lemma 2.1. *Assume that (H1) and (H2) hold. Then, the functionals I and J are well defined, I is weakly lower semicontinuous and coercive, and J is weakly continuous. Moreover, $I, J \in C^1(E, \mathbb{R})$ with the derivatives respectively given by*

$$\langle I'(u,v),(\phi,\psi)\rangle = (\alpha+1)\int_\Omega |\Delta u|^{p-2}\Delta u \Delta\phi\,dx + (\beta+1)\int_\Omega |\Delta v|^{q-2}\Delta v\Delta\psi\,dx \tag{2.3}$$

and

$$\begin{aligned}\langle J'(u,v),(\phi,\psi)\rangle &= (\alpha+1)\left(\int_\Omega a(x)|u|^{p-2}u\phi\,dx + \int_\Omega c(x)|u|^{\alpha-1}|v|^{\beta+1}u\phi\,dx\right)\\ &\quad +(\beta+1)\left(\int_\Omega b(x)|v|^{q-2}v\psi\,dx + \int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta-1}v\psi\,dx\right)\end{aligned} \tag{2.4}$$

for all $(u,v),(\phi,\psi) \in E$, where $\langle\cdot,\cdot\rangle$ denotes the duality between E and its dual space E^*.

Definition 2.1. *We say that $\lambda \in \mathbb{R}$ is an eigenvalue of system (1.1) if there exists $(u,v) \in E$ with $(u,v) \not\equiv (0,0)$ such that $\langle I'(u,v),(\phi,\psi)\rangle = \lambda\langle J'(u,v),(\phi,\psi)\rangle$ for all $(\phi,\psi) \in E$. The pair (u,v) is called an eigenfunction corresponding to the eigenvalue λ and the pair $(\lambda,(u,v))$ is called an eigenpair. The smallest eigenvalue $\lambda_1 > 0$ of system (1.1) is usually called the principal eigenvalue.*

Remark 2.1. From (2.3) and (2.4), we see that if $(\lambda,(u,v))$ is an eigenpair of system (1.1), then

$$\int_\Omega |\Delta u|^{p-2}\Delta u\Delta\phi\,dx = \lambda\int_\Omega a(x)|u|^{p-2}u\phi\,dx + \lambda\int_\Omega c(x)|u|^{\alpha-1}|v|^{\beta+1}u\phi\,dx$$

and

$$\int_\Omega |\Delta v|^{q-2}\Delta v\Delta\psi\,dx = \lambda\int_\Omega b(x)|v|^{q-2}v\psi\,dx + \lambda\int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta-1}v\psi\,dx$$

for all $(\phi,\psi) \in E$.

We now state the first result of this paper.

Theorem 2.1. *Assume that (H1) and (H2) hold. Then, system (1.1) admits an eigenpair $(\lambda_1,(u_1,v_1))$ such that $\lambda_1 > 0$ is the principal eigenvalue, u_1 and v_1 are nonnegative in Ω, and*

$$\lambda_1 = I(u_1,v_1) = \inf_{J(u,v)=1} I(u,v). \tag{2.5}$$

Moreover, the following properties hold:

(a) *the set of all eigenfunctions corresponding to λ_1 forms a one-dimensional manifold $E_1 \subset E$ given by $E_1 = \left\{(c_1u_1, c_1^{p/q}v_1) \;:\; c_1 \in \mathbb{R}\right\}$;*

(b) *λ_1 is the only eigenvalue of system (1.1) to which corresponds a component nonnegative eigenfunction.*

Replacing (H2) with a stronger assumption, we can further prove that λ_1 is isolated.

Theorem 2.2. *Assume that (H1) and the following condition hold:*

(H3) *$p = q$ and $a, b, c \in L^\infty(\Omega)$.*

Then, the principal eigenvalue λ_1 of system (1.1), given by (2.5), is isolated in the sense that there exists $\zeta > 0$ such that λ_1 is the only eigenvalue of system (1.1) contained in the interval $(0, \lambda_1 + \zeta)$.

3. Proof of Theorem 2.1

We need the following weak form of Picone's identity for the p-biharmonic operator in the sense that the C^2-regularity of the involved functions are not required. With C^2-regularity, a Picone-type identity for the p-biharmonic operator was derived in [11, Lemma 2.2] and two Picone-type identities for the weighted p-biharmonic operator were established in [17, Lemmas 2.1 and 2.2]. Motivated by these results, we can prove the following lemma.

Lemma 3.1. *Assume that $u, v \in W^{2,p}(\Omega)$ with $u, v \geq 0$ and $\Delta v \leq 0$ in Ω. Let $\Omega_\delta = \{x \in \Omega : v(x) \geq \delta\}$ for some $\delta > 0$. Define*

$$R(u,v) = |\Delta u|^p - \Delta\left(\frac{u^p}{v^{p-1}}\right)|\Delta v|^{p-2}\Delta v$$

and

$$\begin{aligned} L(u,v) &= |\Delta u|^p + (p-1)\frac{u^p}{v^p}|\Delta v|^p - p\frac{u^{p-1}}{v^{p-1}}|\Delta v|^{p-2}\Delta u \Delta v \\ &\quad - p(p-1)\frac{u^{p-2}}{v^{p-2}}|\Delta v|^{p-2}\Delta v\left(\nabla u - \frac{u}{v}\nabla v\right)^2. \end{aligned}$$

Then, we have

(a) $\int_{\Omega_\delta} L(u,v)dx = \int_{\Omega_\delta} R(u,v)dx$;
(b) $\int_{\Omega_\delta} L(u,v)dx \geq 0$.

Proof. By expanding $R(u,v)$, we see that part (a) holds. We now show part (b). Rewrite $L(u,v)$ as

$$L(u,v) = L_1(u,v) + L_2(u,v) + L_3(u,v),$$

where

$$L_1(u,v) = |\Delta u|^p + (p-1)\frac{u^p}{v^p}|\Delta v|^p - p\frac{u^{p-1}}{v^{p-1}}|\Delta v|^{p-2}|\Delta u|\,|\Delta v|,$$

$$L_2(u,v) = p\frac{u^{p-1}}{v^{p-1}}|\Delta v|^{p-2}\left(|\Delta u|\,|\Delta v| - \Delta u \Delta v\right),$$

and

$$L_3(u,v) = -p(p-1)\frac{u^{p-2}}{v^{p-2}}|\Delta v|^{p-2}\Delta v\left(\nabla u - \frac{u}{v}\nabla v\right)^2.$$

Let $x \in \Omega_\delta$. By Young's inequality, we have

$$p\frac{u^{p-1}}{v^{p-1}}|\Delta v|^{p-2}|\Delta u|\,|\Delta v| = p\left(\frac{u}{v}|\Delta v|\right)^{p-1}|\Delta u| \leq |\Delta u|^p + (p-1)\frac{u^p}{v^p}|\Delta v|^p.$$

The equal sign holds if and only if $|\Delta u|^p = |\Delta v|^p \frac{u^p}{v^p}$. Thus,

$$L_1(u,v) \geq 0 \quad \text{in } \Omega_\delta. \tag{3.1}$$

Obviously,

$$L_2(u,v) \geq 0 \quad \text{in } \Omega_\delta, \tag{3.2}$$

In view of $\Delta v \leq 0$ in Ω, we have

$$L_3(u,v) \geq 0 \quad \text{in } \Omega_\delta. \tag{3.3}$$

Then, part (b) follows readily from (3.1)–(3.3). This completes the proof of the lemma. □

The following lemma plays a key role in the proof of Theorem 2.1 and is taken from [6, Theorem 6.3.2].

Lemma 3.2. *Suppose that the C^1 functionals Φ and Ψ defined on the reflexive Banach space Y have the following properties:*

(i) *Φ is weakly lower semicontinuous and coercive in $Y \cap \{\Psi(y) \leq const.\}$;*

(ii) *Ψ is weakly continuous and $\Psi'(y) = 0$ only at $y = 0$.*

Then, the equation $\Phi'(y) = \lambda\Psi'(y)$ has a one-parameter family of nontrivial solution (y_R, λ_R) for all R in the range of Ψ such that $\Psi(y_R) = R$ and y_R is characterized as the minimum of $\Phi(y)$ over the set $\Psi(y) = R$.

Now, we prove Theorem 2.1.

Proof of Theorem 2.1. In view of Lemma 2.1, we see that all the assumptions of Lemma 3.2 with $Y = E$, $\Phi = I$, and $\Psi = J$ are satisfied. Then, Lemma 3.2 implies that system (1.1) admits an eigenpair $(\lambda_1, (u_1, v_1))$, where the pair $(u_1, v_1) \in E$ satisfies

$$I(u_1, v_1) = \inf_{J(u,v)=1} I(u, v).$$

Thus,

$$\langle I'(u_1, v_1), (\phi, \psi)\rangle = \lambda_1 \langle J'(u_1, v_1), (\phi, \psi)\rangle \quad \text{for all } (\phi, \psi) \in E.$$

Choosing $\phi = \frac{1}{p}u_1$ and $\psi = \frac{1}{q}v_1$. Then, from (2.1)–(2.4) and the fact that $J(u_1, v_1) = 1$, we have

$$\lambda_1 = \frac{\langle I'(u_1, v_1), (u_1/p, v_1/q)\rangle}{\langle J'(u_1, v_1), (u_1/p, v_1/p)\rangle} = \frac{I(u_1, v_1)}{J(u_1, v_1)} = I(u_1, v_1) = \inf_{J(u,v)=1} I(u, v). \tag{3.4}$$

Then, (2.5) holds and $\lambda_1 > 0$. Note that when (u_1, v_1) is a minimizer of (2.5), $(|u_1|, |v_1|)$ is also a minimizer. Then, we may assume that u_1 and v_1 are nonnegative in Ω.

Now, we prove that λ_1 is the smallest eigenvalue of system (1.1). Assume that $(\bar{\lambda}, (\bar{u}, \bar{v}))$ is another eigenpair of system (1.1). Then, as in (3.4), we can get that

$$\bar{\lambda} = \frac{I(\bar{u}, \bar{v})}{J(\bar{u}, \bar{v})}. \tag{3.5}$$

Let

$$\hat{u} = \frac{|\bar{u}|}{|J(\bar{u}, \bar{v})|^{1/p}} \quad \text{and} \quad \hat{v} = \frac{|\bar{v}|}{|J(\bar{u}, \bar{v})|^{1/q}}.$$

Then, we have

$$I(\hat{u}, \hat{v}) = \frac{I(\bar{u}, \bar{v})}{J(\bar{u}, \bar{v})} \quad \text{and} \quad J(\hat{u}, \hat{v}) = 1. \tag{3.6}$$

From (3.4)–(3.6), it follows that $\bar{\lambda} = I(\hat{u}, \hat{v}) \geq \lambda_1$. Thus, λ_1 is the principal eigenvalue.

Next, we show properties (a) and (b). Let (μ, ν) be an eigenfunction of system (1.1) corresponding to λ_1. From Remark 2.1, we have

$$\int_\Omega |\Delta\mu|^p dx = \lambda_1 \int_\Omega a(x)|\mu|^p dx + \lambda_1 \int_\Omega c(x)|\mu|^{\alpha+1}|\nu|^{\beta+1} dx \tag{3.7}$$

and

$$\int_\Omega |\Delta\nu|^q dx = \lambda_1 \int_\Omega b(x)|\nu|^q dx + \lambda_1 \int_\Omega c(x)|\mu|^{\alpha+1}|\nu|^{\beta+1} dx. \tag{3.8}$$

Without loss of generality, we may assume that $\mu(x) \geq 0$ and $\nu(x) \geq 0$ in Ω. Choose sequences $\{\mu_n\}, \{\nu_n\} \subset C_0^\infty(\Omega)$ such that $(\mu_n, \nu_n) \to (\mu, \nu)$ in E. Let $w(x) = |\Delta u_1|^{p-2}\Delta u_1$. From (1.1), we see that

$$\Delta w = \lambda_1 a(x)|u_1|^{p-2}u_1 + \lambda_1 c(x)|u_1|^{\alpha-1}|v_1|^{\beta+1}u_1 \geq 0 \text{ in } \Omega, \quad \text{and} \quad w = 0 \text{ on } \partial\Omega.$$

Then, by the maximum principle (see, for example, [13, Theorem 1, p. 344] or [16, Theorem 8.1]), we obtain that $w \leq 0$, and so $\Delta u_1 \leq 0$ in Ω. Hence, for any fixed $n \in \mathbb{N}$ and $\epsilon > 0$, Lemma 3.1 implies that

$$\begin{aligned} 0 &\leq \int_\Omega R(\mu_n, u_1+\epsilon)dx = \int_\Omega |\Delta\mu_n|^p dx - \int_\Omega \Delta\left(\frac{\mu_n^p}{(u_1+\epsilon)^{p-1}}\right)|\Delta u_1|^{p-2}\Delta u_1 dx \\ &= \int_\Omega |\Delta\mu_n|^p dx - \int_\Omega \left(\frac{\mu_n^p}{(u_1+\epsilon)^{p-1}}\right)\Delta(|\Delta u_1|^{p-2}\Delta u_1)dx \\ &= \int_\Omega |\Delta\mu_n|^p dx - \lambda_1\int_\Omega a(x)|u_1|^{p-1}\frac{\mu_n^p}{(u_1+\epsilon)^{p-1}}dx - \lambda_1\int_\Omega c(x)|u_1|^\alpha|v_1|^{\beta+1}\frac{\mu_n^p}{(u_1+\epsilon)^{p-1}}dx. \end{aligned}$$

Letting $n \to \infty$, we have

$$0 \leq \int_\Omega |\Delta\mu|^p dx - \lambda_1\int_\Omega a(x)|u_1|^{p-1}\frac{\mu^p}{(u_1+\epsilon)^{p-1}}dx - \lambda_1\int_\Omega c(x)|u_1|^\alpha|v_1|^{\beta+1}\frac{\mu^p}{(u_1+\epsilon)^{p-1}}dx.$$

Then, from (3.7),

$$\begin{aligned} 0 \leq\ & \int_\Omega a(x)|\mu|^p dx + \int_\Omega c(x)|\mu|^{\alpha+1}|\nu|^{\beta+1}dx \\ & - \int_\Omega a(x)|u_1|^{p-1}\frac{\mu^p}{(u_1+\epsilon)^{p-1}}dx - \int_\Omega c(x)|u_1|^\alpha|v_1|^{\beta+1}\frac{\mu^p}{(u_1+\epsilon)^{p-1}}dx. \end{aligned} \tag{3.9}$$

Let $\epsilon \to 0$. By the Lebesgue dominated convergence theorem,

$$\int_\Omega a(x)|\mu|^p dx - \int_\Omega a(x)|u_1|^{p-1}\frac{\mu^p}{(u_1+\epsilon)^{p-1}}dx \to 0$$

and by Fatou's lemma,

$$\liminf_{\epsilon\to 0}\int_\Omega c(x)|u_1|^\alpha|v_1|^{\beta+1}\frac{\mu^p}{(u_1+\epsilon)^{p-1}}dx \geq \int_\Omega c(x)|u_1|^{\alpha+1-p}|v_1|^{\beta+1}\mu^p dx.$$

Thus, (3.9) yields that

$$\int_\Omega c(x)|u_1|^{\alpha+1-p}|v_1|^{\beta+1}\mu^p dx \leq \int_\Omega c(x)|\mu|^{\alpha+1}|\nu|^{\beta+1}dx. \tag{3.10}$$

Using Lemma 3.1 for $R(\nu_n, v_1+\epsilon)$ and arguing as above, we obtain that

$$\int_\Omega c(x)|u_1|^{\alpha+1}|v_1|^{\beta+1-q}\nu^q dx \leq \int_\Omega c(x)|\mu|^{\alpha+1}|\nu|^{\beta+1}dx. \tag{3.11}$$

Multiplying (3.10) and (3.11) by $\frac{\alpha+1}{p}$ and $\frac{\beta+1}{q}$, respectively, and then adding them together, we have

$$0 \leq \int_\Omega c(x)\left(|\mu|^{\alpha+1}|\nu|^{\beta+1} - \frac{\alpha+1}{p}|u_1|^{\alpha+1-p}|v_1|^{\beta+1}\mu^p - \frac{\beta+1}{q}|u_1|^{\alpha+1}|v_1|^{\beta+1-q}\nu^q\right)dx. \tag{3.12}$$

Now, with $\xi_1 = \frac{(\alpha+1)(\beta+1)}{q}$ and $\xi_2 = \frac{(\alpha+1)(\beta+1)}{p}$, by Young's inequality, it follows that

$$\begin{aligned}
&\int_\Omega c(x)|\mu|^{\alpha+1}|\nu|^{\beta+1}dx = \int_\Omega c(x)|\mu|^{\alpha+1}|\nu|^{\beta+1}\frac{v_1^{\xi_2}}{u_1^{\xi_1}}\frac{u_1^{\xi_1}}{v_1^{\xi_2}}dx \\
= &\int_\Omega \left(c^{\frac{\alpha+1}{p}}(x)|\mu|^{\alpha+1}\frac{v_1^{\xi_2}}{u_1^{\xi_1}}\right)\left(c^{\frac{\beta+1}{q}}(x)|\nu|^{\beta+1}\frac{u_1^{\xi_1}}{v_1^{\xi_2}}\right)dx \\
\leq &\int_\Omega \left[\frac{\alpha+1}{p}\left(c^{\frac{\alpha+1}{p}}(x)|\mu|^{\alpha+1}\frac{v_1^{\xi_2}}{u_1^{\xi_1}}\right)^{\frac{p}{\alpha+1}} + \frac{\beta+1}{q}\left(c^{\frac{\beta+1}{q}}(x)|\nu|^{\beta+1}\frac{u_1^{\xi_1}}{v_1^{\xi_2}}\right)^{\frac{q}{\beta+1}}\right]dx \\
= &\int_\Omega c(x)\left[\frac{\alpha+1}{p}|u_1|^{\alpha+1-p}|v_1|^{\beta+1}\mu^p + \frac{\beta+1}{q}|u_1|^{\alpha+1}|v_1|^{\beta+1-q}\nu^q\right]dx.
\end{aligned}$$

However, this contradicts (3.12) unless $(\mu,\nu) = (c_1u_1, c_2v_1)$. Substituting $\mu = c_1u_1$ and $\nu = c_2v_1$ into (3.7) and (3.8), we get that $c_2 = c_1^{p/q}$. This proves property (a).

Assume that system (1.1) has an eigenpair $(\lambda^*, (\mu^*, \nu^*))$ such that μ^* and ν^* are nonnegative in Ω. Then, by the same argument as above where (μ, ν) is replaced by (μ^*, ν^*) and λ_1 is replaced by λ^*, we can show that $(\mu^*, \nu^*) = (c_1u_1, c_1^{p/q}v_1)$. Hence, $\lambda^* = \frac{I(\mu^*,\nu^*)}{J(\mu^*,\nu^*)} = I(u_1, v_1) = \lambda_1$. This shows property (b). This completes the proof of the theorem. □

4. Proof of Theorem 2.2

In this section, we assume (H1) and (H3). For any $r > 1$, let r' be the conjugate of r, i.e., $r' = \frac{r}{r-1}$, and let the function $\phi_r : \mathbb{R} \to \mathbb{R}$ be defined by

$$\phi_r(s) = \begin{cases} |s|^{r-2}s & s \neq 0, \\ 0 & s = 0. \end{cases}$$

Then, $z = \phi_r(s)$ if and only if $s = \phi_{r'}(z)$, and system (1.1) can be written as

$$\begin{cases} \Delta\left(\phi_p(\Delta u)\right) = \lambda a(x)\phi_p(u) + \lambda c(x)|u|^{\alpha-1}|v|^{\beta+1}u & \text{in } \Omega, \\ \Delta\left(\phi_p(\Delta v)\right) = \lambda b(x)\phi_p(v) + \lambda c(x)|u|^{\alpha+1}|v|^{\beta-1}v & \text{in } \Omega, \\ u = \Delta u = 0, \ v = \Delta v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

In the sequel, we recall some properties of the Dirichlet problem for Poisson's equation

$$\begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

For any $r > 1$ and $f \in L^r(\Omega)$, it is well known that (4.2) has a unique solution $w \in W_0^{1,r}(\Omega) \cap W^{2,r}(\Omega)$. Let the linear operator $\Lambda : L^r(\Omega) \to W_0^{1,r}(\Omega) \cap W^{2,r}(\Omega)$ be defined by $\Lambda f = w$.

The following lemma can be found in [10, 16].

Lemma 4.1. *The operator Λ satisfies the following properties:*

(a) *(Continuity) There exists a constant $c_r > 0$ such that*

$$\|\Lambda f\|_{W^{2,r}(\Omega)} \leq c_r \|f\|_r \quad \textit{for any } r \in (1, \infty) \textit{ and } f \in L^r(\Omega).$$

(b) *(Regularity) Assume that $f \in L^\infty(\Omega)$. Then, $\Lambda f \in C^{1,\alpha}(\overline{\Omega})$ for all $\alpha \in (0,1)$. Moreover, there exists $c_\alpha > 0$ such that*

$$\|\Lambda f\|_{C^{1,\alpha}(\overline{\Omega})} \leq c_\alpha \|f\|_\infty.$$

(c) *(Regularity and Hopf-type maximum principle) Assume that $f \in C(\overline{\Omega})$, $f \geq 0$, and $f \not\equiv 0$. Then, $w = \Lambda f \in C^{1,\alpha}(\overline{\Omega})$ for all $\alpha \in (0,1)$ and w satisfies that $w > 0$ in Ω and $\frac{\partial w}{\partial n} < 0$ on $\partial\Omega$, where n denotes the outer unit normal to $\partial\Omega$.*

Let $w_1 = -\Delta u$ and $w_2 = -\Delta v$ in (4.1). Then, (4.1) can be restated as follows

$$\begin{cases} \phi_p(w_1) = \lambda\Lambda\left(a(x)\phi_p(\Lambda w_1) + c(x)|\Lambda w_1|^{\alpha-1}|\Lambda w_2|^{\beta+1}\Lambda w_1\right) & \text{in } \Omega, \\ \phi_p(w_2) = \lambda\Lambda\left(b(x)\phi_p(\Lambda w_2) + c(x)|\Lambda w_1|^{\alpha+1}|\Lambda w_2|^{\beta-1}\Lambda w_2\right) & \text{in } \Omega, \\ w_1 = w_2 = 0, \ \Lambda w_1 = \Lambda w_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

or as

$$\begin{cases} w_1 = \lambda^{\frac{1}{p-1}}\phi_{p'}\left(\Lambda\left(a(x)\phi_p(\Lambda w_1) + c(x)|\Lambda w_1|^{\alpha-1}|\Lambda w_2|^{\beta+1}\Lambda w_1\right)\right) & \text{in } \Omega, \\ w_2 = \lambda^{\frac{1}{p-1}}\phi_{p'}\left(\Lambda\left(b(x)\phi_p(\Lambda w_2) + c(x)|\Lambda w_1|^{\alpha+1}|\Lambda w_2|^{\beta-1}\Lambda w_2\right)\right) & \text{in } \Omega, \\ w_1 = w_2 = 0, \ \Lambda w_1 = \Lambda w_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Assume now that $(w_1, w_2) \in L^p(\Omega) \times L^p(\Omega)$ solves (4.3). Then, from Lemma 4.1 (a) and the properties of the Nemytskii operator induced by ϕ_p, we have

$$\begin{aligned} & (u,v) = (\Lambda w_1, \Lambda w_2) \in \left(W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)\right) \times \left(W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)\right) \\ \Longrightarrow \quad & (\phi_p(\Lambda w_1), \phi_p(\Lambda w_2)) \in L^{p'}(\Omega) \times L^{p'}(\Omega). \end{aligned} \tag{4.5}$$

Note that $p = q$ from (H3). Then, in view of (H1), we see that

$$\frac{\alpha}{p-1} + \frac{\beta+1}{p-1} = \frac{\alpha+\beta+1}{p-1} = 1.$$

This, together with Young's inequality and (H3), implies that

$$\begin{aligned} \left(c(x)|\Lambda w_1|^\alpha|\Lambda w_2|^{\beta+1}\right)^{p'} &= c^{p'}(x)|\Lambda w_1|^{\frac{p\alpha}{p-1}}|\Lambda w_2|^{\frac{p(\beta+1)}{p-1}} \\ &\leq c^{p'}(x)\left(\frac{\alpha}{p-1}|\Lambda w_1|^p + \frac{\beta+1}{p-1}|\Lambda w_2|^p\right) \in L^1(\Omega). \end{aligned}$$

Thus,

$$c(x)|\Lambda w_1|^\alpha|\Lambda w_2|^{\beta+1} \in L^{p'}(\Omega). \tag{4.6}$$

By a similar argument, we show that

$$c(x)|\Lambda w_1|^{\alpha+1}|\Lambda w_2|^{\beta} \in L^{p'}(\Omega). \tag{4.7}$$

From (4.5), (4.6), and (4.7), it follows that

$$\begin{aligned} &\left(a(x)\phi_p(\Lambda w_1) + c(x)|\Lambda w_1|^{\alpha-1}|\Lambda w_2|^{\beta+1}\Lambda w_1\right) \in L^{p'}(\Omega), \\ &\left(b(x)\phi_p(\Lambda w_2) + c(x)|\Lambda w_1|^{\alpha+1}|\Lambda w_2|^{\beta-1}\Lambda w_2\right) \in L^{p'}(\Omega). \end{aligned}$$

Then, by Lemma 4.1 (a), we have

$$\Lambda\left(a(x)\phi_p(\Lambda w_1)+c(x)|\Lambda w_1|^{\alpha-1}|\Lambda w_2|^{\beta+1}\Lambda w_1\right)\in W_0^{1,p'}(\Omega)\cap W^{2,p'}(\Omega),$$
$$\Lambda\left(b(x)\phi_p(\Lambda w_2)+c(x)|\Lambda w_1|^{\alpha+1}|\Lambda w_2|^{\beta-1}\Lambda w_2\right)\in W_0^{1,p'}(\Omega)\cap W^{2,p'}(\Omega).$$

Thus, in view of (4.3), we obtain that

$$(\phi_p(w_1),\phi_p(w_2))\in\left(W_0^{1,p'}(\Omega)\cap W^{2,p'}(\Omega)\right)\times\left(W_0^{1,p'}(\Omega)\cap W^{2,p'}(\Omega)\right).$$

Therefore,

$$-\Delta(\phi_p(-\Delta u))=\lambda\left(a(x)\phi_p(u)+c(x)|u|^{\alpha-1}|v|^{\beta+1}u\right)\quad\text{holds in } L^{p'}(\Omega)$$
$$-\Delta(\phi_p(-\Delta v))=\lambda\left(b(x)\phi_p(v)+c(x)|u|^{\alpha+1}|v|^{\beta-1}v\right)\quad\text{holds in } L^{p'}(\Omega).$$

From the above discussion, the following result is true.

Lemma 4.2. *The pair $(\lambda,(u,v))\in\mathbb{R}\times E$ is an eigenpair of system (1.1) if and only if, with the same eigenvalue λ, the pair $(w_1,w_2)=(-\Delta u,-\Delta v)\in L^p(\Omega)\times L^p(\Omega)$ solves (4.3) in $L^{p'}(\Omega)\times L^{p'}(\Omega)$.*

Now, motivated by [10, Lemma 3.1], we prove the following lemma.

Lemma 4.3. *Assume that $(w_1,w_2)\in L^p(\Omega)\times L^p(\Omega)$ solves (4.3) in $L^{p'}(\Omega)\times L^{p'}(\Omega)$. Then, $(w_1,w_2)\in C(\overline{\Omega})\times C(\overline{\Omega})$.*

Proof. We first prove the following claim:
Claim: If $(w_1,w_2)\in L^{p_0}(\Omega)\times L^{p_0}(\Omega)$, then we have

(a) $(w_1,w_2)\in L^{p_1}(\Omega)\times L^{p_1}(\Omega)$ with $\frac{1}{p_1}=\frac{1}{p_0}-\frac{2p'}{N}$ if $p_0<\frac{N}{2p'}$;
(b) $(w_1,w_2)\in C(\overline{\Omega})\times C(\overline{\Omega})$ if $p_0>\frac{N}{2p'}$.

In fact, assume that $(w_1,w_2)\in L^{p_0}(\Omega)\times L^{p_0}(\Omega)$ and $p_0<\frac{N}{2p'}$. Then, Lemma 4.1 (a) implies that $(\Lambda w_1,\Lambda w_2)\in W^{2,p_0}(\Omega)\times W^{2,p_0}(\Omega)$. Note that $p_0<\frac{N}{2p'}<\frac{N}{2}$. Then, by Sobolev's embedding theorem and the property of the Nemytskii operator induced by ϕ_p, we have

$$(\Lambda w_1,\Lambda w_2)\in L^{s_0}(\Omega)\times L^{s_0}(\Omega)\quad\text{and}\quad(\phi_p(\Lambda w_1),\phi_p(\Lambda w_2))\in L^{\frac{s_0}{p-1}}(\Omega)\times L^{\frac{s_0}{p-1}}(\Omega),\tag{4.8}$$

where $s_0=\frac{Np_0}{N-2p_0}$. By an argument similar to the one used to prove (4.6) and (4.7), we can show that

$$c(x)|\Lambda w_1|^{\alpha}|\Lambda w_2|^{\beta+1}\in L^{\frac{s_0}{p-1}}(\Omega)$$

and

$$c(x)|\Lambda w_1|^{\alpha+1}|\Lambda w_2|^{\beta}\in L^{\frac{s_0}{p-1}}(\Omega).$$

Then, in view of (H3) and (4.8), we have

$$\begin{cases}\left(a(x)\phi_p(\Lambda w_1)+c(x)|\Lambda w_1|^{\alpha-1}|\Lambda w_2|^{\beta+1}\Lambda w_1\right)\in L^{\frac{s_0}{p-1}}(\Omega),\\ \left(b(x)\phi_p(\Lambda w_2)+c(x)|\Lambda w_1|^{\alpha+1}|\Lambda w_2|^{\beta-1}\Lambda w_2\right)\in L^{\frac{s_0}{p-1}}(\Omega).\end{cases}\tag{4.9}$$

It is clear that

$$s_0=\frac{Np_0}{N-2p_0}\Longleftrightarrow p_0=\frac{Ns_0}{N+2s_0}.$$

Then,

$$p_0 < \frac{N}{2p'} = \frac{N(p-1)}{2p} \iff \frac{Ns_0}{N+2s_0} < \frac{N(p-1)}{2p} \iff \frac{N}{2} > \frac{s_0}{p-1}.$$

Then, again from Lemma 4.1 (a), Sobolev's embedding theorem, and the property of the Nemytskii operator induced by $\phi_{p'}$, it follows that

$$\Lambda\left(a(x)\phi_p(\Lambda w_1) + c(x)|\Lambda w_1|^{\alpha-1}|\Lambda w_2|^{\beta+1}\Lambda w_1\right) \in W^{2,\frac{s_0}{p-1}}(\Omega) \hookrightarrow L^{s_1}(\Omega),$$
$$\Lambda\left(b(x)\phi_p(\Lambda w_2) + c(x)|\Lambda w_1|^{\alpha+1}|\Lambda w_2|^{\beta-1}\Lambda w_2\right) \in W^{2,\frac{s_0}{p-1}}(\Omega) \hookrightarrow L^{s_1}(\Omega),$$

and

$$\phi_{p'}\left(\Lambda\left(a(x)\phi_p(\Lambda w_1) + c(x)|\Lambda w_1|^{\alpha-1}|\Lambda w_2|^{\beta+1}\Lambda w_1\right)\right) \in L^{\frac{s_1}{p'-1}}(\Omega) = L^{s_1(p-1)}(\Omega),$$
$$\phi_{p'}\left(\Lambda\left(b(x)\phi_p(\Lambda w_2) + c(x)|\Lambda w_1|^{\alpha+1}|\Lambda w_2|^{\beta-1}\Lambda w_2\right)\right) \in L^{\frac{s_1}{p'-1}}(\Omega) = L^{s_1(p-1)}(\Omega),$$

where $s_1 = \frac{Ns_0}{N(p-1)-2s_0}$. Thus, (4.4) implies that $(w_1, w_2) \in L^{p_1}(\Omega)\times L^{p_1}(\Omega)$ with $p_1 = s_1(p-1)$, i.e., $\frac{1}{p_1} = \frac{1}{p_0} - \frac{2p'}{N}$. This shows part (a) of the claim.

We now show part (b) of the claim. If $\frac{N}{2} < p_0$, then Lemma 4.1 (a) and Sobolev's embedding theorem imply that $(\Lambda w_1, \Lambda w_2) \in W^{2,p_0}(\Omega) \times W^{2,p_0}(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega}) \times C(\overline{\Omega})$. Then, in view of (H3), for any $s_2 > 1$, we have

$$\left(a(x)\phi_p(\Lambda w_1) + c(x)|\Lambda w_1|^{\alpha-1}|\Lambda w_2|^{\beta+1}\Lambda w_1\right) \in L^{s_2}(\Omega),$$
$$\left(b(x)\phi_p(\Lambda w_2) + c(x)|\Lambda w_1|^{\alpha+1}|\Lambda w_2|^{\beta-1}\Lambda w_2\right) \in L^{s_2}(\Omega).$$

This, together with Lemma 4.1 (a), yields that

$$\Lambda\left(a(x)\phi_p(\Lambda w_1) + c(x)|\Lambda w_1|^{\alpha-1}|\Lambda w_2|^{\beta+1}\Lambda w_1\right) \in W^{2,s_2}(\Omega),$$
$$\Lambda\left(b(x)\phi_p(\Lambda w_2) + c(x)|\Lambda w_1|^{\alpha+1}|\Lambda w_2|^{\beta-1}\Lambda w_2\right) \in W^{2,s_2}(\Omega).$$

By Sobolev's embedding theorem, we have $W^{2,s_2}(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega})$ for all $s_2 > \frac{N}{2}$. Hence, from (4.4), we see that $(w_1, w_2) \in C(\overline{\Omega}) \times C(\overline{\Omega})$.

If $\frac{N}{2p'} < p_0 < \frac{N}{2}$, then $\frac{N}{2} < \frac{s_0}{p-1}$. From Sobolev's embedding theorem, $W^{2,\frac{s_0}{p-1}}(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega})$. Note that

$$\Lambda\left(a(x)\phi_p(\Lambda w_1) + c(x)|\Lambda w_1|^{\alpha-1}|\Lambda w_2|^{\beta+1}\Lambda w_1\right) \in W^{2,\frac{s_0}{p-1}}(\Omega),$$
$$\Lambda\left(b(x)\phi_p(\Lambda w_2) + c(x)|\Lambda w_1|^{\alpha+1}|\Lambda w_2|^{\beta-1}\Lambda w_2\right) \in W^{2,\frac{s_0}{p-1}}(\Omega).$$

Then, from (4.4), we have $(w_1, w_2) \in C(\overline{\Omega}) \times C(\overline{\Omega})$.

If $p_0 = \frac{N}{2}$, then, from Lemma 4.1 (a), Sobolev's embedding theorem, and the property of the Nemytskii operator induced by ϕ_p, we have

$$(\Lambda w_1, \Lambda w_2) \in W^{2,p_0}(\Omega) \times W^{2,p_0}(\Omega) \hookrightarrow\hookrightarrow L^s(\Omega) \times L^s(\Omega)$$

and

$$(\phi_p(\Lambda w_1), \phi_p(\Lambda w_2)) \in L^{\frac{s}{p-1}}(\Omega) \times L^{\frac{s}{p-1}}(\Omega),$$

where $s > 1$ can be taken arbitrarily large. Then, as in showing (4.9), we have

$$\big(a(x)\phi_p(\Lambda w_1) + c(x)|\Lambda w_1|^{\alpha-1}|\Lambda w_2|^{\beta+1}\Lambda w_1\big) \in L^{\frac{s}{p-1}}(\Omega),$$
$$\big(b(x)\phi_p(\Lambda w_2) + c(x)|\Lambda w_1|^{\alpha+1}|\Lambda w_2|^{\beta-1}\Lambda w_2\big) \in L^{\frac{s}{p-1}}(\Omega).$$

Then, from Lemma 4.1 (a), we see that

$$\Lambda\left(a(x)\phi_p(\Lambda w_1) + c(x)|\Lambda w_1|^{\alpha-1}|\Lambda w_2|^{\beta+1}\Lambda w_1\right) \in W^{2,\frac{s}{p-1}}(\Omega),$$
$$\Lambda\left(b(x)\phi_p(\Lambda w_2) + c(x)|\Lambda w_1|^{\alpha+1}|\Lambda w_2|^{\beta-1}\Lambda w_2\right) \in W^{2,\frac{s}{p-1}}(\Omega).$$

For $s > 1$ large enough, by Sobolev's embedding theorem, we have $W^{2,\frac{s}{p-1}}(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega})$. This, together with (4.4), again implies that $(w_1, w_2) \in C(\overline{\Omega}) \times C(\overline{\Omega})$. This shows part (b) of the claim. Therefore, the claim is true.

Now, choose suitable $p_0 \in (1, p]$ and $k \in \mathbb{N}$ such that

$$p_{k-1} < \frac{N}{2p'} < p_k \quad \text{with } \frac{1}{p_k} = \frac{1}{p_0} - \frac{2p'}{N}k,$$

which further implies that

$$\frac{1}{p_j} = \frac{1}{p_{j-1}} - \frac{2p'}{N}, \quad j = 1, \ldots, k.$$

Then, applying part (a) of the claim with $p_0 = p_j$, $j = 1, \ldots, k-1$, successively, we deduce that $(w_1, w_2) \in L^{p_k}(\Omega) \times L^{p_k}(\Omega)$. Hence, by part (b) of the claim, $(w_1, w_2) \in C(\overline{\Omega}) \times C(\overline{\Omega})$. This completes the proof of the lemma. □

Remark 4.1. Let $\{\mu_n\} \subset (0, \infty)$ be a bounded sequence and $\{(w_{1,n}, w_{2,n})\} \subset L^p(\Omega) \times L^p(\Omega)$ be a sequence of pairs solving (4.3) with $\lambda = \mu_n$ in $L^{p'}(\Omega) \times L^{p'}(\Omega)$. Assume that $\mu_n \to \mu_0$ and the pair $\{(w_{1,n}, w_{2,n})\}$ is normalized by $\|w_{1,n}\|_p + \|w_{2,n}\|_p = 1$. Then, up to a subsequence, we may assume that

$$(w_{1,n}, w_{2,n}) \rightharpoonup (w_{1,0}, w_{2,0}) \in L^p(\Omega) \times L^p(\Omega).$$

Then, from the proof of Lemma 4.2, we can derive that

$$(w_{1,n}, w_{2,n}) \to (w_{1,0}, w_{2,0}) \in C(\overline{\Omega}) \times C(\overline{\Omega}),$$

i.e.,

$$\|w_{1,n} - w_{1,01}\|_\infty + \|w_{2,n} - w_{2,0}\|_\infty \to 0.$$

Moreover, $\|w_{1,0}\|_p + \|w_{2,0}\|_p = 1$ and $(w_{1,0}, w_{2,0}) \in L^p(\Omega) \times L^p(\Omega)$ solves (4.3) with $\lambda = \mu_0$ in $L^{p'}(\Omega) \times L^{p'}(\Omega)$.

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. Let the eigenpair $(\lambda_1, (u_1, v_1))$ be given as in Theorem 2.1. Assume that the conclusion is false. Then, there exists a sequence of eigenpairs $\{(\lambda_n, (u_n, v_n))\}_{n=2}^\infty$ of system (1.1) such that $\lambda_n \to \lambda_1$. As λ_1 is the smallest eigenvalue of system (1.1), we have $\lambda_n > \lambda_1$ for $n \geq 2$. So $\lambda_n \in (\lambda_1, \lambda_1 + \zeta)$ for some $\zeta > 0$ and $n > 2$. For $n \in \mathbb{N}$, let

$$w_{1,n} = -\Delta u_n \quad \text{and} \quad w_{2,n} = -\Delta v_n.$$

Without loss of generality, we may assume that $\|w_{1,n}\|_p + \|w_{2,n}\|_p = 1$. By Remark 4.1, passing to a subsequence if necessary, we have

$$\|w_{1,n} - w_{1,1}\|_\infty + \|w_{2,n} - w_{2,1}\|_\infty \to 0.$$

Then, from Lemma 4.1 (b), we obtain that

$$(\Lambda w_{1,n}, \Lambda w_{2,n}) \to (\Lambda w_{1,1}, \Lambda w_{2,1}) \quad \text{in } C^{1,\alpha}(\overline{\Omega}) \text{ for some } \alpha \in (0,1). \tag{4.10}$$

For any $n \in \mathbb{N}$, by Lemmas 4.2 and 4.3, we see that $(w_{1,n}, w_{2,n}) \in C(\overline{\Omega}) \times C(\overline{\Omega})$. Note that $\Lambda w_{1,1} = u_1 \geq 0$ and $\Lambda w_{2,1} = v_1 \geq 0$ in Ω. We first assume that $u_1 \not\equiv 0$ and $v_1 \not\equiv 0$. Then, from (4.4), $w_{1,1}$ and $w_{2,1}$ are nonnegative. This, together with Lemma 4.1 (c), further implies that

$$\Lambda w_{1,1} > 0 \text{ and } \Lambda w_{2,1} > 0 \text{ in } \Omega \quad \text{and} \quad \frac{\partial \Lambda w_{1,1}}{\partial n} < 0 \text{ and } \frac{\partial \Lambda w_{2,1}}{\partial n} < 0 \text{ on } \partial\Omega. \tag{4.11}$$

Moreover, since u_n changes sign in Ω for any $n \geq 2$, by Lemma 4.1 (c), we see that $w_{1,n}$, $\Lambda w_{1,n}$, $w_{2,n}$, and $\Lambda w_{2,n}$ all change sign in Ω. This contradicts (4.10) and (4.11). If one of u_1 and v_1 is identically zero, a similar contradiction can be derived. Hence, λ_1 is isolated. This completes the proof of the theorem. □

5. The principal eigenvalue of system (1.3)

In this section, we extend our results in Section 2 to system (1.3). Let the functionals I and J be defined by (2.1) and (2.2), respectively, and let the functional $K : E \to \mathbb{R}$ be defined by

$$K(u,v) = \frac{\alpha+1}{p} \int_\Omega a(x)|u|^p dx + \frac{\beta+1}{q} \int_\Omega b(x)|v|^q dx + \int_\Omega c(x)|u|^\alpha |v|^\beta uv dx, \tag{5.1}$$

where $(u,v) \in E$. Then, under (H1) and (H2), $K \in C^1(E, \mathbb{R})$ with the derivative given by

$$\begin{aligned}\langle K'(u,v), (\phi,\psi)\rangle &= (\alpha+1)\left(\int_\Omega a(x)|u|^{p-2}u\phi\, dx + \int_\Omega c(x)|u|^\alpha |v|^\beta v\phi\, dx\right) \\ &\quad +(\beta+1)\left(\int_\Omega b(x)|v|^{q-2}v\psi\, dx + \int_\Omega c(x)|u|^\alpha|v|^\beta u\psi\, dx\right)\end{aligned} \tag{5.2}$$

for all $(u,v), (\phi,\psi) \in E$.

Parallel to Definition 2.1 and Remark 2.1, we have the following:

Definition 5.1. *We say that $\lambda \in \mathbb{R}$ is an eigenvalue of system (1.3) if there exists $(u,v) \in E$ with $(u,v) \not\equiv (0,0)$ such that $\langle I'(u,v), (\phi,\psi)\rangle = \lambda \langle K'(u,v), (\phi,\psi)\rangle$ for all $(\phi,\psi) \in E$. The pair (u,v) is called an eigenfunction corresponding to the eigenvalue λ and the pair $(\lambda,(u,v))$ is called an eigenpair. The smallest eigenvalue $\lambda_1 > 0$ of system (1.3) is usually called the principal eigenvalue.*

Remark 5.1. From (2.3) and (5.2), we see that if $(\lambda,(u,v))$ is an eigenpair of system (1.3), then

$$\int_\Omega |\Delta u|^{p-2}\Delta u \Delta\phi\, dx = \lambda \int_\Omega a(x)|u|^{p-2}u\phi\, dx + \lambda \int_\Omega c(x)|u|^\alpha|v|^\beta v\phi\, dx$$

and

$$\int_\Omega |\Delta v|^{q-2}\Delta v \Delta\psi\, dx = \lambda \int_\Omega b(x)|v|^{q-2}v\psi\, dx + \lambda \int_\Omega c(x)|u|^\alpha|v|^\beta u\psi\, dx$$

for all $(\phi,\psi) \in E$.

Theorem 5.1. *Assume that (H1) and (H2) hold and let* $(\lambda_1, (u_1, v_1))$ *be given as in Theorem 2.1. Then,* $(\lambda_1, (u_1, v_1))$ *is an eigenpair of system* (1.3) *such that* $\lambda_1 > 0$ *is the principal eigenvalue of system* (1.3) *and the following properties hold:*

(a) *the set of all eigenfunctions corresponding to* λ_1 *forms a one-dimensional manifold* $E_1 \subset E$ *given by* $E_1 = \left\{(c_1 u_1, c_1^{p/q} v_1) \ : \ c_1 \in \mathbb{R}\right\}$*;*

(b) λ_1 *is the only eigenvalue of system* (1.1) *to which corresponds a component nonnegative eigenfunction.*

Proof. Since systems (1.1) and (1.3) coincide for nonnegative eigenfunctions, $(\lambda_1, (u_1, v_1))$ is an eigenpair of system (1.3). Assume that there exists another eigenpair $(\lambda_*, (u_*, v_*))$ of system (1.3) such that $0 < \lambda_* < \lambda_1$. Then,

$$\langle I'(u_*, v_*), (\phi, \psi)\rangle = \lambda_* \langle K'(u_*, v_*), (\phi, \psi)\rangle \quad \text{for all } (\phi, \psi) \in E.$$

Choosing $\phi = \frac{1}{p} u_*$ and $\psi = \frac{1}{q} v_*$. From (2.1), (2.3), (5.1), and (5.2), we have

$$\lambda_* = \frac{\langle I'(u_*, v_*), (u_*/p, v_*/q)\rangle}{\langle K'(u_*, v_*), (u_*/p, v_*/p)\rangle} = \frac{I(u_*, v_*)}{K(u_*, v_*)}. \tag{5.3}$$

Then, $K(u_*, v_*) > 0$ since $\lambda_* > 0$. Note from (2.2) and (5.1) that

$$\frac{J(u_*, v_*)}{K(u_*, v_*)} \geq 1.$$

Then, (5.3) yields that

$$\lambda_* = \frac{I(u_*, v_*)}{J(u_*, v_*)} \frac{J(u_*, v_*)}{K(u_*, v_*)} \geq \frac{I(u_*, v_*)}{J(u_*, v_*)}. \tag{5.4}$$

Let

$$\tilde{u} = \frac{|u_*|}{|J(u_*, v_*)|^{1/p}} \quad \text{and} \quad \tilde{v} = \frac{|v_*|}{|J(u_*, v_*)|^{1/q}}.$$

Then, we have

$$I(\tilde{u}, \tilde{v}) = \frac{I(u_*, v_*)}{J(u_*, v_*)} \quad \text{and} \quad J(\tilde{u}, \tilde{v}) = 1. \tag{5.5}$$

From (2.5), (5.4) and (5.5), it follows that $\lambda_* = I(\tilde{u}, \tilde{v}) \geq \lambda_1$, which is a contradiction. Thus, λ_1 is the principal eigenvalue of system (1.3).

Property (a) is clearly true. It can also be proved by the same argument as in showing property (a) of Theorem 2.1.

Property (b) follows from the equivalence of systems (1.1) and (1.3) for nonnegative eigenfunctions. This completes the proof of the theorem. □

Remark 5.2. Unlike in the proof of Theorem 2.1, Lemma 3.2 cannot be applied to system (1.3) to show the existence of principal eigenvalue. This is because condition (ii) of Lemma 3.2 is not satisfied with $\Psi = K$.

Finally, with trivial modifications of the proof of Theorem 2.2, we can prove the following theorem.

Theorem 5.2. *Assume that (H1) and (H3) hold. Then, the principal eigenvalue* λ_1 *of system* (1.3)*, given in Theorem 5.1, is isolated in the sense that there exists* $\zeta > 0$ *such that* λ_1 *is the only eigenvalue of system* (1.1) *contained in the interval* $(0, \lambda_1 + \zeta)$*.*

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