

PRINCIPAL SOLUTIONS REVISITED

STEPHEN CLARK, FRITZ GESZTESY, AND ROGER NICHOLS

Dedicated with admiration to Ludwig Streit on the occasion of his 75th birthday

ABSTRACT. The main objective of this paper is to identify principal solutions associated with Sturm–Liouville operators on arbitrary open intervals $(a, b) \subseteq \mathbb{R}$, as introduced by Leighton and Morse in the scalar context in 1936 and by Hartman in the matrix-valued situation in 1957, with Weyl–Titchmarsh solutions, as long as the underlying Sturm–Liouville differential expression is nonoscillatory (resp., disconjugate or bounded from below near an endpoint) and in the limit point case at the endpoint in question. In addition, we derive an explicit formula for Weyl–Titchmarsh functions in this case (the latter appears to be new in the matrix-valued context).

1. INTRODUCTION

We dedicate this paper to Ludwig Streit in great appreciation of the tremendous influence he exerted on all those who were permitted a glimpse at his boundless curiosity and approach to all aspects of science. We hope this modest contribution will create some joy for him.

The main focus of this paper centers around principal and Weyl–Titchmarsh solutions for general Sturm–Liouville operators (associated with three coefficients) on arbitrary open intervals $(a, b) \subseteq \mathbb{R}$. We will discuss in great detail the case of scalar coefficients p, q, r associated with the differential expression

$$\ell = \frac{1}{r} \left(-\frac{d}{dx} p \frac{d}{dx} + q \right), \quad -\infty \leq a < x < b \leq \infty, \quad (1.1)$$

and corresponding operator realizations in the Hilbert space $L^2((a, b); rdx)$, as well as the case of $m \times m$ matrix-valued coefficients P, Q, R , $m \in \mathbb{N}$, associated with the differential expression

$$L = R^{-1} \left(-\frac{d}{dx} P \frac{d}{dx} + Q \right), \quad -\infty \leq a < x < b \leq \infty, \quad (1.2)$$

and corresponding operator realizations in the Hilbert space $L^2((a, b); Rdx; \mathbb{C}^m)$.

Focusing in this introduction for reasons of brevity exclusively on the right end point b , if ℓ is nonoscillatory at b , (real-valued) *principal solutions* $u_b(\lambda, \cdot)$ of $\ell u =$

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λu , $\lambda \in \mathbb{R}$, are characterized by the condition that $u_b(\lambda, \cdot)$ does not vanish in a neighborhood $[c, b)$ of b (with $c \in (a, b)$) and that

$$\int_c^b dx p(x)^{-1} u_b(\lambda, x)^{-2} = \infty. \quad (1.3)$$

As discussed in Lemma 2.7, $u_b(\lambda, \cdot)$ is unique up to constant (possibly, λ -dependent) multiples and, in a certain sense (made precise in Lemma 2.7), also characterized as the smallest (minimal) possible solution of $\ell u = \lambda u$ near the endpoint b .

In contrast to (1.3), if ℓ is in the limit point case at b , *Weyl–Titchmarsh solutions* $\psi_+(z, \cdot)$ of $\ell u = zu$, $z \in \mathbb{C} \setminus \mathbb{R}$, are characterized by the condition that for some (and hence for all) $c \in (a, b)$,

$$\psi_+(z, \cdot) \in L^2((c, b); r dx) \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (1.4)$$

Again, $\psi_+(z, \cdot)$ is unique up to constant (generally, z -dependent) multiples.

Our main result, Theorem 2.13 in Section 2, then proves equality of these solutions (up to constant, possibly spectral parameter dependent multiples) under appropriate assumptions. More precisely, assuming ℓ to be nonoscillatory and in the limit point case at b , there exists $\lambda_b \in \mathbb{R}$, such that for all $\lambda < \lambda_b$, $x, x_0 \in (a, b)$, with x, x_0 beyond the last zero of $\psi_+(\lambda, \cdot)$, $u_b(\lambda, \cdot)$ (if any),

$$\psi_+(\lambda, x) \psi_+(\lambda, x_0)^{-1} = u_b(\lambda, x) u_b(\lambda, x_0)^{-1}. \quad (1.5)$$

Here, $\psi_+(\lambda, \cdot)$, $\lambda < \lambda_b$, denotes the extension of $\psi_+(z, \cdot)$, defined initially only for $z \in \mathbb{C} \setminus \mathbb{R}$, to real values $z < \lambda_b$. This extension is permitted on the basis that ℓ is assumed to be nonoscillatory and in the limit point case at b (cf. Remark 2.12).

We also recall Green's function formulas in terms of principal solutions and an explicit formula for the Weyl–Titchmarsh function at the end of Section 2, supposing the underlying limit point assumptions on ℓ .

In Section 3, the main new section in this paper, we prove the analogous results in the matrix-valued setting. We will be primarily concerned with self-conjugate solutions $U(\lambda, \cdot)$ of $LU = \lambda U$, $\lambda \in \mathbb{R}$, defined by the vanishing of the underlying $m \times m$ matrix-valued Wronskian,

$$W(U(\lambda, \cdot)^*, U(\lambda, \cdot)) = 0, \quad \lambda \in \mathbb{R}. \quad (1.6)$$

Focusing again exclusively on the endpoint b , a self-conjugate solution $U_b(\lambda, \cdot)$ of $LU = \lambda U$ that is invertible on $[c, b)$ for some $c \in (a, b)$ is called a *principal solution* of $LU = \lambda U$ at b if

$$\lim_{x \uparrow b} \left[\int_c^x dx' U_b(\lambda, x')^{-1} P(x')^{-1} [U_b(\lambda, x')^{-1}]^* \right]^{-1} = 0. \quad (1.7)$$

Again, by Lemma 3.6, $U_b(\lambda, \cdot)$ is unique up to right multiplication by invertible (possibly, λ -dependent) constant $m \times m$ matrices, and in a certain sense (detailed in Lemma 3.7) it represents the smallest (minimal) solution of $LU = \lambda U$ near the endpoint b .

In analogy to (1.4), if L is in the limit point case at b , *Weyl–Titchmarsh solutions* $\Psi_+(z, \cdot)$ of $LU = zU$, $z \in \mathbb{C} \setminus \mathbb{R}$, are then characterized by the condition that for some (and hence for all) $c \in (a, b)$, there exists an invertible $m \times m$ matrix-valued solution $\Psi_+(z, \cdot)$ of $LU = zU$ such that the $m \times m$ matrices

$$\int_c^b dx \Psi_+(z, x)^* R(x) \Psi_+(z, x), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (1.8)$$

exist. As in the context of principal solutions, $\Psi_+(z, \cdot)$ is unique up to right multiplication by (generally, z -dependent) invertible $m \times m$ matrices and it can be shown that $\Psi_+(z, \cdot)$ is self-conjugate.

Our main result, Theorem 3.11 in Section 3, once again proves equality of these solutions (up to right multiplication by possibly, spectral parameter dependent invertible $m \times m$ matrices) under appropriate assumptions. More precisely, assuming the existence of $\lambda_b \in \mathbb{R}$, such that $L - \lambda_b I$ is disconjugate on $[c, b)$ for all $c \in (a, b)$, and supposing L to be in the limit point case at b , then for all $\lambda < \lambda_b$, $x, x_0 \in (a, b)$, with x, x_0 beyond the last zero of $\det_{\mathbb{C}^m}(\Psi_+(\lambda, \cdot))$, $\det_{\mathbb{C}^m}(U_b(\lambda, \cdot))$ (if any),

$$\Psi_+(\lambda, x)\Psi_+(\lambda, x_0)^{-1} = U_b(\lambda, x)U_b(\lambda, x_0)^{-1}. \quad (1.9)$$

In addition, with the normalized $m \times m$ matrix-valued solutions $\Theta(z, \cdot, x_0)$ of $LU = zU$ defined by

$$\Theta(z, x_0, x_0) = I_m, \quad [P(x)\Theta'(z, x, x_0)](x_0)|_{x=x_0} = 0, \quad (1.10)$$

we will show the following formula for the $m \times m$ matrix-valued Weyl–Titchmarsh function associated with L ,

$$M_+(z, x_0) = -\lim_{x \uparrow b} \left[\int_{x_0}^x dx' \Theta(z, x', x_0)^{-1} P(x')^{-1} [\Theta(\bar{z}, x', x_0)^{-1}]^* \right]^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (1.11)$$

assuming L to be in the limit point case at b . If in addition, $Lu = \lambda_b u$ is disconjugate for some $\lambda_b \in \mathbb{R}$, then also

$$M_+(\lambda, x_0) = -\left[\int_{x_0}^b dx' \Theta(\lambda, x', x_0)^{-1} P(x')^{-1} [\Theta(\lambda, x', x_0)^{-1}]^* \right]^{-1}, \quad \lambda < \lambda_b, \quad (1.12)$$

holds, and

$$\begin{aligned} (\xi, M_+(\lambda, x_0)^{-1} \eta)_{\mathbb{C}^m} &= -\int_{x_0}^b dx' (\xi, \Theta(\lambda, x', x_0)^{-1} P(x')^{-1} [\Theta(\lambda, x', x_0)^{-1}]^* \eta)_{\mathbb{C}^m}, \\ &\lambda < \lambda_b, \quad \xi, \eta \in \mathbb{C}^m, \end{aligned} \quad (1.13)$$

exists as a Lebesgue integral. Both formulas, (1.11) and (1.12), are of independent interest and we know of no previous source that recorded them.

Concluding this introduction, we briefly summarize some of the notation used in this paper. If \mathcal{H} is a separable complex Hilbert space the symbol $(\cdot, \cdot)_{\mathcal{H}}$ denotes the scalar product in \mathcal{H} (linear in the second entry). If T is a linear operator mapping (a subspace of) a Hilbert space into another, $\text{dom}(T)$ denotes the domain of T . The spectrum and resolvent set of a closed linear operator in \mathcal{H} will be denoted by $\sigma(\cdot)$ and $\rho(\cdot)$, respectively. The closure of a closable operator S in \mathcal{H} is denoted by \bar{S} .

The Banach spaces of bounded and compact linear operators on \mathcal{H} are denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_{\infty}(\mathcal{H})$, respectively.

The symbol I_m , $m \in \mathbb{N}$, represents the identity operator in \mathbb{C}^m . The set of $m \times m$ matrices with complex-valued (resp., real-valued) entries is abbreviated by $\mathbb{C}^{m \times m}$ (resp., $\mathbb{R}^{m \times m}$), and similarly, $L^s((c, d); dx)^{m \times m}$ (resp., $L_{loc}^s((c, d); dx)^{m \times m}$) denotes the set of $m \times m$ matrices with entries in $L^s((c, d); dx)$ (resp., $L_{loc}^s((c, d); dx)$), where $s > 0$ and $a \leq c < d \leq b$. For notational simplicity, I represents the identity operator in $L^2((a, b); rdx)$ and also in $L^2((a, b); rdx; \mathbb{C}^m)$.

Finally, \mathbb{C}_+ (resp., \mathbb{C}_-) denotes the open complex upper (resp., lower) half-plane, and we will use the abbreviation ‘‘a.e.’’ for ‘‘Lebesgue almost everywhere.’’

2. BASIC FACTS ON SCALAR PRINCIPAL SOLUTIONS

In this preparatory section we recall some of the basic facts on oscillation theory with particular emphasis on principal solutions, a notion originally due to Leighton and Morse [70], in connection with scalar Sturm–Liouville operators on arbitrary open intervals $(a, b) \subseteq \mathbb{R}$.

We start by summarizing a few key results in the one-dimensional scalar case, whose extension to the matrix-valued context we are particularly interested in.

Our basic hypothesis in this section will be the following (however, we emphasize that all results in this section have been proved under more general conditions on the coefficients p, q , and for more general differential expressions ℓ , in [16]).

Hypothesis 2.1. *Let $-\infty \leq a < b \leq \infty$ and suppose that p, q, r are (Lebesgue) measurable on (a, b) , and that*

$$\begin{aligned} p > 0, r > 0 \text{ a.e. on } (a, b), \quad q \text{ is real-valued,} \\ 1/p, q, r \in L^1_{loc}((a, b); dx). \end{aligned} \quad (2.1)$$

Given Hypothesis 2.1, we consider the differential expression

$$\ell = \frac{1}{r} \left(-\frac{d}{dx} p \frac{d}{dx} + q \right), \quad -\infty \leq a < x < b \leq \infty, \quad (2.2)$$

and define the minimal operator T_{min} and maximal operator T_{max} in $L^2((a, b); rdx)$ associated with ℓ by

$$\begin{aligned} T_{min}u &= \ell u, \\ u \in \text{dom}(T_{min}) &= \{v \in L^2((a, b); rdx) \mid v, pv' \in AC_{loc}((a, b)); \\ &\quad \text{supp}(v) \subset (a, b) \text{ compact; } \ell v \in L^2((a, b); rdx)\}, \end{aligned} \quad (2.3)$$

$$\begin{aligned} T_{max}u &= \ell u, \\ u \in \text{dom}(T_{max}) &= \{v \in L^2((a, b); rdx) \mid v, pv' \in AC_{loc}((a, b)); \ell v \in L^2((a, b); rdx)\}, \end{aligned} \quad (2.4)$$

respectively. Here $AC_{loc}((a, b))$ denotes the set of locally absolutely continuous functions on (a, b) .

Then T_{min} is densely defined and [75, p. 64, 88]

$$T_{min}^* = T_{max}, \quad T_{max}^* = \overline{T_{min}}. \quad (2.5)$$

Remark 2.2. (i) In obvious notation, we will occasionally write $[p(x_0)u'(x_0)]$ for the quasi-derivative $pu'|_{x=x_0}$.

(ii) In the following we will frequently invoke solutions $u(z, \cdot)$ of $\ell u = zu$ for some $z \in \mathbb{C}$. Such solutions are always assumed to be distributional solutions, that is, we tacitly assume

$$u(z, \cdot), p(\cdot)u'(z, \cdot) \in AC_{loc}((a, b)) \quad (2.6)$$

in such a case.

Lemma 2.3 (cf., e.g., [33]). *Assume Hypothesis 2.1.*

(i) *Suppose $\ell u = \lambda u$ for some $\lambda \in \mathbb{R}$ with $u(\lambda, \cdot) \geq 0$ ($u(\lambda, \cdot) \not\equiv 0$) on (a, b) . Then $u(\lambda, \cdot) > 0$ on (a, b) .*

(ii) (*Harnack's inequality*). *Let $\mathcal{K} \subset (a, b)$ be compact and $\lambda \in \mathbb{R}$. Then there exists a $C_{\mathcal{K}, \lambda} > 0$ such that for all solutions $u(\lambda, \cdot) \geq 0$ satisfying $\ell u = \lambda u$, one has*

$$\sup_{x \in \mathcal{K}} (u(\lambda, x)) \leq C_{\mathcal{K}, \lambda} \inf_{x \in \mathcal{K}} (u(\lambda, x)). \quad (2.7)$$

Definition 2.4. Assume Hypothesis 2.1.

(i) Fix $c \in (a, b)$. Then ℓ is called *nonoscillatory near a* (resp., *b*) for some $\lambda \in \mathbb{R}$ if and only if every solution $u(\lambda, \cdot)$ of $\ell u = \lambda u$ has finitely many zeros in (a, c) (resp., (c, b)). Otherwise, ℓ is called *oscillatory near a* (resp., *b*).

(ii) Let $\lambda_0 \in \mathbb{R}$. Then T_{min} is bounded from below by λ_0 , and one writes $T_{min} \geq \lambda_0 I$, if

$$(u, [T_{min} - \lambda_0 I]u)_{L^2((a,b);rdx)} \geq 0, \quad u \in \text{dom}(T_{min}). \quad (2.8)$$

The following is a key result.

Theorem 2.5 ([39], [55], [82], [92]). Assume Hypothesis 2.1. Then the following assertions are equivalent:

- (i) T_{min} (and hence any symmetric extension of T_{min}) is bounded from below.
- (ii) There exists a $\lambda_0 \in \mathbb{R}$ such that ℓ is nonoscillatory near a and b for all $\lambda < \lambda_0$.
- (iii) For fixed $c \in (a, b)$, there exists a $\lambda_0 \in \mathbb{R}$ such that for all $\lambda < \lambda_0$, $\ell u = \lambda u$ has (real-valued) nonvanishing solutions $u_a(\lambda, \cdot) \neq 0$, $\hat{u}_a(\lambda, \cdot) \neq 0$ in a neighborhood $(a, c]$ of a , and (real-valued) nonvanishing solutions $u_b(\lambda, \cdot) \neq 0$, $\hat{u}_b(\lambda, \cdot) \neq 0$ in a neighborhood $[c, b)$ of b , such that

$$W(u_a(\lambda, \cdot), \hat{u}_a(\lambda, \cdot)) = 1, \quad u_a(\lambda, x) = o(\hat{u}_a(\lambda, x)) \text{ as } x \downarrow a, \quad (2.9)$$

$$W(u_b(\lambda, \cdot), \hat{u}_b(\lambda, \cdot)) = 1, \quad u_b(\lambda, x) = o(\hat{u}_b(\lambda, x)) \text{ as } x \uparrow b, \quad (2.10)$$

$$\int_a^c dx p(x)^{-1} u_a(\lambda, x)^{-2} = \int_c^b dx p(x)^{-1} u_b(\lambda, x)^{-2} = \infty, \quad (2.11)$$

$$\int_a^c dx p(x)^{-1} \hat{u}_a(\lambda, x)^{-2} < \infty, \quad \int_c^b dx p(x)^{-1} \hat{u}_b(\lambda, x)^{-2} < \infty. \quad (2.12)$$

Here

$$W(u, v)(x) = u(x)(pv')(x) - (pu')(x)v(x), \quad x \in (a, b), \quad (2.13)$$

denotes the Wronskian of u and v , assuming $u, (pu'), v, (pv') \in C((a, b))$. In particular, if $\ell u_j = z_j u_j$, $z_j \in \mathbb{C}$, then

$$\frac{d}{dx} W(u_1(z_1, x), u_2(z_2, x)) = (z_1 - z_2)r(x)u_1(z_1, x)u_2(z_2, x), \quad x \in (a, b). \quad (2.14)$$

Definition 2.6. Assume Hypothesis 2.1 and let $\lambda \in \mathbb{R}$. Then $u_a(\lambda, \cdot)$ (resp., $u_b(\lambda, \cdot)$) in Theorem 2.5 (iii) is called a *principal* (or *minimal*) solution of $\ell u = \lambda u$ at a (resp., b). A solution $\tilde{u}_a(\lambda, \cdot)$ (resp., $\tilde{u}_b(\lambda, \cdot)$) of $\ell u = \lambda u$ linearly independent of $u_a(\lambda, \cdot)$ (resp., $u_b(\lambda, \cdot)$) is called *nonprincipal* at a (resp., b).

Principal and nonprincipal solutions are well-defined due to Lemma 2.7 (i) below.

Lemma 2.7 ([39]). Assume Hypothesis 2.1.

(i) $u_a(\lambda, \cdot)$ and $u_b(\lambda, \cdot)$ in Theorem 2.5 (iii) are unique up to (nonvanishing) real constant multiples. Moreover, $u_a(\lambda, \cdot)$ and $u_b(\lambda, \cdot)$ are minimal solutions of $\ell u = \lambda u$ in the sense that

$$u(\lambda, x)^{-1} u_a(\lambda, x) = o(1) \text{ as } x \downarrow a, \quad (2.15)$$

$$u(\lambda, x)^{-1} u_b(\lambda, x) = o(1) \text{ as } x \uparrow b, \quad (2.16)$$

for any other solution $u(\lambda, \cdot)$ of $\ell u = \lambda u$ (which is nonvanishing near a , resp., b) with $W(u_a(\lambda, \cdot), u(\lambda, \cdot)) \neq 0$, respectively, $W(u_b(\lambda, \cdot), u(\lambda, \cdot)) \neq 0$.

(ii) Let $u(\lambda, \cdot)$ be any nonvanishing solution of $\ell u = \lambda u$ near a (resp., b). Then for $c_1 > a$ (resp., $c_2 < b$) sufficiently close to a (resp., b),

$$\hat{u}_a(\lambda, x) = u(\lambda, x) \int_x^{c_1} dx' p(x')^{-1} u(\lambda, x')^{-2} \quad (2.17)$$

$$\left(\text{resp., } \hat{u}_b(\lambda, x) = u(\lambda, x) \int_{c_2}^x dx' p(x')^{-1} u(\lambda, x')^{-2} \right) \quad (2.18)$$

is a nonprincipal solution of $\ell u = \lambda u$ at a (resp., b). If $\hat{u}_a(\lambda, \cdot)$ (resp., $\hat{u}_b(\lambda, \cdot)$) is a nonprincipal solution of $\ell u = \lambda u$ at a (resp., b) then

$$u_a(\lambda, x) = \hat{u}_a(\lambda, x) \int_a^x dx' p(x')^{-1} \hat{u}_a(\lambda, x')^{-2} \quad (2.19)$$

$$\left(\text{resp., } u_b(\lambda, x) = \hat{u}_b(\lambda, x) \int_x^b dx' p(x')^{-1} \hat{u}_b(\lambda, x')^{-2} \right) \quad (2.20)$$

is principal at a (resp., b).

The following two theorems describe a fundamental link between spectral theory and non-oscillation results.

Theorem 2.8 ([39]). *Assume Hypothesis 2.1 and let $\lambda_0 \in \mathbb{R}$. Then the following assertions are equivalent:*

- (i) $T_{min} \geq \lambda_0 I$.
- (ii) There exists a positive (distributional) solution $u > 0$ of $\ell v = \lambda_0 v$ on (a, b) .

For the proof of Theorem 2.8 one notes that Theorems XI.6.1 and XI.6.2 and Corollary XI.6.1 in Hartman's monograph [39] extend to our more general hypotheses on p, q, r without modifications. In particular, item (ii) implies item (i) by Jacobi's factorization identity

$$\begin{aligned} - (pg')' + h^{-1}(ph')'g &= -h^{-1}(ph^2(g/h)')', \\ 0 < h, ph' &\in AC_{loc}((a, b)), g \in \text{dom}(T_{min}). \end{aligned} \quad (2.21)$$

Theorem 2.9 (Dunford–Schwartz [15], Theorem XIII.7.40, [16], Section 11).

Suppose Hypothesis 2.1. Then the following assertions hold:

- (i) T_{min} is not bounded from below if and only if for all $\lambda \in \mathbb{R}$, every solution $u(\lambda, \cdot)$ of $\ell u = \lambda u$ has infinitely many zeros on (a, b) .
- (ii) If T_{min} is bounded from below and $\mu_0 = \inf(\sigma_{ess}(T))$ for some self-adjoint extension T of T_{min} , then, for $\lambda > \mu_0$, every solution $u(\lambda, \cdot)$ of $\ell u = \lambda u$ has infinitely many zeros on (a, b) , while, for $\lambda < \mu_0$, no solution $u(\lambda, \cdot)$ of $\ell u = \lambda u$ has infinitely many zeros on (a, b) .

Thus, the existence of positive solutions on (a, b) can be used to characterize $\inf(\sigma(T))$ while the existence of nonoscillatory solutions can be used to characterize $\inf(\sigma_{ess}(T))$. Without going into further details at this point, we note that under appropriate assumptions on the coefficients, these characterizations extend to elliptic partial differential operators. We also note that eigenvalue counts in essential spectral gaps in terms of (renormalized) oscillation theory in terms of zeros of Wronskians, rather than zeros of eigenfunctions, was established in [29]. For additional work in this direction we refer to [64]–[66].

In order to set up the connection between principal and Weyl–Titchmarsh solutions, we next recall Weyl’s definition of the limit point property of ℓ at the endpoint a (resp., b).

Definition 2.10. *Assume Hypothesis 2.1 and let $z \in \mathbb{C} \setminus \mathbb{R}$. Then ℓ is said to be in the limit point case (l.p.c.) at a (resp., b) if for some (and hence for all) $c \in (a, b)$, there exists a unique solution (up to constant multiples) $\psi_-(z, \cdot)$ (resp., $\psi_+(z, \cdot)$) of $\ell u = zu$ such that*

$$\psi_-(z, \cdot) \in L^2((a, c); rdx) \quad (\text{resp., } \psi_+(z, \cdot) \in L^2((c, b); rdx)). \quad (2.22)$$

The constants permitted in Definition 2.10 (while of course x -independent) are generally z -dependent.

One notes that L^2 -solutions $u_\pm(z, \cdot)$ of $\ell u = zu$ in a neighborhood of a and b always exist. What singles out the limit point case for ℓ at a or b is the uniqueness (up to constant multiples) of the L^2 -solution $\psi_-(z, \cdot)$, respectively, $\psi_+(z, \cdot)$ in Definition 2.10.

Any solution of $\ell u = zu$ satisfying the square integrability in (2.22) in a neighborhood of a (resp., b), independent of whether it is unique up to constant multiples or not, is called a *Weyl–Titchmarsh solution* of $\ell u = zu$ near a (resp., b).

We continue with the fact that nonoscillatory behavior at one end point plus a simple condition on r/p implies the limit point property at that endpoint:

Lemma 2.11 (Hartman [37], see also [16], Section 11, [26], [77], [82]).

Assume Hypothesis 2.1, let $c \in (a, b)$, and suppose that for some $\lambda_0 \in \mathbb{R}$, $\ell - \lambda_0$ is nonoscillatory near $d \in \{a, b\}$. Then, if

$$\left| \int_c^d dx [r(x)/p(x)]^{1/2} \right| = \infty, \quad (2.23)$$

ℓ is in the limit point case at d .

Hartman’s elegant proof of Lemma 2.11 in [37] is based on an application of (non)principal solutions of $\ell u = \lambda u$.

In this context we also mention the following limit point result due to Kurss [69] (for the special case $r = 1$): If for some $c \in (a, b)$, $q(x) \geq q_0(x)$ for a.e. $x \in (a, c]$ (resp., $x \in [c, b)$), and the reference coefficient q_0 gives rise to a differential expression $\ell_0 = r^{-1}[-(d/dx)p(d/dx) + q_0]$ being in the limit point case and nonoscillatory at a (resp., b), then $\ell = r^{-1}[-(d/dx)p(d/dx) + q]$ is in the limit point case at a (resp., b). Kurss uses a Sturm comparison theorem to show that if for some $\lambda \in \mathbb{R}$, $\ell u = \lambda u$ has a solution $u_0 > 0$ on $(a, c]$ (resp., $[c, b)$) dominating a solution $v_0 > 0$ of $\ell_0 v_0 = \lambda v_0$ on $(a, c]$ (resp., $[c, b)$) such that $v_0 \notin L^2((a, c); dx)$ (resp., $v_0 \notin L^2((c, b); dx)$) in the sense that

$$u_0 \geq v_0 > 0 \quad \text{on } (a, c] \quad (\text{resp., } [c, b)). \quad (2.24)$$

Thus, increasing the potential increases the non- L^2 -solution, in particular, $u_0 \notin L^2((a, c); dx)$ (resp., $u_0 \notin L^2((c, b); dx)$). We are indebted to Hubert Kalf [56] for kindly pointing out to us the relevance of [69] and for a detailed discussion of the general case $r \neq 1$.

Remark 2.12. Assuming ℓ to be nonoscillatory near a (resp., b) for some $\lambda_a \in \mathbb{R}$ (resp., $\lambda_b \in \mathbb{R}$), and in the limit point case at a (resp., b), one recalls that ψ_- (resp., ψ_+) in (2.22) analytically extends to $z < \lambda_a$ (resp., $z < \lambda_b$). In particular, for fixed

$x \in (a, b)$, $\psi_-(\cdot, x)$ (resp., $\psi_+(\cdot, x)$) is analytic in $\mathbb{C} \setminus [\lambda_a, \infty)$ (resp., $\mathbb{C} \setminus [\lambda_b, \infty)$). For more details in this context we refer to the comments following [29, Proposition 1.1].

Next, we fix a reference point $x_0 \in (a, b)$, and introduce the normalized solutions $\phi(z, \cdot, x_0)$ and $\theta(z, \cdot, x_0)$ of $\ell u = zu$ by

$$\begin{aligned} \phi(z, x_0, x_0) &= 0, & [p(x)\phi'(z, x, x_0)]_{x=x_0} &= 1, \\ \theta(z, x_0, x_0) &= 1, & [p(x)\theta'(z, x, x_0)]_{x=x_0} &= 0, \end{aligned} \quad (2.25)$$

with prime ' denoting $\partial/\partial x$, one infers (from the z -independence of the initial conditions in (2.25)) that for fixed $x \in (a, b)$, $\phi(\cdot, x, x_0)$ and $\theta(\cdot, x, x_0)$ are entire with respect to $z \in \mathbb{C}$ and that

$$W(\theta(z, \cdot, x_0), \phi(z, \cdot, x_0)) = 1, \quad z \in \mathbb{C}, \quad x_0 \in (a, b). \quad (2.26)$$

Consequently, if $u_{\pm}(z, \cdot)$ denote *any* nontrivial square integrable solutions of $\ell u = zu$ in a neighborhood of a and b , that is, for some (and hence for all) $c \in (a, b)$,

$$u_+(z, \cdot) \in L^2((c, b); rdx), \quad u_-(z, \cdot) \in L^2((a, c); rdx), \quad (2.27)$$

one obtains $u_{\pm}(z, x_0) \neq 0$, and

$$u_{\pm}(z, x)u_{\pm}(z, x_0)^{-1} = \theta(z, x, x_0) + \phi(z, x, x_0)m_{\pm}(z, x_0), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad x, x_0 \in (a, b), \quad (2.28)$$

for some coefficients $m_{\pm}(\cdot, x_0)$, the Weyl–Titchmarsh functions associated with ℓ .

The function $m_-(z, x_0)$ (resp., $m_+(z, x_0)$) is uniquely determined if and only if ℓ is in the limit point case at a (resp., b). In this case $u_-(z, \cdot)$ (resp., $u_+(z, \cdot)$) coincides up to z -dependent constant multiples with $\psi_-(z, \cdot)$ (resp., $\psi_+(z, \cdot)$) in (2.22).

Moreover, $\pm m_{\pm}(\cdot, x_0)$ are Nevanlinna–Herglotz functions, that is, for all $x_0 \in (a, b)$,

$$m_{\pm}(\cdot, x_0) \text{ are analytic in } \mathbb{C} \setminus \mathbb{R}, \quad (2.29)$$

and

$$\pm \operatorname{Im}(m_{\pm}(z, x_0)) > 0, \quad z \in \mathbb{C}_+. \quad (2.30)$$

In addition, for all $x_0 \in (a, b)$, $m_{\pm}(\cdot, x_0)$ satisfy

$$m_{\pm}(z, x_0) = \overline{m_{\pm}(\bar{z}, x_0)}, \quad z \in \mathbb{C}_+. \quad (2.31)$$

Finally, one also infers for all $z \in \mathbb{C} \setminus \mathbb{R}$, $x_0 \in (a, b)$,

$$W(u_+(z, \cdot), u_-(z, \cdot)) = [m_-(z, x_0) - m_+(z, x_0)]u_+(z, x_0)u_-(z, x_0), \quad (2.32)$$

$$m_{\pm}(z, x_0) = [p(x_0)u'_{\pm}(z, x_0)]/u_{\pm}(z, x_0). \quad (2.33)$$

Given these preparations we can finally state the main result of this section which identifies principal and Weyl–Titchmarsh solutions at an endpoint where ℓ is nonoscillatory and in the limit point case:

Theorem 2.13. *Assume Hypothesis 2.1.*

(i) *If ℓ is nonoscillatory and in the limit point case at a , then there exists $\lambda_a \in \mathbb{R}$, such that for all $\lambda < \lambda_a$, $x, x_0 \in (a, b)$, with x, x_0 to the left of the first zero of $\psi_-(\lambda, \cdot)$, $u_a(\lambda, \cdot)$ (if any),*

$$\psi_-(\lambda, x)\psi_-(\lambda, x_0)^{-1} = u_a(\lambda, x)u_a(\lambda, x_0)^{-1}, \quad (2.34)$$

that is, $\psi_-(\lambda, \cdot)$ and $u_a(\lambda, \cdot)$, $\lambda < \lambda_a$ are constant multiples of each other.

(ii) *If ℓ is nonoscillatory and in the limit point case at b , then there exists $\lambda_b \in \mathbb{R}$,*

such that for all $\lambda < \lambda_b$, $x, x_0 \in (a, b)$, with x, x_0 to the right of the last zero of $\psi_+(\lambda, \cdot)$, $u_b(\lambda, \cdot)$ (if any),

$$\psi_+(\lambda, x)\psi_+(\lambda, x_0)^{-1} = u_b(\lambda, x)u_b(\lambda, x_0)^{-1}, \quad (2.35)$$

that is, $\psi_+(\lambda, \cdot)$ and $u_b(\lambda, \cdot)$, $\lambda < \lambda_b$, are constant multiples of each other.

Proof. It suffices to consider the case of ψ_+ and u_b . Then, if ψ_+ is a nonprincipal solution of $\ell u = \lambda u$, Lemma 2.7(i) implies the existence of $C_+ > 0$ and $c \in (a, b)$, such that for all $\lambda < \lambda_b$ and for all $x \in (c, b)$,

$$|u_b(\lambda, x)| \leq C_+ |\psi_+(\lambda, x)|. \quad (2.36)$$

Thus, $u_b(\lambda, \cdot) \in L^2((c, b); rdx)$. But since by hypothesis ψ_+ and u_b are linearly independent, $W(\psi_+(\lambda, \cdot), u_b(\lambda, \cdot)) \neq 0$, this contradicts the limit point hypothesis at b which yields precisely one $L^2((c, b); rdx)$ -solution up to constant (generally, λ -dependent) multiples. \square

In particular, if T_{min} is bounded from below by $\lambda_0 \in \mathbb{R}$ and essentially self-adjoint, then for all $\lambda < \lambda_0$, principal and Weyl–Titchmarsh solutions at an endpoint coincide up to constant (λ -dependent) multiples.

We briefly follow up with the connection between Green’s functions and principal solutions for Sturm–Liouville operators, illustrating once more the relevance of principal solutions.

Lemma 2.14. *Assume Hypothesis 2.1 and suppose that $T_{min} \geq \lambda_0 I$ for some $\lambda_0 \in \mathbb{R}$. In addition, assume that ℓ is in the limit point case at a and b . Then*

$$\overline{T_{min}} = T_{max} := T \quad (2.37)$$

is the unique self-adjoint extension of T_{min} in $L^2((a, b); rdx)$ and for any $x_0 \in (a, b)$,

$$\begin{aligned} 0 < G(\lambda, x, x') &= \left(\int_{x_0}^b dt p(t)^{-1} u_a(\lambda, t)^{-2} \right) u_b(\lambda, x_0)^{-1} u_a(\lambda, x_0) \times \\ &\times \begin{cases} u_a(\lambda, x) u_b(\lambda, x'), & a < x \leq x' < b, \\ u_a(\lambda, x') u_b(\lambda, x), & a < x' \leq x < b, \end{cases} \quad \lambda < \lambda_0, \end{aligned} \quad (2.38)$$

is the positive Green’s function of T . Here we abbreviated

$$G(z, x, x') = (T - zI)^{-1}(x, x'), \quad x, x' \in (a, b), \quad z \in \mathbb{C} \setminus [\lambda_0, \infty). \quad (2.39)$$

As a consequence of Theorem 2.13, $u_a(\lambda, \cdot)$ and $u_b(\lambda, \cdot)$ in the Green’s function representation (2.38) can be replaced by $\psi_-(z, \cdot)$ and $\psi_+(z, \cdot)$. More precisely, an additional analytic continuation with respect to $z \in \mathbb{C} \setminus [\lambda_0, \infty)$ yields

$$\begin{aligned} G(z, x, x') &= \frac{1}{W(\psi_+(z, \cdot), \psi_-(z, \cdot))} \begin{cases} \psi_-(z, x) \psi_+(z, x'), & a < x \leq x' < b, \\ \psi_+(z, x) \psi_-(z, x'), & a < x' \leq x < b, \end{cases} \\ &z \in \mathbb{C} \setminus [\lambda_0, \infty), \end{aligned} \quad (2.40)$$

where for all $z \in \mathbb{C} \setminus [\lambda_0, \infty)$, $x_0 \in (a, b)$,

$$W(\psi_+(z, \cdot), \psi_-(z, \cdot)) = [m_-(z, x_0) - m_+(z, x_0)] \psi_+(z, x_0) \psi_-(z, x_0), \quad (2.41)$$

$$m_{\pm}(z, x_0) = [p(x_0) \psi'_{\pm}(z, x_0)] / \psi_{\pm}(z, x_0). \quad (2.42)$$

The material in Lemma 2.3–Theorem 2.9, Lemma 2.11, and Lemma 2.14 (and considerably more) is discussed in great detail in [33] (with special emphasis on the

Friedrichs extension T_F of T_{min}), and under more general conditions on ℓ and its coefficients in [16].

We conclude this section by recalling a known formula for $m_-(\cdot, x_0)$ (resp., $m_+(\cdot, x_0)$) whenever ℓ is in the limit point case and nonoscillatory at a (resp., b): Assuming the limit point case of ℓ at a (resp., b), it is well-known that

$$\begin{aligned} m_-(z, x_0) &= -\lim_{x \downarrow a} \theta(z, x, x_0) / \phi(z, x, x_0), \quad z \in \mathbb{C} \setminus \mathbb{R}, \\ (\text{resp., } m_+(z, x_0) &= -\lim_{x \uparrow b} \theta(z, x, x_0) / \phi(z, x, x_0), \quad z \in \mathbb{C} \setminus \mathbb{R}). \end{aligned} \quad (2.43)$$

Next, fix $z \in \mathbb{C}$ and suppose that $v(z, \cdot, x_0)$ satisfies $\ell u = zu$ and $v(z, x) \neq 0$ for $x \in [x_0, b)$, then clearly $w(z, \cdot, x_0)$ defined by

$$w(z, x) = v(z, x, x_0) \left[C_1 + C_2 \int_{x_0}^x dx' p(x')^{-1} v(z, x', x_0)^{-2} \right], \quad x \in [x_0, b), \quad (2.44)$$

is a solution of $\ell u = zu$ satisfying $W(v(z, \cdot, x_0), w(z, \cdot, x_0)) = C_2$. An elementary application of these facts to $\phi(z, \cdot, x_0)$ and $\theta(z, \cdot, x_0)$, taking into account that $\theta(z, x, x_0) \neq 0$, $z \in \mathbb{C} \setminus \mathbb{R}$, $x, x_0 \in (a, b)$, yields

$$\phi(z, x, x_0) = \theta(z, x, x_0) \int_{x_0}^x dx' p(x')^{-1} \theta(z, x', x_0)^{-2}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad x, x_0 \in (a, b). \quad (2.45)$$

Insertion of (2.45) into (2.43) yields the interesting formula,

$$m_+(z, x_0) = -\lim_{x \uparrow b} \left[\int_{x_0}^x dx' p(x')^{-1} \theta(z, x', x_0)^{-2} \right]^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (2.46)$$

If in addition, ℓ is nonoscillatory at b , analytic continuation of both sides in (2.46) with respect to z permits one to extend (2.46) to all $z \in \mathbb{C} \setminus [\lambda_b, \infty)$, with λ_b as in Theorem 2.13 (ii). We also note that for $\lambda < \lambda_b$, the expression

$$\left[\int_{x_0}^x dx' p(x')^{-1} \theta(\lambda, x', x_0)^{-2} \right]^{-1}, \quad \lambda < \lambda_b, \quad (2.47)$$

is strictly monotonically decreasing with respect to x and hence the existence of the limit of the integral in (2.47) as $x \uparrow b$ is guaranteed and one obtains

$$m_+(\lambda, x_0) = - \left[\int_{x_0}^b dx' p(x')^{-1} \theta(\lambda, x', x_0)^{-2} \right]^{-1}, \quad \lambda < \lambda_b, \quad (2.48)$$

with $\int_{x_0}^b dx' \dots$ in (2.48) representing a Lebesgue integral.

We first found (2.48) mentioned without proof in a paper by Kotani [59]. Kotani kindly alerted us to a paper by Kac and Krein [54], where such a formula is discussed near the end of their section 2, but the precise history of (2.48) is unknown to us at this point. We will provide a detailed derivation of (2.46), (2.48) in the matrix-valued context in Section 3.

Next, replacing $\phi(z, \cdot, x_0), \theta(z, \cdot, x_0)$ satisfying $\ell u = zu$ and (2.25) by the more general $\phi_\alpha(z, \cdot, x_0), \theta_\alpha(z, \cdot, x_0)$ satisfying $\ell u = zu$ and

$$\begin{aligned} \phi_\alpha(z, x_0, x_0) &= -\sin(\alpha), \quad [p(x)\phi'_\alpha(z, x, x_0)]|_{x=x_0} = \cos(\alpha), \\ \theta_\alpha(z, x_0, x_0) &= \cos(\alpha), \quad [p(x)\theta'_\alpha(z, x_0, x_0)]|_{x=x_0} = \sin(\alpha), \end{aligned} \quad (2.49)$$

for some $\alpha \in [0, \pi)$, and hence replacing (2.28) by

$$u_{\pm, \alpha}(z, x, x_0) = \theta_\alpha(z, x, x_0) + \phi_\alpha(z, x, x_0)m_{\pm, \alpha}(z, x_0), \quad (2.50)$$

$$z \in \mathbb{C} \setminus \mathbb{R}, \quad x, x_0 \in (a, b),$$

for appropriate Weyl–Titchmarsh coefficients $m_{\pm, \alpha}(\cdot, x_0)$, one obtains along the lines leading to (2.45) for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\phi_\alpha(z, x, x_0) = -\tan(\alpha)\theta_\alpha(z, x, x_0) + \theta_\alpha(z, x, x_0) \int_{x_0}^x \frac{dx'}{p(x')\theta_\alpha(z, x', x_0)^2}, \quad (2.51)$$

$$\alpha \in [0, \pi) \setminus \{\pi/2\},$$

$$\theta_\alpha(z, x, x_0) = -\cot(\alpha)\phi_\alpha(z, x, x_0) - \phi_\alpha(z, x, x_0) \int_{x_0}^x \frac{dx'}{p(x')\phi_\alpha(z, x', x_0)^2}, \quad (2.52)$$

$$\alpha \in (0, \pi),$$

and using

$$m_{+, \alpha}(z, x_0) = -\lim_{x \uparrow b} \theta_\alpha(z, x, x_0) / \phi_\alpha(z, x, x_0), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (2.53)$$

one now obtains

$$m_{+, \alpha}(z) = \begin{cases} \left[\tan(\alpha) - \lim_{x \uparrow b} \int_{x_0}^x dx' p(x')^{-1} \theta_\alpha(z, x', x_0)^{-2} \right]^{-1}, & \alpha \in [0, \pi) \setminus \{\pi/2\}, \\ \cot(\alpha) + \lim_{x \uparrow b} \int_{x_0}^x dx' p(x')^{-1} \phi_\alpha(z, x', x_0)^{-2}, & \alpha \in (0, \pi), \end{cases}$$

$$z \in \mathbb{C} \setminus \mathbb{R}, \quad (2.54)$$

whenever ℓ is in the limit point case at b (and similarly for $z < \lambda_{b, \alpha}$ for an appropriate $\lambda_{b, \alpha} \in \mathbb{R}$, and a Lebesgue integral $\int_{x_0}^b dx' \dots$ if ℓ is also nonoscillatory at b).

Replacing $\lim_{x \uparrow b} \int_{x_0}^x dx' \dots$ by $-\lim_{x \downarrow a} \int_x^{x_0} dx' \dots$, all formulas for $m_+(\cdot, x_0)$ and $m_{+, \alpha}(\cdot, x_0)$ immediately extend to $m_-(\cdot, x_0)$ and $m_{-, \alpha}(\cdot, x_0)$, assuming ℓ to be in the limit point case at a (and analogously if ℓ is also nonoscillatory at a).

3. MATRIX-VALUED PRINCIPAL SOLUTIONS

This section is devoted to an extension of some of the basic results on principal solutions of the previous Section 2 to those associated with matrix-valued singular Sturm–Liouville operators.

Matrix oscillation theory relevant to this paper originated with Hartman [38] and Reid [78]. The literature on oscillation theory for systems of differential equations is so rich by now that we cannot possibly offer a comprehensive list of references. Hence, we restrict ourselves primarily to a number of monographs by Coppel [14, Ch. 2], Hartman [39, Sects. X.10, X.11], Hille [41, Sect. 9.6], [42], Kratz [63, Chs. 4, 7], Reid [80, Ch. VII], [81, Ch. V], Rofe-Beketov and Kholkin [86, Chs. 1–4], and a few additional such as [4], [6], [9], [17], [21], [22], [23], [24], [36], [51], [79], [85], and [97].

The basic Weyl–Titchmarsh theory and general spectral theory for matrix-valued singular Sturm–Liouville operators as well as the more general case of singular Hamiltonian systems has been derived in detail by Hinton and Shaw [46]–[50] (we also refer to [8, Ch. 10], [10], [11], [12], [13], [22], [27], [30], [41, Sect. 10.7], [43], [44], [45], [52], [53], [60], [61], [62], [58], [71], [76], [83], [91], [96] for pertinent spectral results in this connection).

In the following we take these developments for granted and only focus on the required changes in Section 2 in connection with principal solutions which are implied by inherent noncommutativity issues due to the matrix-valued setting.

The basic assumptions for this section then read as follows:

Hypothesis 3.1. *Let $-\infty \leq a < b \leq \infty$ and suppose that $P, Q, R \in \mathbb{C}^{m \times m}$, $m \in \mathbb{N}$, have (Lebesgue) measurable entries on (a, b) , and that*

$$\begin{aligned} P > 0, R > 0 \text{ a.e. on } (a, b), \quad Q = Q^* \text{ is self-adjoint,} \\ P^{-1}, Q, R \in L^1_{loc}((a, b); dx)^{m \times m}. \end{aligned} \quad (3.1)$$

In addition, we introduce the Hilbert space of \mathbb{C}^m -valued elements,

$$\begin{aligned} L^2((a, b); Rdx; \mathbb{C}^m) = \left\{ U = (U_1, \dots, U_m)^\top, U_k \text{ (Lebesgue) measurable,} \right. \\ \left. 1 \leq k \leq m \mid \int_{(a, b)} dx (U(x), R(x)U(x))_{\mathbb{C}^m} < \infty \right\}, \end{aligned} \quad (3.2)$$

with associated scalar product

$$\begin{aligned} (U, V)_{L^2((a, b); Rdx; \mathbb{C}^m)} = \int_{(a, b)} dx (U(x), R(x)V(x))_{\mathbb{C}^m}, \\ U, V \in L^2((a, b); Rdx; \mathbb{C}^m). \end{aligned} \quad (3.3)$$

Here $(\dots)^\top$ indicates a column vector in \mathbb{C}^m and $(\cdot, \cdot)_{\mathbb{C}^m}$ represents the standard scalar product in \mathbb{C}^m , that is,

$$(w_1, w_2)_{\mathbb{C}^m} = \sum_{k=1}^m \overline{w_{1,k}} w_{2,k}, \quad w_j = (w_{j,1}, \dots, w_{j,m})^\top \in \mathbb{C}^m, \quad j = 1, 2. \quad (3.4)$$

Given Hypothesis 3.1, we consider the differential expression

$$L = R^{-1} \left(- \frac{d}{dx} P \frac{d}{dx} + Q \right), \quad -\infty \leq a < x < b \leq \infty, \quad (3.5)$$

and once more define the minimal operator T_{min} and maximal operator T_{max} in $L^2((a, b); Rdx; \mathbb{C}^m)$ associated with L by

$$\begin{aligned} T_{min} u &= Lu, \\ u \in \text{dom}(T_{min}) &= \{ v \in L^2((a, b); Rdx; \mathbb{C}^m) \mid v, Pv' \in AC_{loc}((a, b))^{m \times 1}; \\ &\quad \text{supp}(v) \subset (a, b) \text{ compact; } Lv \in L^2((a, b); Rdx; \mathbb{C}^m) \}, \end{aligned} \quad (3.6)$$

$$\begin{aligned} T_{max} u &= Lu, \\ u \in \text{dom}(T_{max}) &= \{ v \in L^2((a, b); Rdx; \mathbb{C}^m) \mid v, Pv' \in AC_{loc}((a, b))^{m \times 1}; \\ &\quad Lv \in L^2((a, b); Rdx; \mathbb{C}^m) \}, \end{aligned} \quad (3.7)$$

respectively. Here $AC_{loc}((a, b))^{m \times n}$ denotes the set of $m \times n$ matrices, $m, n \in \mathbb{N}$, with locally absolutely continuous entries on (a, b) (we will use the analogous in connection with $C((a, b))^{m \times n}$ below).

Again, T_{min} is densely defined and

$$T_{min}^* = T_{max}, \quad T_{max}^* = \overline{T_{min}}. \quad (3.8)$$

In the following, matrix-valued solutions $U(z, \cdot)$ of $LU = zU$ for some $z \in \mathbb{C}$, are always assumed to be distributional solutions, in addition, we either assume the vector-valued

$$u(z, \cdot), Pu'(z, \cdot) \in AC_{loc}((a, b))^{m \times 1}, \quad (3.9)$$

or the $m \times m$ matrix-valued case

$$U(z, \cdot), PU'(z, \cdot) \in AC_{loc}((a, b))^{m \times m}, \quad (3.10)$$

in this context. In fact, assuming $U, (PU'), V, (PV') \in C((a, b))^{m \times m}$, one introduces the matrix-valued Wronskian of u and v by

$$W(U, V)(x) = U(x)(PV')(x) - (PU')(x)V(x), \quad x \in (a, b), \quad (3.11)$$

and if U_j are $m \times m$ matrix solutions of $LU_j = z_j U_j$, $z_j \in \mathbb{C}$, then

$$\frac{d}{dx} W(U_1(\bar{z}_1, x)^*, U_2(z_2, x)) = (z_1 - z_2) U_1(\bar{z}_1, x)^* R(x) U_2(z_2, x), \quad x \in (a, b). \quad (3.12)$$

Definition 3.2. Assume Hypothesis 3.1 and let $z \in \mathbb{C} \setminus \mathbb{R}$. Then L is said to be in the limit point case (l.p.c.) at a (resp., b) if for some (and hence for all) $c \in (a, b)$, there exists a unique invertible $m \times m$ matrix-valued solution (up to constant multiples by right multiplication with invertible $m \times m$ matrices) $\Psi_-(z, \cdot)$ (resp., $\Psi_+(z, \cdot)$) of $LU = zU$ such that the $m \times m$ matrices

$$\int_a^c dx \Psi_-(z, x)^* R(x) \Psi_-(z, x) \quad \left(\text{resp., } \int_c^b dx \Psi_+(z, x)^* R(x) \Psi_+(z, x) \right) \quad (3.13)$$

exist.

Again, the constant invertible $m \times m$ matrices permitted in connection with right multiplication in Definition 3.2 (while of course x -independent) are generally z -dependent.

Given the analogy to the scalar case $m = 1$, any solution of $LU = zU$ satisfying the square integrability condition (3.13) in a neighborhood of a (resp., b), independent of uniqueness up to right multiplication by constant invertible matrices, will be called a ($m \times m$ matrix-valued) Weyl–Titchmarsh solution of $LU = zU$ near a (resp., b).

Remark 3.3. Assuming there exists a $\lambda_a \in \mathbb{R}$ (resp., $\lambda_b \in \mathbb{R}$) for which $(u, [T_{min} - \lambda_a I]u)_{L^2((a, b); Rdx; \mathbb{C}^m)} \geq 0$ for all $u \in \text{dom}(T_{min})$ with $u = 0$ in a neighborhood of b (resp., $(u, [T_{min} - \lambda_b I]u)_{L^2((a, b); Rdx; \mathbb{C}^m)} \geq 0$ for all $u \in \text{dom}(T_{min})$ with $u = 0$ in a neighborhood of a) and that L is in the limit point case at a (resp., b), then Ψ_- (resp., Ψ_+) in (3.13) analytically extends to $z < \lambda_a$ (resp., $z < \lambda_b$). In particular, for fixed $x \in (a, b)$, $\Psi_-(\cdot, x)$ (resp., $\Psi_+(\cdot, x)$) is analytic in $\mathbb{C} \setminus [\lambda_a, \infty)$ (resp., $\mathbb{C} \setminus [\lambda_b, \infty)$) (cf. the analogous Remark 2.12 in the scalar context).

Next, we turn to a brief summary of the principal facts of Weyl–Titchmarsh theory in the present matrix-valued context. Again, we fix a reference point $x_0 \in (a, b)$, and introduce the normalized $m \times m$ matrix-valued solutions $\Phi(z, \cdot, x_0)$ and $\Theta(z, \cdot, x_0)$ of $LU = zU$ by

$$\begin{aligned} \Phi(z, x_0, x_0) &= 0, & [P(x)\Phi'(z, x, x_0)]|_{x=x_0} &= I_m, \\ \Theta(z, x_0, x_0) &= I_m, & [P(x)\Theta'(z, x, x_0)]|_{x=x_0} &= 0, \end{aligned} \quad (3.14)$$

and note again that for fixed $x \in (a, b)$, $\Phi(\cdot, x, x_0)$ and $\Theta(\cdot, x, x_0)$ are entire with respect to $z \in \mathbb{C}$. Moreover, one verifies (cf., e.g., [11, Sect. 2], [31, Sect. 2]) that for any $z \in \mathbb{C}$, $x_0 \in (a, b)$,

$$W(\Theta(\bar{z}, \cdot, x_0)^*, \Phi(z, \cdot, x_0)) = I_m, \quad (3.15)$$

$$W(\Phi(\bar{z}, \cdot, x_0)^*, \Theta(z, \cdot, x_0)) = I_m, \quad (3.16)$$

$$W(\Phi(\bar{z}, \cdot, x_0)^*, \Phi(z, \cdot, x_0)) = 0, \quad (3.17)$$

$$W(\Theta(\bar{z}, \cdot, x_0)^*, \Theta(z, \cdot, x_0)) = 0, \quad (3.18)$$

as well as,

$$\Phi(z, x, x_0)\Theta(\bar{z}, x, x_0)^* - \Theta(z, x, x_0)\Phi(\bar{z}, x, x_0)^* = 0, \quad (3.19)$$

$$[P(x)\Phi'(z, x, x_0)][P(x)\Theta'(\bar{z}, x, x_0)]^* - [P(x)\Theta'(z, x, x_0)][P(x)\Phi'(\bar{z}, x, x_0)]^* = 0, \quad (3.20)$$

$$[P(x)\Phi'(z, x, x_0)]\Theta(\bar{z}, x, x_0)^* - [P(x)\Theta'(z, x, x_0)]\Phi(\bar{z}, x, x_0)^* = I_m, \quad (3.21)$$

$$\Theta(z, x, x_0)[P(x)\Phi'(\bar{z}, x, x_0)]^* - \Phi(z, x, x_0)[P(x)\Theta'(\bar{z}, x, x_0)]^* = I_m. \quad (3.22)$$

$$(3.23)$$

Consequently, if $U_{\pm}(z, \cdot)$ denote *any* invertible square integrable $m \times m$ matrix-valued solutions of $LU = zU$ in a neighborhood of a and b in the sense that for some (and hence for all) $c \in (a, b)$, the $m \times m$ matrices

$$\int_a^c dx U_-(z, x)^* R(x) U_-(z, x), \quad \int_c^b dx U_+(z, x)^* R(x) U_+(z, x), \quad (3.24)$$

exist, one obtains

$$U_{\pm}(z, x)U_{\pm}(z, x_0)^{-1} = \Theta(z, x, x_0) + \Phi(z, x, x_0)M_{\pm}(z, x_0), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad x, x_0 \in (a, b), \quad (3.25)$$

for some $m \times m$ matrix-valued coefficients $M_{\pm}(z, x_0) \in \mathbb{C}^{m \times m}$, the Weyl–Titchmarsh matrices associated with L .

Again, the matrix $M_-(z, x_0)$ (resp., $M_+(z, x_0)$) is uniquely determined if and only if L is in the limit point case at a (resp., b). In this case $U_-(z, \cdot)$ (resp., $U_+(z, \cdot)$) coincides up to right multiplication by z -dependent constant matrices with $\Psi_-(z, \cdot)$ (resp., $\Psi_+(z, x_0)$) in (3.13).

Moreover, $\pm M_{\pm}(\cdot, x_0)$ are $m \times m$ Nevanlinna–Herglotz matrices, that is, for all $x_0 \in (a, b)$,

$$M_{\pm}(\cdot, x_0) \text{ are analytic in } \mathbb{C} \setminus \mathbb{R}, \quad \text{rank}(M_{\pm}(z, x_0)) = m, \quad z \in \mathbb{C}_+, \quad (3.26)$$

and

$$\pm \text{Im}(M_{\pm}(z, x_0)) > 0, \quad z \in \mathbb{C}_+. \quad (3.27)$$

(Here, in obvious notation, $\text{Im}(M) = (2i)^{-1}(M - M^*)$, $M \in \mathbb{C}^{m \times m}$.) In addition, for all $x_0 \in (a, b)$, $M_{\pm}(\cdot, x_0)$ satisfy

$$M_{\pm}(z, x_0) = M_{\pm}(\bar{z}, x_0)^*, \quad z \in \mathbb{C}_+. \quad (3.28)$$

Finally, one also infers for all $z \in \mathbb{C} \setminus \mathbb{R}$, $x_0 \in (a, b)$,

$$W(U_+(\bar{z}, \cdot)^*, U_-(z, \cdot)) = U_+(\bar{z}, x_0)^*[M_-(z, x_0) - M_+(z, x_0)]U_-(z, x_0), \quad (3.29)$$

$$M_{\pm}(z, x_0) = P(x_0)U'_{\pm}(z, x_0)U_{\pm}(z, x_0)^{-1}. \quad (3.30)$$

Unraveling the crucial identities (3.28) and (3.30) results in the fundamental fact

$$U_{\pm}(\bar{z}, x)^*[P(x)U'_{\pm}(z, x)] = [P(x)U'_{\pm}(\bar{z}, x)]^*U_{\pm}(z, x), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (3.31)$$

for $x \in (a, b)$. In particular,

$$W(U_{\pm}(\bar{z}, \cdot)^*, U_{\pm}(z, \cdot)) = 0. \quad (3.32)$$

In other words, invertible square integrable $m \times m$ matrix-valued solutions of $LU = zU$ in the sense of (3.24) closely resemble *prepared* solutions in the sense of Hartman [38]. We use the term “closely resemble” as Hartman avoids the use of a complex spectral parameter and focuses on $z = 0$ instead. The term “prepared” did not stick as one finds also the notions of *conjoined*, *isotropic*, and *self-conjugate* solutions in the literature in connection with the property (3.31) (resp., (3.32)). Be that as it may, isolating property (3.31) was definitely a crucial step in the spectral analysis of systems of differential equations as the following observations will demonstrate.

Definition 3.4. *Assume Hypothesis 3.1, let $\lambda \in \mathbb{R}$, and suppose that $U(\lambda, \cdot)$ is an $m \times m$ matrix-valued solution of $LU = \lambda U$. Then $U(\lambda, \cdot)$ is called *self-conjugate* if*

$$W(U(\lambda, \cdot)^*, U(\lambda, \cdot)) = 0, \quad (3.33)$$

equivalently, if

$$U(\lambda, x)^*[P(x)U'(\lambda, x)] = [P(x)U'(\lambda, x)]^*U(\lambda, x), \quad x \in (a, b). \quad (3.34)$$

That is, $U(\lambda, x)^*[P(x)U'(\lambda, x)]$ is self-adjoint in (3.34) for all $x \in (a, b)$, and this is why we thought it most natural to follow those who adopted the term “self-conjugate” in connection with Definition 3.4. While we could have extended Definition 3.4 immediately to $\lambda \in \mathbb{C}$ along the lines of (3.31), (3.32), we are eventually aiming at principal matrix-valued solutions which are typically considered for $\lambda \in \mathbb{R}$.

Next, let $V_{-}(z, \cdot, c)$ (resp., $V_{+}(z, \cdot, c)$) be $m \times m$ matrix-valued solutions of $LU = zU$, invertible on the interval $(a, c]$ (resp., $[c, b)$) for some $c \in (a, b)$, satisfying property (3.31) on $(a, c]$ (resp., $[c, b)$). In particular,

$$W(V_{\pm}(\bar{z}, \cdot, c)^*, V_{\pm}(z, \cdot, c)) = 0. \quad (3.35)$$

We introduce

$$\begin{aligned} W_{-}(z, x, c) &= V_{-}(z, x, c) \left[C_{-,1} \right. \\ &\quad \left. - \int_x^c dx' V_{+}(z, x', c)^{-1} P(x')^{-1} [V_{+}(\bar{z}, x', c)^{-1}]^* C_{-,2} \right], \quad x \in (a, c], \end{aligned} \quad (3.36)$$

and

$$\begin{aligned} W_{+}(z, x, c) &= V_{+}(z, x, c) \left[C_{+,1} \right. \\ &\quad \left. + \int_c^x dx' V_{+}(z, x', c)^{-1} P(x')^{-1} [V_{+}(\bar{z}, x', c)^{-1}]^* C_{+,2} \right], \quad x \in [c, b), \end{aligned} \quad (3.37)$$

where $C_{\pm, j} \in \mathbb{C}^{m \times m}$, $j = 1, 2$. Then straightforward computations yield

$$LW_{-}(z, \cdot, c) = zW_{-}(z, \cdot, c) \quad \text{on } (a, c], \quad (3.38)$$

$$LW_{+}(z, \cdot, c) = zW_{+}(z, \cdot, c) \quad \text{on } [c, b), \quad (3.39)$$

$$W(V_{\pm}(\bar{z}, \cdot, c)^*, W_{\pm}(z, \cdot, c)) = C_{\pm, 2}, \quad (3.40)$$

$$W(W_{\pm}(\bar{z}, \cdot, c)^*, W_{\pm}(z, \cdot, c)) = C_{\pm,1}^* C_{\pm,2} - C_{\pm,2}^* C_{\pm,1}. \quad (3.41)$$

At this point we introduce the notion of matrix-valued principal solutions of $LU = \lambda U$, $\lambda \in \mathbb{R}$.

Definition 3.5. *Assume Hypothesis 3.1 and let $\lambda \in \mathbb{R}$.*

(i) *Suppose that $U_a(\lambda, \cdot)$ is a self-conjugate solution of $LU = \lambda U$ that is invertible on $(a, c]$ for some $c \in (a, b)$. Then $U_a(\lambda, \cdot)$ is called a principal solution of $LU = \lambda U$ at a if*

$$\lim_{x \downarrow a} \left[\int_x^c dx' U_a(\lambda, x')^{-1} P(x')^{-1} [U_a(\lambda, x')^{-1}]^* \right]^{-1} = 0. \quad (3.42)$$

(ii) *Suppose that $U_b(\lambda, \cdot)$ is a self-conjugate solution of $LU = \lambda U$ that is invertible on $[c, b)$ for some $c \in (a, b)$. Then $U_b(\lambda, \cdot)$ is called a principal solution of $LU = \lambda U$ at b if*

$$\lim_{x \uparrow b} \left[\int_c^x dx' U_b(\lambda, x')^{-1} P(x')^{-1} [U_b(\lambda, x')^{-1}]^* \right]^{-1} = 0. \quad (3.43)$$

Principal solutions, if they exist, are unique up to right multiplication with invertible constant $m \times m$ matrices:

Lemma 3.6. *Assume Hypothesis 3.1 and let $\lambda \in \mathbb{R}$. Then if a principal solution $U_a(\lambda, \cdot)$ at a (resp., $U_b(\lambda, \cdot)$ at b) of $LU = \lambda U$ exists, it is unique up to right multiplication with an invertible (generally, λ -dependent) constant $m \times m$ matrix.*

This follows from [14, Theorem 2.3], or [39, Theorem 10.5 (ii)].

Lemma 3.7. *Assume Hypothesis 3.1 and let $\lambda \in \mathbb{R}$. Suppose that $U_0(\lambda, \cdot)$ is a self-conjugate solution of $LU = \lambda U$ that is invertible on $(a, c]$ (resp., $[c, b)$) and let $V(\lambda, \cdot)$ be any $m \times m$ matrix-valued solution of $LU = \lambda U$. Then $U_0(\lambda, \cdot)$ is a principal solution at a (resp., b) and*

$$W(U_0(\lambda, \cdot)^*, V(\lambda, \cdot)) \text{ is invertible,} \quad (3.44)$$

if and only if $V(\lambda, \cdot)$ is invertible near a (resp., b) and

$$\lim_{x \downarrow a} V(\lambda, x)^{-1} U_0(\lambda, x) = 0 \quad (\text{resp., } \lim_{x \uparrow b} V(\lambda, x)^{-1} U_0(\lambda, x) = 0). \quad (3.45)$$

If (3.45) holds, then, for appropriate $c_a, c_b \in (a, b)$,

$$\left[\lim_{x \downarrow a} \int_x^{c_a} dx' V(\lambda, x')^{-1} P(x')^{-1} [V(\lambda, x')^{-1}]^* \right]^{-1} \text{ exists and is invertible} \quad (3.46)$$

$$\left(\text{resp., } \lim_{x \uparrow b} \left[\int_{c_b}^x dx' V(\lambda, x')^{-1} P(x')^{-1} [V(\lambda, x')^{-1}]^* \right]^{-1} \text{ exists and is invertible} \right).$$

Again, this follows from [14, Proposition 2.4], or [39, Theorem 10.5 (iii)].

Definition 3.8. *Assume Hypothesis 3.1. The equation $Lu = zu$, $z \in \mathbb{C}$, is called disconjugate on the interval $J \subseteq (a, b)$ if every nontrivial (i.e., not identically vanishing) solution v of $Lu = zu$ vanishes at most once on J .*

It is well known (cf., e.g., [14, Sect. 2.1], [39, Theorem XI.10.1]) that $Lu = zu$, $z \in \mathbb{C}$, is disconjugate on $J \subseteq (a, b)$ if and only if

$$\text{for any } x_j \in J \text{ and any } \eta_j \in \mathbb{C}^m, Lu = zu \text{ has a unique solution } v(z, \cdot) \quad (3.47)$$

$$\text{satisfying } v(z, x_j) = \eta_j, j = 1, 2.$$

Equivalently, $Lu = zu$, $z \in \mathbb{C}$, is disconjugate on $J \subseteq (a, b)$ if and only if for any $x_j \in J$, $j = 1, 2$, $x_1 \neq x_2$, $Lu = zu$ has a no nontrivial solution $v(z, \cdot)$ satisfying $v(z, x_j) = 0$, $j = 1, 2$.

(3.48)

We also recall the following useful result:

Theorem 3.9 ([14], Sect. 2.1–2.2, [39], Theorem XI.10.2). *Assume Hypothesis 3.1. (i) If J is a closed half-line (i.e., $J = [c, b)$ or $J = (a, c]$ for some $c \in (a, b)$), then $Lu = zu$, $z \in \mathbb{C}$, is disconjugate on J if and only if there exists a self-conjugate solution $U(z, \cdot)$ of $LU = zU$ such that $U(z, \cdot)$ is invertible on the interior of J .*

(ii) If J is a closed bounded subinterval of (a, b) or $J \subseteq (a, b)$ is an open interval, then $Lu = zu$, $z \in \mathbb{C}$, is disconjugate on J if and only if there exists a self-conjugate solution $U(z, \cdot)$ of $LU = zU$ such that $U(z, \cdot)$ is invertible on J .

Next, we derive the analog of (2.45)–(2.48) in the present matrix-valued context. Combining (3.14)–(3.18) and (3.35)–(3.36) yields the analog of (2.45),

$$\Phi(z, x, x_0) = \Theta(z, x, x_0) \int_{x_0}^x dx' \Theta(z, x', x_0)^{-1} P(x')^{-1} [\Theta(\bar{z}, x', x_0)^{-1}]^*, \quad (3.49)$$

$z \in \mathbb{C} \setminus \mathbb{R}$, $x, x_0 \in (a, b)$.

Moreover, assuming that L is in the limit point case at a (resp., b), it is known (cf., [46]) that the analog of (2.43) also holds in the form,

$$M_-(z, x_0) = -\lim_{x \downarrow a} \Phi(z, x, x_0)^{-1} \Theta(z, x, x_0), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (3.50)$$

(resp., $M_+(z, x_0) = -\lim_{x \uparrow b} \Phi(z, x, x_0)^{-1} \Theta(z, x, x_0)$, $z \in \mathbb{C} \setminus \mathbb{R}$).

Hence, we obtain the following formulas for $M_{\pm}(\cdot, x_0)$, the analogs of (2.46) and (2.48):

Theorem 3.10. *Assume Hypothesis 3.1 and suppose that L is in the limit point case at a (resp., b). Then*

$$M_-(z, x_0) = \lim_{x \downarrow a} \left[\int_x^{x_0} dx' \Theta(z, x', x_0)^{-1} P(x')^{-1} [\Theta(\bar{z}, x', x_0)^{-1}]^* \right]^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (3.51)$$

$$\left(\text{resp., } M_+(z, x_0) = -\lim_{x \uparrow b} \left[\int_{x_0}^x dx' \Theta(z, x', x_0)^{-1} P(x')^{-1} [\Theta(\bar{z}, x', x_0)^{-1}]^* \right]^{-1}, \right. \\ \left. z \in \mathbb{C} \setminus \mathbb{R} \right). \quad (3.52)$$

If, in addition, $Lu = \lambda_a u$ is disconjugate on $(a, x_0]$ (resp., $[x_0, b)$) for some $\lambda_a \in \mathbb{R}$ (resp., $\lambda_b \in \mathbb{R}$), then

$$M_-(\lambda, x_0) = \left[\int_a^{x_0} dx' \Theta(\lambda, x', x_0)^{-1} P(x')^{-1} [\Theta(\lambda, x', x_0)^{-1}]^* \right]^{-1}, \quad \lambda < \lambda_a, \quad (3.53)$$

$$\left(\text{resp., } M_+(\lambda, x_0) = -\left[\int_{x_0}^b dx' \Theta(\lambda, x', x_0)^{-1} P(x')^{-1} [\Theta(\lambda, x', x_0)^{-1}]^* \right]^{-1}, \right. \\ \left. \lambda < \lambda_b \right), \quad (3.54)$$

and

$$\begin{aligned} (\xi, M_-(\lambda, x_0)^{-1}\eta)_{\mathbb{C}^m} &= \int_a^{x_0} dx' (\xi, \Theta(\lambda, x', x_0)^{-1}P(x')^{-1}[\Theta(\lambda, x', x_0)^{-1}]^*\eta)_{\mathbb{C}^m}, \\ &\lambda < \lambda_a, \xi, \eta \in \mathbb{C}^m, \end{aligned} \quad (3.55)$$

$$\begin{aligned} \left(\text{resp., } (\xi, M_+(\lambda, x_0)^{-1}\eta)_{\mathbb{C}^m} &= - \int_{x_0}^b dx' (\xi, \Theta(\lambda, x', x_0)^{-1}P(x')^{-1} \right. \\ &\left. \times [\Theta(\lambda, x', x_0)^{-1}]^*\eta)_{\mathbb{C}^m}, \quad \lambda < \lambda_b, \xi, \eta \in \mathbb{C}^m \right), \end{aligned} \quad (3.56)$$

exists as a Lebesgue integral.

Proof. It suffices to focus on the endpoint b . Combining (3.49) and (3.50) (employing that L is l.p.c. at b) yields relation (3.52).

For the remainder of this proof we thus assume that $Lu = \lambda_b u$ is disconjugate on $[x_0, b)$ for some $\lambda_b \in \mathbb{R}$ (in addition to L being l.p.c. at b). Then,

$$\int_{x_0}^x dx' \Theta(\lambda, x', x_0)^{-1}P(x')^{-1}[\Theta(\lambda, x', x_0)^{-1}]^*, \quad \lambda < \lambda_b, \quad (3.57)$$

is strictly monotone increasing with respect to $x > x_0$. Recalling the well-known fact (cf. [7, Lemma 2.1]),

$$\begin{aligned} \text{If } 0 \leq C_1 \leq C_2 \leq \dots \leq C_\infty, \text{ with } C_n, C_\infty \in \mathcal{B}_\infty(\mathcal{H}), n \in \mathbb{N}, \\ \text{then } \lim_{n \rightarrow \infty} \|C_n - C\|_{\mathcal{B}(\mathcal{H})} = 0 \text{ for some } C \in \mathcal{B}_\infty(\mathcal{H}), \end{aligned} \quad (3.58)$$

one infers convergence of the $m \times m$ matrix $-\int_{x_0}^x dx' \dots$ to $-\int_{x_0}^b dx' \dots$ on the right-hand-side in (3.54) as $x \uparrow b$. In addition, the monotone convergence theorem implies the existence of

$$\int_{x_0}^b dx' (\xi, \Theta(\lambda, x', x_0)^{-1}P(x')^{-1}[\Theta(\lambda, x', x_0)^{-1}]^*\xi)_{\mathbb{C}^m}, \quad \lambda < \lambda_b, \xi \in \mathbb{C}^m, \quad (3.59)$$

as a Lebesgue integral. The general case depicted in (3.56) for $\xi, \eta \in \mathbb{C}^m$ then follows by polarization.

It remains to prove equality of $M_+(\lambda, x_0)$ with the right-hand side of (3.54) for $\lambda < \lambda_b$. We start by noting that disconjugacy of $Lu = \lambda_b u$ implies analyticity of $M_+(\cdot, x_0)$ on $\mathbb{C} \setminus [\lambda_b, \infty)$ and hence the fact that the $m \times m$ matrix-valued measure $\Omega(\cdot, x_0)$ in the Nevanlinna–Herglotz representation for $M_+(\cdot, x_0)$ is supported on $[\lambda_b, \infty)$, that is, one infers the representation,

$$\begin{aligned} M_+(z, x_0) &= A + \int_{[\lambda_b, \infty)} d\Omega(\lambda, x_0)[(\lambda - z)^{-1} - \lambda(1 + \lambda^2)^{-1}], \quad z \in \mathbb{C} \setminus [\lambda_b, \infty), \\ A = A^* \in \mathbb{C}^{m \times m}, \quad \int_{[\lambda_b, \infty)} d(\xi, \Omega(\lambda, x_0)\xi)_{\mathbb{C}^m}(1 + \lambda^2)^{-1} &< \infty, \quad \xi \in \mathbb{C}^m. \end{aligned} \quad (3.60)$$

Similarly, one infers that for each $x \in [x_0, b)$ the $m \times m$ matrix-valued function, $M_{+,x}(\cdot, x_0)$, defined by

$$\begin{aligned} M_{+,x}(z, x_0) &:= -\Phi(z, x, x_0)^{-1}\Theta(z, x, x_0) \\ &= - \left[\int_{x_0}^x dx' \Theta(z, x', x_0)^{-1}P(x')^{-1}[\Theta(\bar{z}, x', x_0)^{-1}]^* \right]^{-1}, \quad (3.61) \\ &z \in \mathbb{C} \setminus [\lambda_b, \infty), x \in (x_0, b), \end{aligned}$$

is meromorphic on \mathbb{C} and also analytic on $\mathbb{C} \setminus [\lambda_b, \infty)$. General Weyl–Titchmarsh theory in connection with the interval $[x_0, x]$, $x \in (x_0, b)$, where x_0, x are regular endpoints for L , yields that for fixed $x_0, y \in (a, b)$, $M_{+,y}(\cdot, x_0)$, and hence the $m \times m$ matrix-valued integral in (3.61), represents a matrix-valued meromorphic Herglotz–Nevanlinna function (cf. [46]). Indeed, employing (3.19) yields

$$M_{+,y}(z, x_0)^* = M_{+,y}(\bar{z}, x_0), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (3.62)$$

and introducing

$$U_y(z, x, x_0) = \Theta(z, x, x_0) + \Phi(z, x, x_0)[- \Phi(z, y, x_0)^{-1} \Theta(z, y, x_0)], \quad (3.63)$$

$$z \in \mathbb{C} \setminus \mathbb{R}, \quad x \in [x_0, y], \quad y \in (x_0, b),$$

a combination of (3.12) (for $z = \bar{z}_1 = z_2$), $U_y(z, y, x_0) = 0$, (3.15)–(3.18) imply the identity

$$\begin{aligned} \operatorname{Im}(M_{+,y}(z, x_0)) &= \operatorname{Im}(-\Phi(z, y, x_0)^{-1} \Theta(z, y, x_0)) \\ &= \operatorname{Im}(z) \int_{x_0}^y dx' U_y(z, x', x_0)^* R(x') U_y(z, x', x_0), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad y \in (x_0, b). \end{aligned} \quad (3.64)$$

Again, disconjugacy of $Lu = \lambda_b u$ implies that the $m \times m$ matrix-valued measure Ω_x associated with $M_{+,x}(\cdot, x_0)$ in (3.61), is again supported on $[\lambda_b, \infty)$, that is, for each $x \in (x_0, b)$,

$$\begin{aligned} M_{+,x}(z, x_0) &= -\Phi(z, x, x_0)^{-1} \Theta(z, x, x_0) \\ &= -\left[\int_{x_0}^x dx' \Theta(z, x', x_0)^{-1} P(x')^{-1} [\Theta(\bar{z}, x', x_0)^{-1}]^* \right]^{-1} \\ &= A_x + \int_{[\lambda_b, \infty)} d\Omega_x(\lambda, x_0) [(\lambda - z)^{-1} - \lambda(1 + \lambda^2)^{-1}], \quad z \in \mathbb{C} \setminus [\lambda_b, \infty), \end{aligned} \quad (3.65)$$

$$A_x = A_x^* \in \mathbb{C}^{m \times m}, \quad \int_{[\lambda_b, \infty)} d(\xi, \Omega_x(\lambda, x_0) \xi)_{\mathbb{C}^m} (1 + \lambda^2)^{-1} < \infty, \quad \xi \in \mathbb{C}^m.$$

In accordance with the limiting relation (3.50), the finite measures $d\Omega_x(\lambda, x_0)(1 + \lambda^2)^{-1}$ converge to $d\Omega(\lambda, x_0)(1 + \lambda^2)^{-1}$ as $x \uparrow b$ in the weak-* sense (cf. also [67]), that is,

$$\lim_{x \uparrow b} \int_{[\lambda_b, \infty)} d\Omega_x(\lambda, x_0) (1 + \lambda^2)^{-1} f(\lambda) = \int_{[\lambda_b, \infty)} d\Omega(\lambda, x_0) (1 + \lambda^2)^{-1} f(\lambda) \quad (3.66)$$

for all $f \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}; d\lambda)$. The Nevanlinna–Herglotz representation (3.65) for $M_{+,x}(\cdot, x_0)$ demonstrates that for any compact $K \subset \mathbb{C} \setminus [\lambda_b, \infty)$, there exists a constant $C(K) > 0$ such that $\|M_{+,x}(z, x_0)\|_{\mathcal{B}(\mathbb{C}^m)} \leq C(K)$ uniformly with respect to $z \in K$ and $x \in (x_0, b)$. An application of Vitali’s Theorem (see, e.g., [84, Sect. 7.3]) then proves that the convergence in (3.50) extends to

$$M_+(z, x_0) = -\lim_{x \uparrow b} \Phi(z, x, x_0)^{-1} \Theta(z, x, x_0), \quad z \in \mathbb{C} \setminus [\lambda_b, \infty), \quad (3.67)$$

in particular, it applies to $z < \lambda_b$ and hence yields (3.54). \square

We are not aware of any source containing formulas of the type (3.51)–(3.56). Naturally, these formulas extend to the more general self-adjoint boundary conditions at the regular endpoint $x_0 \in (a, b)$ discussed in detail in [46] (cf. also [10], [11]) in the matrix-valued context, extending the scalar case described in (2.49)–(2.54). We omit further details at this point.

The main result of this section then reads as follows.

Theorem 3.11. *Assume Hypothesis 3.1.*

(i) *Suppose that for some $\lambda_a \in \mathbb{R}$, $(u, [T_{min} - \lambda_a I]u)_{L^2((a,b); Rdx; \mathbb{C}^m)} \geq 0$ for all $u \in \text{dom}(T_{min})$ with $u = 0$ in a neighborhood of b . In addition, assume that L is in the limit point case at a . Then for all $\lambda < \lambda_a$, the Weyl–Titchmarsh solution $\Psi_-(\lambda, \cdot)$ is also a principal solution of $LU = \lambda U$ at a , that is, for x, x_0 to the left of the first zero of $\det_{\mathbb{C}^m}(\Psi_-(\lambda, \cdot)), \det_{\mathbb{C}^m}(U_a(\lambda, \cdot))$ (if any),*

$$\Psi_-(\lambda, x)\Psi_-(\lambda, x_0)^{-1} = U_a(\lambda, x)U_a(\lambda, x_0)^{-1}. \quad (3.68)$$

(ii) *Suppose that for some $\lambda_b \in \mathbb{R}$, $(u, [T_{min} - \lambda_b I]u)_{L^2((a,b); Rdx; \mathbb{C}^m)} \geq 0$ for all $u \in \text{dom}(T_{min})$ with $u = 0$ in a neighborhood of a . In addition, assume that L is in the limit point case at b . Then for all $\lambda < \lambda_b$, the Weyl–Titchmarsh solution $\Psi_+(\lambda, \cdot)$ is also a principal solution of $LU = \lambda U$ at b , that is, for x, x_0 to the right of the last zero of $\det_{\mathbb{C}^m}(\Psi_+(\lambda, \cdot)), \det_{\mathbb{C}^m}(U_b(\lambda, \cdot))$ (if any),*

$$\Psi_+(\lambda, x)\Psi_+(\lambda, x_0)^{-1} = U_b(\lambda, x)U_b(\lambda, x_0)^{-1}. \quad (3.69)$$

Proof. It suffices to consider item (ii). By [39, Theorem XI.10.3], the assumption on $T_{min} - \lambda_b I$ implies that for all $\lambda < \lambda_b$ and all $c \in (a, b)$, $Lu = \lambda u$ is disconjugate on $[c, b)$. By [14, Theorem 2.3] or [39, Theorem XI.10.5], $LU = \lambda U$ has a principal solution $U_b(\lambda, \cdot)$ for all $\lambda < \lambda_b$. Without loss of generality we may uniquely determine $U_b(\lambda, \cdot, x_0)$ by demanding the normalization $U_b(\lambda, x_0; x_0) = I_m$. As proved in [14, p. 44–45], it is possible to approximate $U_b(\lambda, \cdot, x_0)$ as follows: For $y \in (x_0, b)$, consider the unique solution $U_y(\lambda, \cdot; x_0)$, of $LU = \lambda U$, $\lambda < \lambda_b$, satisfying

$$U_y(\lambda, x_0, x_0) = I_m, \quad U_y(\lambda, y, x_0) = 0. \quad (3.70)$$

Then

$$U_b(\lambda, \cdot, x_0) = \lim_{y \uparrow b} U_y(\lambda, \cdot, x_0), \quad \lambda < \lambda_b. \quad (3.71)$$

In addition, one obtains that

$$U_y(\lambda, \cdot, x_0) = \Theta(\lambda, \cdot, x_0) + \Phi(\lambda, \cdot, x_0)M_{+,y}(\lambda; x_0), \quad (3.72)$$

with $M_{+,y}(\lambda; x_0)$ introduced in (3.61). Employing the convergence result (3.53), that is, $\lim_{y \uparrow b} M_{+,y}(\lambda, x_0) = M_+(\lambda, x_0)$, $\lambda < \lambda_b$, in (3.72) thus also yields

$$\lim_{y \uparrow b} U_y(\lambda, \cdot, x_0) = \Psi_+(\lambda, x)\Psi_+(\lambda, x_0)^{-1}, \quad \lambda < \lambda_b. \quad (3.73)$$

A comparison of (3.71) and (3.73) then proves

$$U_b(\lambda, \cdot, x_0) = \Psi_+(\lambda, x)\Psi_+(\lambda, x_0)^{-1}, \quad \lambda < \lambda_b, \quad (3.74)$$

completing the proof. \square

We emphasize that the continuity assumptions on the coefficients in L made in the context of oscillation theory in [14, Sect. 2.1], [39, Sect. XI.10] are not necessary and the quoted results in this section all extend to our current Hypothesis 3.1.

We also note that while we focused on Sturm–Liouville operators with matrix-valued coefficients, a treatment of more general singular Hamiltonian systems (along the lines of [11], [14, Ch. 2], [43]–[50], [78], [79], [80, Ch. VII], [81, Chs. V, VI], is clearly possible.

Emboldened by the results in Theorem 3.11 in the matrix context, one might guess that if $T_{min} \geq \lambda_0 I$ for some $\lambda_0 \in \mathbb{R}$, positivity of the solution $u(\lambda_0, \cdot)$ of $\ell u = \lambda_0 u$, or alternatively, $u \neq 0$ in Theorem 2.8 could be translated to the

matrix-valued case in a multitude of different ways. Let $U(\lambda_0, \cdot) \in \mathbb{C}^{m \times m}$ denote a matrix-valued solution of $LU = \lambda_0 U$, then here is a possible list of “positivity results” one could imagine in the matrix context from the outset:

- (I) $U \in \mathbb{C}^{m \times m}$ is invertible.
- (II) $U \in \mathbb{C}^{m \times m}$ is positive definite.
- (III) $U \in \mathbb{C}^{m \times m}$ is positivity preserving.
- (IV) $U \in \mathbb{C}^{m \times m}$ is positivity improving.

For completeness we briefly recall the notions of positivity preserving (resp., improving) matrices:

Definition 3.12. Let $A = (A_{j,k})_{1 \leq j, k \leq m} \in \mathbb{R}^{m \times m}$ for some $m \in \mathbb{N}$.

- (i) A is called positivity preserving if $A_{j,k} \geq 0$ for all $1 \leq j, k \leq m$.
- (ii) A is called positivity improving if $A_{j,k} > 0$ for all $1 \leq j, k \leq m$.

However, item (II) implies self-adjointness of $U(\lambda_0, \cdot)$ and hence upon invoking the equation adjoint to $LU = \lambda_0 U$, commutativity of $U(\lambda_0, \cdot)$ and $Q(\cdot)$. Our next example, a matrix-valued Schrödinger operator (i.e., $P(\cdot) = R(\cdot) = I_m$ in L), provides a simple counter-example to positive definiteness.

Example 3.13. Let $m = 2$, $(a, b) = \mathbb{R}$, $P(\cdot) = R(\cdot) = I_2$ a.e. on \mathbb{R} , and

$$Q(x) = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \quad (3.75)$$

in L . One verifies that

$$T_{min} \geq -2. \quad (3.76)$$

Taking $E = -2$, the general solution to $LU = -2U$ has the form

$$U(x) = \begin{pmatrix} U_{1,1}(x) & U_{1,2}(x) \\ U_{2,1}(x) & U_{2,2}(x) \end{pmatrix}, \quad x \in \mathbb{R}, \quad (3.77)$$

where

$$U_{1,1}(x) = c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x}, \quad (3.78)$$

$$U_{2,1}(x) = \tilde{c}_1 e^{2x} + \tilde{c}_2 e^{-2x} - (c_1/2)e^{\sqrt{2}x} - (c_2/2)e^{-\sqrt{2}x}, \quad (3.79)$$

$$U_{2,2}(x) = d_1 \cos(x) + d_2 \sin(x) + d_3 e^{\sqrt{7}x} + d_4 e^{-\sqrt{7}x}, \quad (3.80)$$

$$U_{1,2}(x) = \tilde{d}_1 e^{\sqrt{2}x} + \tilde{d}_2 e^{-\sqrt{2}x} - (d_1/3) \cos(x) - (d_2/3) \sin(x) \quad (3.81)$$

$$+ (d_3/5)e^{\sqrt{7}x} + (d_4/5)e^{-\sqrt{7}x}, \quad (3.82)$$

and $c_j, \tilde{c}_j, \tilde{d}_j$, $j \in \{1, 2\}$, and d_k , $k \in \{1, 2, 3, 4\}$ are arbitrary parameters. No solution of the form (3.77)–(3.82) is positive definite for all $x \in \mathbb{R}$. If such a solution were positive definite, it would commute with Q , so it suffices to show that solutions that commute with Q are not positive definite. By writing out $UQ = QU$, and equating corresponding matrix entries, one infers that U commutes with Q if and only if

$$\tilde{c}_1 = \tilde{c}_2 = d_1 = d_2 = d_3 = d_4 = 0, \quad c_1 = -2\tilde{d}_1, \quad c_2 = -2\tilde{d}_2, \quad (3.83)$$

with \tilde{d}_1 and \tilde{d}_2 arbitrary. (One could just as well arrive at (3.83) using self-adjointness of $U(x)$.) Taking (3.83) for granted, for a fixed choice of constants \tilde{d}_1 and \tilde{d}_2 , $U(x)$ has the form

$$U(x) = \begin{pmatrix} -2A(x) & A(x) \\ A(x) & 0 \end{pmatrix}, \quad x \in \mathbb{R}, \quad (3.84)$$

where we have set

$$A(x) = \tilde{d}_1 e^{\sqrt{2}x} + \tilde{d}_2 e^{-\sqrt{2}x}, \quad x \in \mathbb{R}. \quad (3.85)$$

One then computes the eigenvalues of $U(x)$ in (3.84) to be

$$\lambda_{\pm}(x) = -A(x) \pm \sqrt{2}|A(x)|, \quad x \in \mathbb{R}. \quad (3.86)$$

Since $\lambda_-(x) \leq 0$ for all $x \in \mathbb{R}$, $U(x)$ is not positive definite for any value of x , let alone for all $x \in \mathbb{R}$.

In addition, items (III) and (IV) are ruled out by the following elementary constant coefficient example:

Example 3.14. Let $m = 2$, $(a, b) = \mathbb{R}$, $P(\cdot) = R(\cdot) = I_2$ a.e. on \mathbb{R} , $q_0 \in \mathbb{R} \setminus \{0\}$, and

$$Q(x) = \begin{pmatrix} 0 & q_0 \\ q_0 & 0 \end{pmatrix}. \quad (3.87)$$

One verifies that

$$T_{min} \geq -|q_0|. \quad (3.88)$$

Assuming that $E \leq -|q_0| < 0$, let

$$\delta_{\pm}(E) = \sqrt{|E \pm q_0|}. \quad (3.89)$$

We claim that $U_{\infty}(E, \cdot)$, defined by

$$U_{\infty}(E, x) = \begin{pmatrix} e^{-\delta_-(E)x} & -e^{-\delta_+(E)x} \\ e^{-\delta_-(E)x} & e^{-\delta_+(E)x} \end{pmatrix}, \quad E \leq -|q_0| < 0, \quad x \in \mathbb{R}, \quad (3.90)$$

is a principal solution of $LU = EU$ at ∞ . That $U_{\infty}(E, \cdot)$ is self-conjugate follows from the observation that

$$(U_{\infty}(E, x'))^* U_{\infty}(E, x) = \begin{pmatrix} -2\delta_-(E)e^{-\delta_-(E)x} & 0 \\ 0 & -2\delta_+(E)e^{-\delta_+(E)x} \end{pmatrix}. \quad (3.91)$$

Since $\det(U_{\infty}(E, x)) = 2e^{-(\delta_-(E)+\delta_+(E))x}$, one infers that $U_{\infty}(E, \cdot)$ is invertible on \mathbb{R} . That this particular solution is principal at ∞ follows from the fact that

$$\begin{aligned} & \left[\int_0^x dx' U_{\infty}(E, x')^{-1} [U_{\infty}(E, x')^{-1}]^* \right]^{-1} \\ &= \begin{pmatrix} 4\delta_-(E)/(e^{2\delta_-(E)x} - 1) & 0 \\ 0 & 4\delta_+(E)/(e^{2\delta_+(E)x} - 1) \end{pmatrix} \xrightarrow{x \uparrow \infty} 0. \end{aligned} \quad (3.92)$$

Next, we turn to all principal solutions of this example and hence consider

$$\begin{aligned} \tilde{U}_{\infty}(E, x) &= \begin{pmatrix} e^{-\delta_-(E)x} & -e^{-\delta_+(E)x} \\ e^{-\delta_-(E)x} & e^{-\delta_+(E)x} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \epsilon \end{pmatrix} \\ &= \begin{pmatrix} \alpha e^{-\delta_-(E)x} - \gamma e^{-\delta_+(E)x} & \beta e^{-\delta_-(E)x} - \epsilon e^{-\delta_+(E)x} \\ \alpha e^{-\delta_-(E)x} + \gamma e^{-\delta_+(E)x} & \beta e^{-\delta_-(E)x} + \epsilon e^{-\delta_+(E)x} \end{pmatrix} \end{aligned} \quad (3.93)$$

with $\begin{pmatrix} \alpha & \beta \\ \gamma & \epsilon \end{pmatrix} \in \mathbb{C}^{2 \times 2}$ a nonsingular constant matrix.

By inspection, $\tilde{U}_{\infty}(-|q_0|, \cdot)$ is never positivity preserving (let alone, improving).

The question of positive vector solutions of $Lu = \lambda_0 u$ has been studied in the literature and we refer, for instance to [1], [2], [3], [5], [25], [93].

We conclude with the remark that the results presented in this section extend from the case of $m \times m$ matrix-valued coefficients to the situation of operator-valued coefficients in an infinite-dimensional, complex, separable Hilbert space. For instance, basic Weyl–Titchmarsh theory for the infinite-dimensional case has been derived by Gorbachuk [34], Gesztesy, Weikard, and Zinchenko [31], [32], Saito [87], [88], [89], [90] (see also [28], [35, Chs. 3, 4], [72], [73], [74], [86, Chs. 1–4], [94], [95]). For oscillation theoretic results in the infinite-dimensional context we refer, for example, to [18], [19], [20], [40], [57], [68]. A detailed treatment of this circle of ideas will appear elsewhere.

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REFERENCES

- [1] S. Ahmad, *On Sturmian theory for second order systems*, Proc. Amer. Math. Soc. **87**, 661–665 (1983).
- [2] S. Ahmad and A. C. Lazar, *Component properties of second order linear systems*, Bull. Amer. Math. Soc. **82**, 287–289 (1976).
- [3] S. Ahmad and A. C. Lazar, *On the components of extremal solutions of second order systems*, SIAM J. Math. Anal. **8**, 16–23 (1977).
- [4] S. Ahmad and A. C. Lazar, *An N -dimensional extension of the Sturm separation and comparison theory to a class of nonselfadjoint systems*, SIAM J. Math. Anal. **9**, 1137–1150 (1978).
- [5] S. Ahmad and A. C. Lazar, *Positive operators and Sturmian theory of nonselfadjoint second-order systems*, in *Nonlinear Equations in Abstract Spaces*, V. Lakshmikantham (ed.), Academic Press, New York, 1978, p 25–42.
- [6] S. Ahmad and J. A. Salazar, *Conjugate points and second order systems*, J. Math. Anal. Appl. **84**, 63–72 (1981).
- [7] G. D. Allen and F. J. Narcowich, *R -operators I. Representation Theory and Applications*, Indiana Univ. Math. J. **25**, 945–963 (1976).
- [8] F. V. Atkinson, *Discrete and Continuous Boundary Problems*, Academic Press, New York, 1964.
- [9] G. J. Butler, L. H. Erbe, and A. B. Mingarelli, *Riccati techniques and variational principles in oscillation theory for linear systems*, Trans. Amer. Math. Soc. **303**, 263–282 (1987).
- [10] S. Clark and F. Gesztesy, *Weyl–Titchmarsh M -function asymptotics for matrix-valued Schrödinger operators*, Proc. London Math. Soc. (3) **82**, 701–724 (2001).
- [11] S. Clark and F. Gesztesy, *Weyl–Titchmarsh M -function asymptotics and Borg-type theorems for Dirac-type operators*, Trans. Amer. Math. Soc. **354**, 3475–3534 (2002).
- [12] S. Clark, F. Gesztesy, *On Povzner–Wienholtz-type Self-Adjointness Results for Matrix-Valued Sturm–Liouville Operators*, Proc. Math. Soc. Edinburgh **133A**, 747–758 (2003).
- [13] S. Clark, F. Gesztesy, H. Holden, and B. M. Levitan, *Borg-type theorems for matrix-valued Schrödinger operators*, J. Diff. Eq. **167**, 181–210 (2000).
- [14] W. A. Coppel, *Disconjugacy*, Lecture Notes in Mathematics **220**, Springer, Berlin, 1971.
- [15] N. Dunford and J. T. Schwartz, *Linear Operators, Part II*, Wiley, Interscience Publ., New York, 1988.
- [16] J. Eckhardt, F. Gesztesy, R. Nichols, and G. Teschl, *Weyl–Titchmarsh theory for Sturm–Liouville operators with distributional potentials*, Opuscula Math. **33**, 467–563 (2013).
- [17] G. J. Etgen, *On the oscillation of solutions of second order self-adjoint matrix differential equations*, J. Diff. Eq. **6**, 187–195 (1969).

- [18] G. J. Etgen and R. T. Lewis, *A Hille-Wintner comparison theorem for second order differential systems*, Czech. Math. J. **30**, 98–107 (1980).
- [19] G. J. Etgen and J. F. Pawlowski, *Oscillation criteria for second order self adjoint differential systems*, Pac. J. Math. **66**, 99–110 (1976).
- [20] G. J. Etgen and J. F. Pawlowski, *A comparison theorem and oscillation criteria for second order differential systems*, Pac. J. Math. **72**, 59–69 (1977).
- [21] R. Fabbri, R. Johnson, and C. Núñez, *Rotation number for non-autonomous linear Hamiltonian systems I: Basic properties*, Z. angew. Math. Phys. **54**, 484–502 (2003).
- [22] R. Fabbri, R. Johnson, and C. Núñez, *Rotation number for non-autonomous linear Hamiltonian systems II: The Floquet coefficient*, Z. angew. Math. Phys. **54**, 652–676 (2003).
- [23] R. Fabbri, R. Johnson, and C. Núñez, *Disconjugacy and the rotation number for linear, non-autonomous Hamiltonian systems*, Ann. Mat. **185**, S3–S21 (2006).
- [24] R. Fabbri, R. Johnson, S. Novo, and C. Núñez, *Some remarks concerning weakly disconjugate linear Hamiltonian systems*, J. Math. Anal. Appl. **380**, 853–864 (2011).
- [25] F. Finster Zirker, *Node theorem for matrix Schrödinger operators*, Ph.D. Thesis, ETH-Zürich, 1995.
- [26] F. Gesztesy, *A complete spectral characterization of the double commutation method*, J. Funct. Anal. **117**, 401–446 (1993).
- [27] F. Gesztesy, A. Kiselev, and K. A. Makarov, *Uniqueness results for matrix-valued Schrödinger, Jacobi, and Dirac-type operators*, Math. Nachr. **239–240**, 103–145 (2002).
- [28] F. Gesztesy, N.J. Kalton, K.A. Makarov, and E. Tsekanovskii, *Some applications of operator-valued Herglotz functions*, in “Operator Theory, System Theory and Related Topics,” Oper. Theory Adv. Appl., Vol. 123, Birkhäuser, Basel, 2001, pp. 271–321.
- [29] F. Gesztesy, B. Simon, and G. Teschl, *Zeros of the Wronskian and renormalized oscillation theory*, Amer. J. Math. **118**, 571–594 (1996).
- [30] F. Gesztesy and E. Tsekanovskii, *On matrix-valued Herglotz functions*, Math. Nachr. **218**, 61–138 (2000).
- [31] F. Gesztesy, R. Weikard, and M. Zinchenko, *Initial value problems and Weyl–Titchmarsh theory for Schrödinger operators with operator-valued potentials*, Operators and Matrices **7**, 241–283 (2013).
- [32] F. Gesztesy, R. Weikard, and M. Zinchenko, *On Spectral Theory for Schrödinger Operators with Operator-Valued Potentials*, J. Diff. Eq. **255**, 1784–1827 (2013).
- [33] F. Gesztesy and Z. Zhao, *On critical and subcritical Sturm-Liouville Operators*, J. Funct. Anal. **98**, 311–345 (1991).
- [34] M. L. Gorbachuk, *On spectral functions of a second order differential operator with operator coefficients*, Ukrain. Math. J. **18**, No. 2, 3–21 (1966). (Russian.) Engl. transl. in Amer. Math. Soc. Transl. (2), **72**, 177–202 (1968).
- [35] V. I. Gorbachuk and M. L. Gorbachuk, *Boundary Value Problems for Operator Differential Equations*, Kluwer, Dordrecht, 1991.
- [36] E. Gutkin and R. Johnson, *Intersection theory for linear eigenvalue problems*, J. reine angew. Math. **401**, 1–24 (1989).
- [37] P. Hartman, *Differential equations with non-oscillatory eigenfunctions*, Duke Math. J. **15**, 697–709 (1948).
- [38] P. Hartman, *Self-adjoint, non-oscillatory systems of ordinary, second order, linear differential equations*, Duke Math. J. **24**, 25–35 (1957).
- [39] P. Hartman, *Ordinary Differential Equations*, Birkhäuser, Boston, 1982.
- [40] T. L. Hayden and H. C. Howard, *Oscillation of differential equations in Banach spaces*, Ann. Mat. Pura Appl. **85**, 385–394 (1970).
- [41] E. Hille, *Lectures on Ordinary Differential Equations*, Addison-Wesley, Reading, Ma, 1969.
- [42] D. Hinton, *Principal solutions of positive linear Hamiltonian systems*, J. Austral. Math. Soc. **22A**, 411–420 (1976).
- [43] D. B. Hinton and A. Schneider, *On the Titchmarsh-Weyl coefficients for singular S-Hermitian Systems I*, Math. Nachr. **163**, 323–342 (1993).
- [44] D. B. Hinton and A. Schneider, *On the Titchmarsh-Weyl coefficients for singular S-Hermitian Systems II*, Math. Nachr. **185**, 67–84 (1997).
- [45] D. Hinton and A. Schneider, *On the spectral representation for singular selfadjoint boundary eigenvalue problems*, in *Contributions to Operator Theory in Spaces with an Indefinite Metric. The Heinz Langer Anniversary Volume*, A. Dijksma, I. Gohberg, M. A. Kaashoek,

- R. Mennicken (eds.), *Operator Theory: Advances and Applications*, Vol. 106, Birkhäuser–Springer, Basel, 1998, pp. 217–251.
- [46] D. B. Hinton and J. K. Shaw, *On Titchmarsh-Weyl $M(\lambda)$ -functions for linear Hamiltonian systems*, *J. Diff. Eqs.* **40**, 316–342 (1981).
- [47] D. B. Hinton and J. K. Shaw, *On the spectrum of a singular Hamiltonian system*, *Quaest. Math.* **5**, 29–81 (1982).
- [48] D. B. Hinton and J. K. Shaw, *Hamiltonian systems of limit point or limit circle type with both endpoints singular*, *J. Diff. Eqs.* **50**, 444–464 (1983).
- [49] D. B. Hinton and J. K. Shaw, *On boundary value problems for Hamiltonian systems with two singular points*, *SIAM J. Math. Anal.* **15**, 272–286 (1984).
- [50] D. B. Hinton and J. K. Shaw, *On the spectrum of a singular Hamiltonian system, II*, *Quaest. Math.* **10**, 1–48 (1986).
- [51] V. A. Jakubovič, *Oscillatory properties of the solutions of canonical equations*, *Amer. Math. Soc. Transl., Ser. 2*, Vol. 42, 247–288 (1964).
- [52] R. A. Johnson, *m-Functions and Floquet exponents for linear differential systems*, *Ann. Mat. Pura Appl., Ser. 4*, **147**, 211–248 (1987).
- [53] R. Johnson, S. Novo, and R. Obaya, *Ergodic properties and Weyl M -functions for random linear Hamiltonian systems*, *Proc. Roy. Soc. Edinburgh* **130A**, 1045–1079 (2000).
- [54] I. S. Kac and M. G. Krein, *On the spectral functions of the string*, *Amer. Math. Soc. Transl. (2)* **10**, 19–102 (1974).
- [55] H. Kalf, *A characterization of the Friedrichs extension of Sturm-Liouville operators*, *J. London Math. Soc. (2)* **17**, 511–521 (1978).
- [56] H. Kalf, private communication, April and May 2015.
- [57] A. G. Kartsatos and T. Walters, *Origins of oscillation criteria of operator differential equations in Hilbert space*, *J. Math. Anal. Appl.* **67**, 12–16 (1979).
- [58] V. I. Kogan and F. S. Rofe-Beketov, *On square-integrable solutions of symmetric systems of differential equations of arbitrary order*, *Proc. Roy. Soc. Edinburgh* **74A**, 1–40 (1974).
- [59] S. Kotani, *Support theorems for random Schrödinger operators*, *Commun. Math. Phys.* **97**, 443–452 (1985).
- [60] S. Kotani and B. Simon, *Stochastic Schrödinger operators and Jacobi matrices on a strip*, *Commun. Math. Phys.* **119**, 403–429 (1988).
- [61] A. M. Krall, *$M(\lambda)$ theory for singular Hamiltonian systems with one singular point*, *SIAM J. Math. Anal.* **20**, 664–700 (1989).
- [62] A. M. Krall, *$M(\lambda)$ theory for singular Hamiltonian systems with two singular points*, *SIAM J. Math. Anal.* **20**, 701–715 (1989).
- [63] W. Kratz, *Quadratic Functionals in Variational Analysis and Control Theory*, Akademie Verlag, Berlin, 1995.
- [64] H. Krüger and G. Teschl, *Effective Prüfer angles and relative oscillation criteria*, *J. Diff. Eq.* **245**, 3823–3848 (2008).
- [65] H. Krüger and G. Teschl, *Relative oscillation theory for Sturm-Liouville operators extended*, *J. Funct. Anal.* **254**, 1702–1720 (2008).
- [66] H. Krüger and G. Teschl, *Relative oscillation theory, weighted zeros of the Wronskian, and the spectral shift function*, *Commun. Math. Phys.* **287**, 613–640 (2009).
- [67] D. Krutikov and C. Remling, *Schrödinger operators with sparse potentials: Asymptotics of the Fourier transform of the spectral measure*, *Commun. Math. Phys.* **223**, 509–532 (2001).
- [68] M. R. S. Kulenović, *On a result of Etgen and Lewis*, *Czech. Math. J.* **32**, 373–376 (1982).
- [69] H. Kurss, *A limit-point criterion for nonoscillatory Sturm-Liouville differential operators*, *Proc. Amer. Math. Soc.* **18**, 445–449 (1967).
- [70] W. Leighton and M. Morse, *Singular quadratic functionals*, *Trans. Amer. Math. Soc.* **40**, 252–286 (1936).
- [71] M. Lesch and M. Malamud, *On the deficiency indices and self-adjointness of symmetric Hamiltonian systems*, *J. Diff. Eq.* **189**, 556–615 (2003).
- [72] M. Malamud and H. Neidhardt, *Sturm-Liouville boundary value problems with operator potentials and unitary equivalence*, *J. Diff. Eq.* **252**, 5875–5922 (2012).
- [73] V. I. Mogilevskii, *Description of spectral functions of differential operators with arbitrary deficiency indices*, *Math. Notes* **81**, 553–559 (2007).
- [74] V. Mogilevskii, *Boundary triplets and Titchmarsh-Weyl functions of differential operators with arbitrary deficiency indices*, *Meth. Funct. Anal. Topology* **15**, 280–300 (2009).

- [75] M. A. Naimark, *Linear Differential Operators, Part II*, Ungar, New York, 1968.
- [76] S. A. Orlov, *Nested matrix disks analytically depending on a parameter, and theorems on the invariance of ranks of radii of limiting disks*, Math. USSR Izv. **10**, 565–613 (1976).
- [77] T. T. Read, *Limit-point criteria for polynomials in a non-oscillatory expression*, Proc. Roy. Soc. Edinburgh **76A**, 13–29 (1976).
- [78] W. T. Reid, *Principal solutions of non-oscillatory self-adjoint linear differential systems*, Pac. J. Math. **8**, 147–169 (1958).
- [79] W. T. Reid, *Principal solutions of nonoscillatory linear differential systems*, J. Math. Anal. **9**, 397–423 (1964).
- [80] W. T. Reid, *Ordinary Differential Equations*, Wiley, New York, 1971.
- [81] W. T. Reid, *Sturmian Theory for Ordinary Differential Equations*, Springer, Berlin, 1980.
- [82] F. Rellich, *Halbbeschränkte gewöhnliche Differentialoperatoren zweiter Ordnung*, Math. Ann. **122**, 343–368 (1951).
- [83] C. Remling, *Geometric characterization of singular self-adjoint boundary conditions for Hamiltonian systems*, Appl. Anal. **60**, 49–61 (1996).
- [84] R. Remmert, *Classical Topics in Complex Function Theory*, Graduate Texts in Math., Vol. 172, Springer, 1998.
- [85] F. S. Rofo-Beketov and A. M. Hol'kin, *On the connection between spectral and oscillatory properties of the Sturm-Liouville matrix problem*, Math. USSR Sbornik **31**, 365–378 (1977).
- [86] F. S. Rofo-Beketov and A. M. Kholkin, *Spectral Analysis of Differential Operators. Interplay Between Spectral and Oscillatory Properties*, Monograph Series in Mathematics, Vol. 7, World Scientific, Singapore, 2005.
- [87] Y. Saitō, *Eigenfunction expansions associated with second-order differential equations for Hilbert space-valued functions*, Publ. RIMS, Kyoto Univ. **7**, 1–55 (1971/72).
- [88] Y. Saitō, *The principle of limiting absorption for second-order differential equations with operator-valued coefficients*, Publ. RIMS, Kyoto Univ. **7**, 581–619 (1971/72).
- [89] Y. Saitō, *Spectral and scattering theory for second-order differential operators with operator-valued coefficients*, Osaka J. Math. **9**, 463–498 (1972).
- [90] Y. Saitō, *Spectral Representations for Schrödinger Operators with Long-Range Potentials*, Lecture Notes in Mathematics **727**, Springer, Berlin, 1979.
- [91] A. L. Sakhnovich, *Spectral functions of a canonical system of order $2n$* , Math. USSR Sbornik **71**, 355–369 (1992).
- [92] K. M. Schmidt, *An application of Gesztesy-Simon-Teschl oscillation theory to a problem in differential geometry*, J. Math. Anal. Appl. **261**, 61–71 (2001).
- [93] K. Schmitt and H. L. Smith, *Positive solutions and conjugate points for systems of differential equations*, Nonlin. Anal. Th., Meth. & Appl. **2**, 93–105 (1978).
- [94] I. Trooshin, *Asymptotics for the spectral and Weyl functions of the operator-valued Sturm-Liouville problem*, in “Inverse Problems and Related Topics”, G. Nakamura, S. Saitoh, J. K. Seo, and M. Yamamoto (eds.), Chapman & Hall/CRC, Res. Notes Math. **419**, Boca Raton, FL, 2000, pp. 189–208.
- [95] L. I. Vainerman and M. L. Gorbachuk, *On self-adjoint semibounded abstract differential operators*, Ukrain. Math. J. **22**, 694–696 (1970).
- [96] J. Weidmann, *Spectral Theory of Ordinary Differential Operators*, Lecture Notes in Mathematics **1258**, Springer, Berlin, 1987.
- [97] V. A. Yakubovich, *Nonoscillation of linear periodic Hamiltonian equations, and related topics*, St. Petersburg Math. J. **3**, 1165–1188 (1992).

DEPARTMENT OF MATHEMATICS & STATISTICS, MISSOURI UNIVERSITY OF SCIENCE AND TECHNOLOGY, ROLLA, MO 65409, USA

E-mail address: sclark@mst.edu

URL: <http://web.mst.edu/~sclark/>

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211, USA

E-mail address: gesztesyf@missouri.edu

URL: <http://www.math.missouri.edu/personnel/faculty/gesztesyf.html>

MATHEMATICS DEPARTMENT, THE UNIVERSITY OF TENNESSEE AT CHATTANOOGA, 415 EMCS BUILDING, DEPT. 6956, 615 McCALLIE AVE, CHATTANOOGA, TN 37403, USA

E-mail address: Roger-Nichols@utc.edu

URL: <http://www.utc.edu/faculty/roger-nichols/>