

WEAK CONVERGENCE OF SPECTRAL SHIFT FUNCTIONS FOR ONE-DIMENSIONAL SCHRÖDINGER OPERATORS

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Dedicated with great pleasure to Eduard Tsekanovskii on the occasion of his 75th birthday

ABSTRACT. We study the manner in which spectral shift functions associated with self-adjoint one-dimensional Schrödinger operators on the finite interval $(0, R)$ converge in the infinite volume limit $R \rightarrow \infty$ to the half-line spectral shift function.

Relying on a Fredholm determinant approach combined with certain measure theoretic facts, we show that prior vague convergence results in the literature in the special case of Dirichlet boundary conditions extend to the notion of weak convergence and arbitrary separated self-adjoint boundary conditions at $x = 0$ and $x = R$.

1. INTRODUCTION

In a nutshell, we are interested in the manner in which spectral shift functions converge in an infinite volume limit. More precisely, in this paper we give an exhaustive treatment of the one-dimensional case in which the finite interval $(0, R)$ converges to the half-line $[0, \infty)$ (the case of the entire real line being completely analogous). We explicitly consider the case of all separated self-adjoint boundary conditions at the endpoints 0 and R . The multi-dimensional case, based on an abstract approach to this circle of ideas, will appear elsewhere [19].

Before we focus on the abstract situation discussed in this paper, it is appropriate to briefly survey the known results in this area. Consider self-adjoint Schrödinger operators H_j and $H_{0,j}$ in $L^2((-j, j)^n; d^n x)$, $n \in \mathbb{N}$, $n \geq 2$, generated by the differential expression $-\Delta + V$ and $-\Delta$ on $(-j, j)^n$, respectively, with Dirichlet boundary conditions on $\partial(-j, j)^n$, where $0 \leq V \in L^\infty(\mathbb{R}^n; d^n x)$ is of fixed compact support in $(-j, j)^n$, real-valued, and nonzero a.e. Denoting by $\xi(\lambda; H_j, H_{0,j})$ for a.e. $\lambda \in \mathbb{R}$, the spectral shift function associated with the pair $(H_j, H_{0,j})$ (cf. [53, Ch. 8]), normalized to be zero in a neighborhood of $-\infty$, Kirsch [32] showed in 1987 that (perhaps, somewhat unexpectedly) for any $\lambda > 0$,

$$\sup_{j \in \mathbb{N}} |\xi(\lambda; H_j, H_{0,j})| = \infty. \quad (1.1)$$

Moreover, denoting by H and H_0 the corresponding self-adjoint Schrödinger operators in $L^2(\mathbb{R}^n; d^n x)$ generated by the differential expression $-\Delta + V$ and $-\Delta$

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on \mathbb{R}^n , respectively, one cannot expect pointwise a.e. convergence (or convergence in measure) of $\xi(\cdot; H_j, H_{0,j})$ to $\xi(\lambda; H, H_0)$ in the infinite volume limit $j \rightarrow \infty$ by the following elementary argument: For a.e. $\lambda > 0$, $\xi(\lambda; H, H_0)$ is a continuous function with respect to λ , related to the determinant of the underlying fixed energy scattering matrix. Yet $\xi(\cdot; H_j, H_{0,j})$, as a difference of eigenvalue counting functions corresponding to the number of eigenvalues (counting multiplicity) of H_j and $H_{0,j}$, respectively, is integer-valued and hence cannot possibly converge to a non-constant continuous function as $j \rightarrow \infty$. In particular, this argument applies to the one-dimensional context (in which case $\xi(\lambda; H, H_0) \rightarrow 0$ as $\lambda \rightarrow \infty$).

Having ruled out pointwise a.e. convergence of spectral shift functions in the infinite volume limit $j \rightarrow \infty$ in all space dimensions, it becomes clear that one has to invoke the concept of certain generalized limits. Indeed, in 1995, Geisler, Kostykin, and Schrader [16] proved for potentials $V \in \ell^1(L^2(\mathbb{R}^3; d^3x))$ (a Birman–Solomyak space, cf., e.g., [47, Ch. 4]) that for all $\lambda \in \mathbb{R}$,

$$\lim_{j \rightarrow \infty} \int_{(-\infty, \lambda]} \xi(\lambda'; H_j, H_{0,j}) d\lambda' = \int_{(-\infty, \lambda]} \xi(\lambda'; H, H_0) d\lambda'. \quad (1.2)$$

Since H_j and $H_{0,j}$ are bounded from below uniformly with respect to $j \in \mathbb{N}$, the limiting relation (1.2) involving distribution functions of the spectral shift measures is equivalent to vague convergence of the latter as observed in [29, Prop. 4.3].

In the one-dimensional half-line context, Borovyk and Makarov [8] (see also Borovyk [7]) proved in 2009 that for potentials $V \in L^1((0, \infty); (1 + |x|)dx)$ real-valued, and denoting by H_R the self-adjoint Schrödinger operator in $L^2((0, R); dx)$ and H the corresponding self-adjoint Schrödinger operator in $L^2((0, \infty); dx)$, both with Dirichlet boundary conditions (and otherwise maximally defined or defined in terms of quadratic forms), and analogously for $H_{0,R}$ and H_0 in the unperturbed case $V = 0$, the following vague limit holds:

$$\text{For any } g \in C_0(\mathbb{R}),$$

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}} \xi(\lambda; H_R, H_{0,R}) d\lambda g(\lambda) = \int_{\mathbb{R}} \xi(\lambda; H, H_0) d\lambda g(\lambda). \quad (1.3)$$

In addition, they proved that the following Cesàro limit,

$$\lim_{R \rightarrow \infty} \frac{1}{R} \int_0^R \xi(\lambda; H_r, H_{0,r}) dr = \xi(\lambda; H, H_0), \quad \lambda \in \mathbb{R} \setminus (\sigma_d(H) \cup \{0\}) \quad (1.4)$$

exists (and the limit in (1.4) extends to $\lambda = 0$ if H has no zero energy resonance).

Returning to the case of multi-dimensional boxes $[-R, R]^n$, Hislop and Müller [25] (see also [26]) proved a result going somewhat beyond vague convergence in 2010. More precisely, assuming a real-valued background potential $V^{(0)}$ satisfying $V_-^{(0)} \in K(\mathbb{R}^n)$, $V_+^{(0)} \in K_{\text{loc}}(\mathbb{R}^n)$ and a potential $0 \leq V \in K_{\text{loc}}(\mathbb{R}^n)$, $\text{supp}(V)$ compact (cf. [2], [46] for the definition of (local) Kato classes), they show that

$$\text{For any } f \in C_0(\mathbb{R}), \text{ and for any } f = \chi_J, J \subset \mathbb{R} \text{ a finite interval,}$$

$$\begin{aligned} & \lim_{R \rightarrow \infty} \int_{\mathbb{R}} \xi(\lambda; H_{0,R} + V^{(0)} + V, H_{0,R} + V^{(0)}) d\lambda f(\lambda) \\ &= \int_{\mathbb{R}} \xi(\lambda; H_0 + V^{(0)} + V, H_0 + V^{(0)}) d\lambda f(\lambda). \end{aligned} \quad (1.5)$$

In addition, they derived a weaker version than the Cesàro limit in (1.4) in the multi-dimensional context. More precisely, they proved that for every sequence of lengths $\{L_j\}_{j \in \mathbb{N}} \subset (0, \infty)$ with $\lim_{j \rightarrow \infty} L_j = \infty$, there exists a subsequence $\{j_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ with $\lim_{k \rightarrow \infty} j_k = \infty$, such that for every subsequence $\{k_\ell\}_{\ell \in \mathbb{N}} \subset \mathbb{N}$ with $\lim_{\ell \rightarrow \infty} k_\ell = \infty$,

$$\begin{aligned} \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{\ell=1}^L \xi(\lambda; H_{L^{j_{k_\ell}}}^{(0)} + V^{(0)} + V, H_{L^{j_{k_\ell}}}^{(0)} + V^{(0)}) \\ \leq \xi(\lambda; H_0 + V^{(0)} + V, H_0 + V^{(0)}) \end{aligned} \quad (1.6)$$

for (Lebesgue) a.e. $\lambda \in \mathbb{R}$.

Before describing our results we should mention that spectral shift functions feature prominently in the context of eigenvalue counting functions and hence in the context of the integrated density of states. We refer, for instance, to [11], [12], [24], [27], [28], [33], [34], [35], [41], and the references cited therein. For bounds on the spectral shift function we refer to [12], [27], [28], and [48].

In Section 2 we introduce basic facts on one-dimensional Schrödinger operators, describe associated Green's functions, Krein-type resolvent formulas, compute the integral kernel for the square root of the resolvent of the Dirichlet Laplacian on a finite interval and on a half-line. In addition, we derive Jost–Pais-type [30] (cf. also [18] and the extensive literature therein) reductions of Fredholm determinants corresponding to Birman–Schwinger-type kernels to Wronski determinants of appropriate solutions of the associated Schrödinger equation. Basic convergence results of resolvents and closely related operators (including Birman–Schwinger-type kernels) are discussed in Section 3. Finally, our principal Section 4 provides a new approach to vague, and especially, weak convergence of spectral shift functions in the infinite volume limit based on the convergence of underlying Fredholm determinants combined with certain measure theoretic facts. We refer to Theorem 4.12 for the main result of this paper. The latter considerably extends (1.3) in a variety of ways: First, $g(\cdot)$ in (1.3) can now be replaced by $f(\cdot)/(1 + \lambda^2)$ with f a bounded continuous function (this can further be improved). Second, we can permit any separated self-adjoint boundary conditions (not just Dirichlet boundary conditions) at $x = 0$ and $x = R$. Third, we can permit real-valued (singular) potentials V satisfying $V \in L^1((0, \infty); dx)$ (in the case of Dirichlet boundary conditions at $x = 0$ one can even permit $V \in L^1((0, \infty); [x/(1 + x)]dx)$, cf. Remark 2.8). Appendix A collects basic properties of spectral shift functions used in Section 4, and Appendix B derives a particular decomposition formula relating spectral shift functions for the interval $(0, R_1)$ and $(0, R_2)$, $0 < R_1 < R_2$, associated with Dirichlet boundary conditions at $x = 0, R_1, R_2$.

Finally, we briefly summarize some of the notation used in this paper: Let \mathcal{H} be a separable complex Hilbert space, $(\cdot, \cdot)_{\mathcal{H}}$ the scalar product in \mathcal{H} (linear in the second argument), and $I_{\mathcal{H}}$ the identity operator in \mathcal{H} . Next, let T be a linear operator mapping (a subspace of) a Hilbert space into another, with $\text{dom}(T)$ and $\text{ker}(T)$ denoting the domain and kernel (i.e., null space) of T . The resolvent set of a closed linear operator in \mathcal{H} will be denoted by $\rho(\cdot)$. The Banach spaces of bounded (resp., compact) linear operators on \mathcal{H} is denoted by $\mathcal{B}(\mathcal{H})$ (resp., $\mathcal{B}_{\infty}(\mathcal{H})$). The corresponding ℓ^p -based trace ideals will be denoted by $\mathcal{B}_p(\mathcal{H})$, $p > 0$.

The form sum (resp. difference) of two self-adjoint operators A and W will be denoted by $A +_q W$ (resp., $A -_q W = A +_q (-W)$).

The operator closure of a closable operator T in \mathcal{H} will be denoted by \overline{T} .

Lastly, for E a nonvoid locally compact space, we denote by $C(E)$ the space of continuous functions on E , by $C_\infty(E)$ the space of continuous functions on E vanishing at infinity, by $C_0(E)$ the continuous functions on E with compact support, and by $C_b(E)$ the bounded continuous functions on E .

2. PRELIMINARIES ON ONE-DIMENSIONAL SCHRÖDINGER OPERATORS AND ASSOCIATED FREDHOLM DETERMINANTS

In this section we introduce the basics of one-dimensional Schrödinger operators, including Green's functions, Krein-type resolvent formulas, the integral kernel of the square root of the resolvent of the Dirichlet Laplacian on $(0, R)$ and $(0, \infty)$, and discuss Jost–Pais-type reductions of associated Fredholm determinants to Wronski determinants.

We start with the case of finite interval Schrödinger operators on $(0, R)$.

Let $R > 0$, $\alpha, \beta \in [0, \pi)$,

$$V \in L^1((0, R); dx) \text{ real-valued,} \quad (2.1)$$

and introduce the differential expression τ by

$$\tau = -\frac{d^2}{dx^2} + V(x), \quad x \in J, \quad (2.2)$$

with $J \subseteq \mathbb{R}$ an appropriate interval.

We start by considering the self-adjoint (regular) Schrödinger operator $H_{(0,R),\alpha,\beta}$ in $L^2((0, R); dx)$,

$$\begin{aligned} (H_{(0,R),\alpha,\beta} f)(x) &= (\tau f)(x), \quad x \in (0, R), \\ f \in \text{dom}(H_{(0,R),\alpha,\beta}) &= \{g \in L^2((0, R); dx) \mid g, g' \in AC([0, R]); \\ &\sin(\alpha)g'(0) + \cos(\alpha)g(0) = 0, \sin(\beta)g'(R) + \cos(\beta)g(R) = 0; \\ &\tau f \in L^2((0, R); dx)\}. \end{aligned} \quad (2.3)$$

Denoting by $\psi_{0,\alpha}(z, \cdot)$ and $\psi_{R,\beta}(z, \cdot)$ the Weyl-type solutions for $H_{(0,R),\alpha,\beta}$ satisfying $\tau_{(0,R)}\psi = z\psi$ on $(0, R)$ and the boundary conditions in (2.3),

$$\begin{aligned} \sin(\alpha)\psi'_{0,\alpha}(z, 0) + \cos(\alpha)\psi_{0,\alpha}(z, 0) &= 0, \\ \sin(\beta)\psi'_{R,\beta}(z, R) + \cos(\beta)\psi_{R,\beta}(z, R) &= 0, \end{aligned} \quad (2.4)$$

at 0 and R , respectively, the resolvent of $H_{(0,R),\alpha,\beta}$ is given by

$$\begin{aligned} ((H_{(0,R),\alpha,\beta} - zI_{(0,R)})^{-1}f)(x) &= \int_0^R dx' G_{(0,R),\alpha,\beta}(z, x, x')f(x'), \\ x \in (0, R), z \in \rho(H_{(0,R),\alpha,\beta}), f \in L^2((0, R); dx), \end{aligned} \quad (2.5)$$

with the Green's function $G_{(0,R),\alpha,\beta}$ of $H_{(0,R),\alpha,\beta}$ expressed in terms of the Weyl solutions $\psi_{0,\alpha}$ and $\psi_{R,\beta}$ by

$$\begin{aligned} G_{(0,R),\alpha,\beta}(z, x, x') &= (H_{(0,R),\alpha,\beta} - zI_{(0,R)})^{-1}(x, x') \\ &= \frac{-1}{W(\psi_{0,\alpha}(z, \cdot), \psi_{R,\beta}(z, \cdot))} \begin{cases} \psi_{0,\alpha}(z, x) \psi_{R,\beta}(z, x'), & 0 \leq x \leq x' \leq R, \\ \psi_{0,\alpha}(z, x') \psi_{R,\beta}(z, x), & 0 \leq x' \leq x \leq R. \end{cases} \end{aligned} \quad (2.6)$$

Here the symbol I_J abbreviates the identity operator in $L^2(J; dx)$ and

$$W(f, g)(x) = f(x)g'(x) - f'(x)g(x), \quad x \in J, \quad (2.7)$$

denotes the Wronskian of f and g ($f, g \in C^1(J)$), with $J \subseteq \mathbb{R}$ an appropriate interval.

The corresponding half-line Schrödinger operators on $(0, \infty)$ are defined as follows.

Let $\alpha \in [0, \pi)$,

$$V \in L^1((0, \infty); dx) \text{ real-valued}, \quad (2.8)$$

and consider the self-adjoint Schrödinger operator $H_{(0, \infty), \alpha}$ in $L^2((0, \infty); dx)$,

$$\begin{aligned} (H_{(0, \infty), \alpha} f)(x) &= (\tau f)(x), \quad x \in (0, \infty), \\ f \in \text{dom}(H_{(0, \infty), \alpha}) &= \{g \in L^2((0, \infty); dx) \mid g, g' \in AC([0, R]) \text{ for all } R > 0; \\ &\quad \sin(\alpha)g'(0) + \cos(\alpha)g(0) = 0; \tau g \in L^2((0, \infty); dx)\}. \end{aligned} \quad (2.9)$$

Again, we denote by $\psi_{0, \alpha}(z, \cdot)$ and $\psi_+(z, \cdot)$ the Weyl-type solutions for $H_{(0, \infty), \alpha}$ satisfying $\tau\psi = z\psi$ on $(0, \infty)$ and

$$\psi_+(z, \cdot) \in L^2((0, \infty); dx), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (2.10)$$

with $\psi_{0, \alpha}(z, \cdot)$ satisfying the boundary condition in (2.9)

$$\sin(\alpha)\psi'_{0, \alpha}(z, 0) + \cos(\alpha)\psi_{0, \alpha}(z, 0) = 0. \quad (2.11)$$

One notes that $\psi_+(z, \cdot)$ is unique up to constant (possibly, z -dependent) multiples, and no boundary condition is needed at $+\infty$ due to the fact that assumption (2.8) implies the limit point case of τ at $+\infty$.

The resolvent of $H_{(0, \infty), \alpha}$ is then given by

$$\begin{aligned} ((H_{(0, \infty), \alpha} - zI_{(0, \infty)})^{-1} f)(x) &= \int_0^\infty dx' G_{(0, \infty), \alpha}(z, x, x') f(x'), \\ x \in (0, \infty), z \in \rho(H_{(0, \infty), \alpha}), f \in L^2((0, \infty); dx), \end{aligned} \quad (2.12)$$

with the Green's function $G_{(0, \infty), \alpha}$ of $H_{(0, \infty), \alpha}$ expressed in terms of $\psi_{0, \alpha}$ and ψ_+ by

$$\begin{aligned} G_{(0, \infty), \alpha}(z, x, x') &= (H_{(0, \infty), \alpha} - zI_{(0, \infty)})^{-1}(x, x') \\ &= \frac{-1}{W(\psi_{0, \alpha}(z, \cdot), \psi_+(z, \cdot))} \begin{cases} \psi_{0, \alpha}(z, x) \psi_+(z, x'), & 0 \leq x \leq x' < \infty, \\ \psi_{0, \alpha}(z, x') \psi_+(z, x), & 0 \leq x' \leq x < \infty. \end{cases} \end{aligned} \quad (2.13)$$

We start with some Green's function formulas. First, we compare $G_{(0, R), \alpha, \beta}$ and $G_{(0, \infty), \alpha}$:

Lemma 2.1. *Assume (2.8). Then the Green's functions $G_{(0, R), \alpha, \beta}$ and $G_{(0, \infty), \alpha}$ defined in (2.6) and (2.13) satisfy the relation*

$$\begin{aligned} G_{(0, R), \alpha, \beta}(z, x, x') &= G_{(0, \infty), \alpha}(z, x, x') \\ &+ \frac{[\sin(\beta)\psi'_+(z, R) + \cos(\beta)\psi_+(z, R)]\psi_{0, \alpha}(z, x)\psi_{0, \alpha}(z, x')}{[\sin(\beta)\psi'_{0, \alpha}(z, R) + \cos(\beta)\psi_{0, \alpha}(z, R)]W(\psi_{0, \alpha}(z, \cdot), \psi_+(z, \cdot))}, \\ &\quad z \in \rho(H_{(0, R), \alpha, \beta}) \cap \rho(H_{(0, \infty), \alpha}), \quad 0 \leq x, x' \leq R. \end{aligned} \quad (2.14)$$

Proof. Define

$$g_{\alpha,\beta}(z, x) = \int_0^R dx' G_{(0,R),\alpha,\beta}(z, x, x') f(x'), \quad x \in [0, R], \quad (2.15)$$

$$g_\alpha(z, x) = \int_0^\infty dx' G_{(0,\infty),\alpha}(z, x, x') f(x'), \quad x \geq 0, \quad (2.16)$$

$$z \in \rho(H_{(0,\infty),\alpha}) \cap \rho(H_{(0,R),\alpha,\beta}), \quad f \in L^2((0, \infty); dx).$$

Evidently,

$$\left(-\frac{d^2}{dx^2} + V(x) - z\right) g_{\alpha,\beta}(z, x) = f(x) = \left(-\frac{d^2}{dx^2} + V(x) - z\right) g_\alpha(z, x), \quad x \in [0, R], \quad (2.17)$$

that is,

$$\left(-\frac{d^2}{dx^2} + V(x) - z\right) (g_{\alpha,\beta}(z, x) - g_\alpha(z, x)) = 0, \quad x \in [0, R]. \quad (2.18)$$

Therefore, $g_{\alpha,\beta} - g_\alpha$ is expressible as a linear combination of $\psi_{\alpha,\alpha}(z, \cdot)$ and $\psi_+(z, \cdot)$ on $[0, R]$. More precisely, there exist z -dependent constants $c_1(z)$ and $c_2(z)$ such that

$$g_{\alpha,\beta}(z, x) = g_\alpha(z, x) + c_1(z) \psi_{0,\alpha}(z, x) + c_2(z) \psi_+(z, x), \quad (2.19)$$

$$x \in [0, R], \quad z \in \rho(H_{(0,R),\alpha,\beta}) \cap \rho(H_{(0,\infty),\alpha}).$$

Since $g_{\alpha,\beta}(z, \cdot)$ belongs to the domain of $H_{(0,R),\alpha,\beta}$, it satisfies the boundary conditions in (2.3),

$$\sin(\alpha) g'_{\alpha,\beta}(z, 0) + \cos(\alpha) g_{\alpha,\beta}(z, 0) = 0, \quad (2.20)$$

$$\sin(\beta) g'_{\alpha,\beta}(z, R) + \cos(\beta) g_{\alpha,\beta}(z, R) = 0. \quad (2.21)$$

Equation (2.16) then yields

$$g_{\alpha,\beta}(z, x) = -\frac{\psi_+(z, x)}{W_{0,\alpha,+}} \int_0^x dx' \psi_{0,\alpha}(z, x') f(x') - \frac{\psi_{0,\alpha}(z, x)}{W_{0,\alpha,+}} \int_x^R dx' \psi_+(z, x') f(x') + c_1(z) \psi_{0,\alpha}(z, x) + c_2(z) \psi_+(z, x), \quad (2.22)$$

$$g'_{\alpha,\beta}(z, x) = -\frac{\psi'_+(z, x)}{W_{0,\alpha,+}} \int_0^x dx' \psi_{0,\alpha}(z, x') f(x') - \frac{\psi'_{0,\alpha}(z, x)}{W_{0,\alpha,+}} \int_x^R dx' \psi_+(z, x') f(x') + c_1(z) \psi'_{0,\alpha}(z, x) + c_2(z) \psi'_+(z, x), \quad (2.23)$$

where we abbreviated

$$W_{0,\alpha,+} = W(\psi_{0,\alpha}(z, \cdot), \psi_+(z, \cdot)). \quad (2.24)$$

Thus, (2.20) becomes

$$\sin(\alpha) \left[\frac{-\psi'_{0,\alpha}(z, 0)}{W_{0,\alpha,+}} \int_0^R dx' \psi_+(z, x') f(x') + c_1 \psi'_{0,\alpha}(z, 0) + c_2 \psi'_+(z, 0) \right] + \cos(\alpha) \left[\frac{-\psi_{0,\alpha}(z, 0)}{W_{0,\alpha,+}} \int_0^R dx' \psi_+(z, x') f(x') + c_1 \psi_{0,\alpha}(z, 0) + c_2 \psi_+(z, 0) \right] = 0. \quad (2.25)$$

Together with

$$\sin(\alpha) \psi'_{0,\alpha}(z, 0) + \cos(\alpha) \psi_{0,\alpha}(z, 0) = 0, \quad (2.26)$$

since $\psi_{0,\alpha}$ satisfies the boundary condition at 0, (2.25) implies

$$c_2(z) = 0. \quad (2.27)$$

To obtain (2.27), we have made use of the fact that

$$\sin(\alpha)\psi'_+(z, 0) + \cos(\alpha)\psi_+(z, 0) \neq 0; \quad (2.28)$$

otherwise, $\psi_+(z, 0)$ would satisfy the boundary condition in (2.9), and thus be an eigenfunction of $H_{(0,\infty),\alpha}$, implying $z \in \sigma(H_{(0,\infty),\alpha})$. In view of (2.22) and (2.23), (2.21) can be rewritten as

$$\begin{aligned} & \sin(\beta) \left[\frac{-\psi'_+(z, R)}{W_{0,\alpha,+}} \int_0^R dx' \psi_{0,\alpha}(z, x') f(x') + c_1(z)\psi'_{0,\alpha}(z, R) + c_2(z)\psi'_+(z, R) \right] \\ & + \cos(\beta) \left[\frac{-\psi_+(z, R)}{W_{0,\alpha,+}} \int_0^R dx' \psi_{0,\alpha}(z, x') f(x') + c_1(z)\psi_{0,\alpha}(z, R) \right. \\ & \left. + c_2(z)\psi_+(z, R) \right] = 0, \quad (2.29) \end{aligned}$$

which, together with (2.27) yields

$$c_1(z) = \frac{[\sin(\beta)\psi'_+(z, R) + \cos(\beta)\psi_+(z, R)] \frac{1}{W_{0,\alpha,+}} \int_0^R dx' \psi_{0,\alpha}(z, x') f(x')}{\sin(\beta)\psi'_{0,\alpha}(z, R) + \cos(\beta)\psi_{0,\alpha}(z, R)}. \quad (2.30)$$

One notes that the denominator in (2.30) is always nonzero; otherwise, by (2.26), $\psi_{0,\alpha}(z, x)$ would satisfy (2.3), and thus be an eigenfunction of $H_{(0,R),\alpha,\beta}$, implying $z \in \sigma(H_{(0,R),\alpha,\beta})$.

The result (2.14) now follows by substituting (2.27) and (2.30) into (2.22), noting that $f \in L^2((0, \infty); dx)$ was arbitrary. \square

Next, we compare the half-line Green's functions $G_{(0,\infty),\tilde{\alpha}}$ and $G_{(0,\infty),\alpha}$, that is, we investigate the integral kernels associated with a special case of Krein's formula for resolvents (cf. [3, & 106]):

Lemma 2.2. *Assume (2.8) and $\tilde{\alpha}, \alpha \in [0, \pi)$ with $\tilde{\alpha} \neq \alpha$. Then the Green's functions $G_{(0,\infty),\tilde{\alpha}}$ and $G_{(0,\infty),\alpha}$ defined in (2.13) satisfy the relation*

$$\begin{aligned} G_{(0,\infty),\tilde{\alpha}}(z, x, x') &= G_{(0,\infty),\alpha}(z, x, x') \\ &+ \frac{[\sin(\tilde{\alpha})\psi'_{0,\alpha}(z, 0) + \cos(\tilde{\alpha})\psi_{0,\alpha}(z, 0)]\psi_+(z, x)\psi_+(z, x')}{W_{0,\alpha,+}[\sin(\tilde{\alpha})\psi'_+(z, 0) + \cos(\tilde{\alpha})\psi_+(z, 0)]}, \quad (2.31) \\ &z \in \rho(H_{(0,\infty),\tilde{\alpha}}) \cap \rho(H_{(0,\infty),\alpha}), \quad x, x' \geq 0. \end{aligned}$$

Proof. For $\alpha \in [0, \pi)$, define

$$\begin{aligned} g_\alpha(z, x) &= \int_0^\infty dx' G_{(0,\infty),\alpha}(z, x, x') f(x'), \quad (2.32) \\ x &\geq 0, \quad z \in \rho(H_{(0,\infty),\alpha}), \quad f \in L^2((0, \infty); dx). \end{aligned}$$

If $\tilde{\alpha} \neq \alpha$, by the same argument leading to (2.19) in the proof of Lemma 2.1, there are z -dependent constants, $c_1(z)$ and $c_2(z)$, such that

$$\begin{aligned} g_{\tilde{\alpha}}(z, x) &= g_\alpha(z, x) + c_1(z)\psi_{0,\alpha}(z, x) + c_2(z)\psi_+(z, x), \quad (2.33) \\ x &\in [0, \infty), \quad z \in \rho(H_{(0,\infty),\tilde{\alpha}}) \cap \rho(H_{(0,\infty),\alpha}). \end{aligned}$$

Since $g_{\tilde{\alpha}}, g_{\alpha} \in L^2((0, \infty); dx)$, one concludes that

$$c_1(z) = 0, \quad z \in \rho(H_{(0, \infty), \tilde{\alpha}}) \cap \rho(H_{(0, \infty), \alpha}). \quad (2.34)$$

as $\psi_{0, \alpha}(z, \cdot)$ (resp., $\psi_{0, \tilde{\alpha}}(z, \cdot)$) corresponding to α (resp., $\tilde{\alpha}$) is not $L^2((0, \infty); dx)$ for $z \in \rho(H_{(0, \infty), \alpha})$ (resp., $z \in \rho(H_{(0, \infty), \tilde{\alpha}})$).

Moreover, since

$$g_{\alpha} \in \text{dom}(H_{(0, \infty), \alpha}) \text{ and } g_{\tilde{\alpha}} \in \text{dom}(H_{(0, \infty), \tilde{\alpha}}), \quad (2.35)$$

each must satisfy the boundary condition in (2.9) at 0,

$$\sin(\tilde{\alpha})g'_{\tilde{\alpha}}(z, 0) + \cos(\tilde{\alpha})g_{\tilde{\alpha}}(z, 0) = 0, \quad (2.36)$$

$$\sin(\alpha)g'_{\alpha}(z, 0) + \cos(\alpha)g_{\alpha}(z, 0) = 0. \quad (2.37)$$

Applying (2.32),

$$\begin{aligned} g_{\tilde{\alpha}}(z, x) &= -\frac{1}{W_{0, \alpha, +}} \int_0^x dx' \psi_{0, \alpha}(z, x') \psi_+(z, x') f(x') \\ &\quad - \frac{1}{W_{0, \alpha, +}} \int_x^{\infty} dx' \psi_{0, \alpha}(z, x') \psi_+(z, x') f(x') + c_2(z) \psi_+(z, x) \end{aligned} \quad (2.38)$$

$$\begin{aligned} g'_{\alpha, \beta}(z, x) &= -\frac{\psi'_+(z, x)}{W_{0, \alpha, +}} \int_0^x dx' \psi_{0, \alpha}(z, x') f(x') \\ &\quad - \frac{\psi'_{0, \alpha}(z, x)}{W_{0, \alpha, +}} \int_x^{\infty} dx' \psi_+(z, x') f(x') + c_2(z) \psi_+(z, x), \end{aligned} \quad (2.39)$$

$$x \geq 0, \quad f \in L^2((0, \infty); dx).$$

Therefore, 2.36 becomes

$$\begin{aligned} \sin(\tilde{\alpha}) \left[\frac{\psi'_{0, \alpha}(z, 0)}{W_{0, \alpha, +}} \int_0^{\infty} dx' \psi_+(z, x') f(x') + c_2(z) \psi'_+(z, 0) \right] \\ + \cos(\tilde{\alpha}) \left[\frac{\psi'_{0, \alpha}(z, 0)}{W_{0, \alpha, +}} \int_0^{\infty} dx' \psi_+(z, x') f(x') + c_2(z) \psi_+(z, 0) \right] = 0. \end{aligned} \quad (2.40)$$

Hence,

$$c_2(z) = \frac{[\sin(\tilde{\alpha}) \psi'_{0, \alpha}(z, 0) + \cos(\tilde{\alpha}) \psi_{0, \alpha}(z, 0)] \frac{1}{W_{0, \alpha, +}} \int_0^{\infty} dx' \psi_+(z, x') f(x')}{\sin(\tilde{\alpha}) \psi'_+(z, 0) + \cos(\tilde{\alpha}) \psi_+(z, 0)}. \quad (2.41)$$

One notes that the denominator in (2.41) is nonzero; otherwise, $z \in \sigma(H_{(0, \infty), \tilde{\alpha}})$. Since $f \in L^2((0, \infty); dx)$ is arbitrary, substituting (2.34) and (2.41) into (2.33) yields (2.31). \square

In the following, specializing to the unperturbed case, where $V = 0$ on $(0, R)$, we abbreviate the corresponding quantities with the superscript (0). This notation applies to the unperturbed Schrödinger operator $H_{(0, R), \alpha, \beta}^{(0)}$, the unperturbed differential expression $\tau^{(0)} = -d^2/dx^2$, the unperturbed Green's function $G_{(0, R), \alpha, \beta}^{(0)}$, and to the corresponding unperturbed Weyl-type solutions of $\tau^{(0)}\psi = z\psi$ on $(0, R)$. The latter are given by

$$\begin{aligned} \psi_{0, \alpha}^{(0)}(z, x) &= \cos(\alpha) \frac{\sin(z^{1/2}x)}{z^{1/2}} - \sin(\alpha) \cos(z^{1/2}x) \\ \psi_{R, \beta}^{(0)}(z, x) &= \cos(\beta) \frac{\sin(z^{1/2}(R-x))}{z^{1/2}} + \sin(\beta) \cos(z^{1/2}(R-x)), \end{aligned} \quad (2.42)$$

$$x \in (0, R), \quad \text{Im}(z^{1/2}) \geq 0,$$

implying

$$\begin{aligned} W(\psi_{R,\beta}^{(0)}(z, \cdot), \psi_{0,\alpha}^{(0)}(z, \cdot)) &= \sin(\beta - \alpha) \cos(z^{1/2}R) + \sin(\alpha) \sin(\beta) z^{1/2} \sin(z^{1/2}R) \\ &\quad + \cos(\alpha) \cos(\beta) \frac{\sin(z^{1/2}R)}{z^{1/2}}. \end{aligned} \quad (2.43)$$

For the corresponding perturbed solutions (i.e., for $V \neq 0$) one obtains

$$\psi_{0,\alpha}(z, x) = \psi_{0,\alpha}^{(0)}(z, x) + \int_0^x dx' \frac{\sin(z^{1/2}(x-x'))}{z^{1/2}} V(x') \psi_{0,\alpha}(z, x'), \quad (2.44)$$

$$\psi_{R,\beta}(z, x) = \psi_{R,\beta}^{(0)}(z, x) - \int_x^R dx' \frac{\sin(z^{1/2}(x-x'))}{z^{1/2}} V(x') \psi_{R,\beta}(z, x'), \quad (2.45)$$

$$0 \leq x \leq R, \quad \text{Im}(z^{1/2}) \geq 0,$$

implying

$$\psi'_{0,\alpha}(z, x) = \psi_{0,\alpha}^{(0)'}(z, x) + \int_0^x dx' \cos(z^{1/2}(x-x')) V(x') \psi_{0,\alpha}(z, x'),$$

$$\psi'_{R,\beta}(z, x) = \psi_{R,\beta}^{(0)'}(z, x) - \int_x^R dx' \cos(z^{1/2}(x-x')) V(x') \psi_{R,\beta}(z, x'), \quad (2.46)$$

$$x \in (0, R), \quad \text{Im}(z^{1/2}) \geq 0,$$

and hence,

$$\begin{aligned} \psi_{0,\alpha}(z, 0) &= \psi_{0,\alpha}^{(0)}(z, 0), & \psi'_{0,\alpha}(z, 0) &= \psi_{0,\alpha}^{(0)'}(z, 0). \\ \psi_{R,\beta}(z, R) &= \psi_{R,\beta}^{(0)}(z, R), & \psi'_{R,\beta}(z, R) &= \psi_{R,\beta}^{(0)'}(z, R). \end{aligned} \quad (2.47)$$

In the half-line case $(0, \infty)$ one introduces, in obvious notation, the unperturbed Schrödinger operator $H_{(0,\infty),\alpha}^{(0)}$, the unperturbed Green's function $G_{(0,\infty),\alpha}^{(0)}$, and the corresponding unperturbed Weyl-type solutions of $\tau^{(0)}\psi = z\psi$ on $(0, \infty)$. The latter are given by

$$\begin{aligned} \psi_{0,\alpha}^{(0)}(z, x) &= \cos(\alpha) \frac{\sin(z^{1/2}x)}{z^{1/2}} - \sin(\alpha) \cos(z^{1/2}x), & \psi_+^{(0)}(z, x) &= e^{iz^{1/2}x}, \\ & & x \in (0, \infty), \quad \text{Im}(z^{1/2}) \geq 0, \end{aligned} \quad (2.48)$$

implying

$$W(\psi_+^{(0)}(z, \cdot), \psi_{0,\alpha}^{(0)}(z, \cdot)) = \cos(\alpha) + iz^{1/2} \sin(\alpha). \quad (2.49)$$

For the corresponding perturbed solutions (i.e., for $V \neq 0$) one obtains

$$\psi_{0,\alpha}(z, x) = \psi_{0,\alpha}^{(0)}(z, x) + \int_0^x dx' \frac{\sin(z^{1/2}(x-x'))}{z^{1/2}} V(x') \psi_{0,\alpha}(z, x'),$$

$$\psi_+(z, x) = \psi_+^{(0)}(z, x) - \int_x^\infty dx' \frac{\sin(z^{1/2}(x-x'))}{z^{1/2}} V(x') \psi_+(z, x'), \quad (2.50)$$

$$x \in (0, \infty), \quad \text{Im}(z^{1/2}) \geq 0,$$

implying

$$\psi'_{0,\alpha}(z, x) = \psi_{0,\alpha}^{(0)'}(z, x) + \int_0^x dx' \cos(z^{1/2}(x-x')) V(x') \psi_{0,\alpha}(z, x'),$$

$$\psi'_+(z, x) = \psi_+^{(0)'}(z, x) - \int_x^\infty dx' \cos(z^{1/2}(x-x'))V(x')\psi_+(z, x'), \quad (2.51)$$

$$x \in (0, \infty), \operatorname{Im}(z^{1/2}) \geq 0,$$

and hence,

$$W(\psi_+(z, \cdot), \psi_{0,\alpha}(z, \cdot)) = \cos(\alpha)\psi_+(z, 0) + \sin(\alpha)\psi'_+(z, 0). \quad (2.52)$$

To make the connection between Fredholm determinants and ratios of Wronskians, we first need a technical result on integral kernels of square roots of resolvents in the special case $V = 0$ a.e. on $(0, R)$. For simplicity we state the result in the case of Dirichlet boundary conditions (i.e., for $\alpha = \beta = 0$) only:

We start by recalling the explicit formulas for the Dirichlet Green's functions,

$$\begin{aligned} G_{(0,R),0,0}^{(0)}(z, x, x') &= \frac{1}{z^{1/2} \sin(z^{1/2}R)} \begin{cases} \sin(z^{1/2}x) \sin(z^{1/2}(R-x')), & 0 \leq x \leq x' \leq R, \\ \sin(z^{1/2}x') \sin(z^{1/2}(R-x)), & 0 \leq x' \leq x \leq R, \end{cases} \end{aligned} \quad (2.53)$$

$$G_{(0,\infty),0}^{(0)}(z, x, x') = \frac{1}{z^{1/2}} \begin{cases} \sin(z^{1/2}x)e^{z^{1/2}x'}, & 0 \leq x \leq x' < \infty, \\ \sin(z^{1/2}x')e^{z^{1/2}x}, & 0 \leq x' \leq x < \infty. \end{cases} \quad (2.54)$$

Lemma 2.3. *Let $E > 0$.*

(i) *The integral kernel $R_{(0,R),0,0}^{(0),1/2}(-E, \cdot, \cdot)$ for $(H_{(0,R),0,0}^{(0)} + E)^{-1/2}$ is given by*

$$R_{(0,R),0,0}^{(0),1/2}(-E, x, x') = \begin{cases} 2\pi^{-1} \int_{E^{1/2}}^\infty \frac{\sinh(sx)}{(s^2-E)^{1/2}} \frac{\sinh(s(R-x'))}{\sinh(sR)} ds, & 0 \leq x \leq x' \leq R, \\ 2\pi^{-1} \int_{E^{1/2}}^\infty \frac{\sinh(sx')}{(s^2-E)^{1/2}} \frac{\sinh(s(R-x))}{\sinh(sR)} ds, & 0 \leq x' \leq x \leq R. \end{cases} \quad (2.55)$$

(ii) *The integral kernel $R_{(0,\infty),0}^{(0),1/2}(-E, \cdot, \cdot)$ of $(H_{(0,\infty),0} + E)^{-1/2}$ is given by*

$$R_{(0,\infty),0}^{(0),1/2}(-E, x, x') = \pi^{-1} [K_0(E^{1/2}|x' - x|) - K_0(E^{1/2}(x + x'))], \quad x, x' \in (0, \infty), \quad (2.56)$$

where $K_0(\cdot)$ denotes the modified irregular Bessel function of order zero (cf. [1, Sect. 9.6]).

Proof. If A is a positive-type operator with integral kernel $H(t, x, x')$ such that

$$[(A + tI_J)^{-1}u](x) = \int_J dx' H(t, x, x')u(x'), \quad x \in J, t > 0, \quad (2.57)$$

$J \subset \mathbb{R}$ an appropriate interval, then the operator A^{-q} , $0 < q < 1$, has the integral kernel $R^q(\cdot, \cdot)$ (cf., e.g., [36, Sect. 16])

$$R^q(x, x') = \frac{\sin(\pi q)}{\pi} \int_0^\infty dt t^{-q} H(t, x, x'), \quad x, x' \in J. \quad (2.58)$$

Applying (2.58) to $A = H_{(0,R),0,0}^{(0)} + E$, one calculates for $0 \leq x \leq x' \leq R$,

$$\begin{aligned} R_{(0,R),0,0}^{(0),1/2}(-E, x, x') &= \pi^{-1} \int_0^\infty dt t^{-1/2} G_{(0,R),0,0}^{(0)}(-E-t, x, x') \\ &= \pi^{-1} \int_0^\infty dt t^{-1/2} \psi_{0,0}^{(0)}(-E-t, x) \psi_{R,0}^{(0)}(-E-t, x') \\ &= 2\pi^{-1} \int_{|E|^{1/2}}^\infty ds \frac{\sinh(sx)}{(s^2-E)^{1/2}} \frac{\sinh(s(R-x'))}{\sinh(sR)}. \end{aligned} \quad (2.59)$$

For the case $0 \leq x' \leq x$ one simply interchanges the roles of x and x' .

Similarly, applying (2.58) to $A = H_{(0,\infty),0}^{(0)} + E$, one calculates for $0 \leq x \leq x'$,

$$\begin{aligned}
R_{(0,\infty),0}^{(0),1/2}(-E, x, x') &= \pi^{-1} \int_0^\infty dt t^{-1/2} G_{(0,\infty),0}^{(0)}(-E-t, x, x') \\
&= \pi^{-1} \int_0^\infty dt t^{-1/2} \psi_{0,0}^{(0)}(-E-t, x) \psi_+^{(0)}(-E-t, x') \\
&= 2\pi^{-1} \int_{E^{1/2}}^\infty ds \frac{\sinh(sx)}{(s^2-E)^{1/2}} e^{-sx'} \\
&= \pi^{-1} \int_1^\infty ds \frac{e^{-E^{1/2}s(x'-x)}}{(s^2-1)^{1/2}} - \pi^{-1} \int_1^\infty ds \frac{e^{-E^{1/2}s(x+x')}}{(s^2-1)^{1/2}}.
\end{aligned} \tag{2.60}$$

The last two integrals in (2.60) can be computed in terms of the modified irregular Bessel function of order zero, $K_0(\cdot)$ (see [22, No. 8.432.3]) and one obtains

$$R_{(0,\infty),0}^{(0),1/2}(-E, x, x') = \pi^{-1} (K_0(E^{1/2}(x'-x)) - K_0(E^{1/2}(x+x'))), \quad 0 \leq x \leq x'. \tag{2.61}$$

The case $0 \leq x' \leq x$ is treated analogously. \square

Remark 2.4. The integrand in (2.59) may be expressed as

$$\frac{\sinh(sx)}{(s^2-E)^{1/2}} \frac{\sinh(s(R-x'))}{\sinh(sR)} = \frac{\sinh(sx)}{(s^2-E)^{1/2}} \left(e^{-sx'} - 2 \frac{e^{-2Rs}}{1-e^{-2Rs}} \sinh(sx') \right), \tag{2.62}$$

and hence the integrand in (2.59) has the integrable majorant

$$\frac{\sinh(sx)}{(s^2-E)^{1/2}} \frac{\sinh(s(R-x'))}{\sinh(sR)} \leq \frac{\sinh(sx)}{(s^2-E)^{1/2}} e^{-sx'}, \tag{2.63}$$

implying

$$0 \leq R_{(0,R),0,0}^{(0),1/2}(-E, x, x') \leq R_{(0,\infty),0}^{(0),1/2}(-E, x, x'), \quad 0 \leq x, x' \leq R, \quad E > 0. \tag{2.64}$$

More generally, assuming (2.8) and employing domain monotonicity for Dirichlet Green's functions (see, e.g., [13, Sect. 1.VII.6]), that is,

$$0 \leq G_{(0,R),0,0}(-E, x, x') \leq G_{(0,\infty),0}(-E, x, x'), \quad 0 \leq x, x' \leq R, \quad E > 0, \tag{2.65}$$

and inserting (2.65) into the analog of (2.58) yields

$$0 \leq R_{(0,R),0,0}^q(-E, x, x') \leq R_{(0,\infty),0}^q(-E, x, x'), \quad 0 \leq x, x' \leq R, \quad E > 0, \quad 0 < q < 1. \tag{2.66}$$

Next, we factor V as

$$V(x) = u(x)v(x), \quad v(x) = |V(x)|^{1/2}, \quad u(x) = v(x) \operatorname{sgn}(V(x)), \quad x \in (0, \infty), \tag{2.67}$$

and denote

$$V_R(x) = V(x)|_{(0,R)}, \quad v_R(x) = v(x)|_{(0,R)}, \quad u_R(x) = u(x)|_{(0,R)}, \quad x \in (0, R). \tag{2.68}$$

Theorem 2.5. *Let $R > 0$ and $V_R \in L^1((0, R); dx)$ be real-valued, and assume that $z \in \rho(H_{(0,R),\alpha,\beta}^{(0)})$. Then*

$$\overline{u_R(H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1}v_R} \in \mathcal{B}_1(L^2((0, R); dx)), \quad (2.69)$$

and the determinant $\det_{L^2((0,R);dx)} \left(I_{(0,R)} + \overline{u_R(H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1}v_R} \right)$ is a quotient of Wronskians,

$$\det_{L^2((0,R);dx)} \left(I_{(0,R)} + \overline{u_R(H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1}v_R} \right) = \frac{W(\psi_{R,\beta}(z, \cdot), \psi_{0,\alpha}(z, \cdot))}{W(\psi_{R,\beta}^{(0)}(z, \cdot), \psi_{0,\alpha}^{(0)}(z, \cdot))} \quad (2.70)$$

$$= [\sin(\beta)\psi'_{0,\alpha}(z, R) + \cos(\beta)\psi_{0,\alpha}(z, R)] [\cos(\alpha)\cos(\beta)z^{-1/2}\sin(z^{1/2}R) - \sin(\alpha - \beta)\cos(z^{1/2}R) + \sin(\alpha)\sin(\beta)z^{1/2}\sin(z^{1/2}R)]^{-1} \quad (2.71)$$

$$= [\sin(\alpha)\psi'_{R,\beta}(z, 0) + \cos(\alpha)\psi_{R,\beta}(z, 0)] [\cos(\alpha)\cos(\beta)z^{-1/2}\sin(z^{1/2}R) - \sin(\alpha - \beta)\cos(z^{1/2}R) + \sin(\alpha)\sin(\beta)z^{1/2}\sin(z^{1/2}R)]^{-1}. \quad (2.72)$$

Proof. To get (2.71) and (2.72), observe that

$$\frac{W(\psi_{R,\beta}(z, \cdot), \psi_{0,\alpha}(z, \cdot))}{W(\psi_{R,\beta}^{(0)}(z, \cdot), \psi_{0,\alpha}^{(0)}(z, \cdot))} = \frac{\psi_{R,\beta}(z, x)\psi'_{0,\alpha}(z, x) - \psi'_{R,\beta}(z, x)\psi_{0,\alpha}(z, x)}{W(\psi_{R,\beta}^{(0)}(z, \cdot), \psi_{0,\alpha}^{(0)}(z, \cdot))}, \quad (2.73)$$

$$x \in [0, R], \quad z \in \rho(H_{(0,R),\alpha,\beta}^{(0)}).$$

In view of (2.42), (2.43), and (2.47), the identity (2.71) (resp., (2.72)) is given by evaluating (2.73) at $x = R$ (resp. $x = 0$).

In order to verify (2.70), let $x, x' \in [0, R]$ and $z \in \rho(H_{(0,R),\alpha,\beta}^{(0)})$, $\text{Im}(z^{1/2}) \geq 0$. Define the functions

$$f_R^{(1)}(z, x) = -u_R(x)\psi_{R,\beta}^{(0)}(z, x), \quad f_R^{(2)}(z, x) = \frac{-u_R(x)\psi_{0,\alpha}^{(0)}(z, x)}{W(\psi_{R,\beta}^{(0)}(z, \cdot), \psi_{0,\alpha}^{(0)}(z, \cdot))}, \quad (2.74)$$

$$g_R^{(1)}(z, x) = \frac{v_R(x)\psi_{0,\alpha}^{(0)}(z, x)}{W(\psi_{R,\beta}^{(0)}(z, \cdot), \psi_{0,\alpha}^{(0)}(z, \cdot))}, \quad g_R^{(2)}(z, x) = v_R(x)\psi_{R,\beta}^{(0)}(z, x),$$

where, for brevity, we have written

$$W(\psi_{R,\beta}^{(0)}, \psi_{0,\alpha}^{(0)}) = W(\psi_{R,\beta}^{(0)}(z, \cdot), \psi_{0,\alpha}^{(0)}(z, \cdot)), \quad (2.75)$$

and introduce the Volterra integral kernel

$$H_R(z, x, x') = f_R^{(1)}(z, x)g_R^{(1)}(z, x') - f_R^{(2)}(z, x)g_R^{(2)}(z, x') \\ = u_R(x) \frac{\sin(z^{1/2}(x - x'))}{z^{1/2}} v_R(x'). \quad (2.76)$$

In addition, one introduces the functions $\widehat{f}_R^{(j)}(z, \cdot)$, $j = 1, 2$, as solutions of the Volterra integral equations

$$\widehat{f}_R^{(1)}(z, x) = f_R^{(1)}(z, x) - \int_x^R dx' H(z, x, x') \widehat{f}_R^{(1)}(z, x'), \quad (2.77)$$

$$\widehat{f}_R^{(2)}(z, x) = f_R^{(2)}(z, x) + \int_0^x dx' H(z, x, x') \widehat{f}_R^{(2)}(z, x'). \quad (2.78)$$

Comparing (2.77) to (2.45) and (2.78) to (2.44), one infers

$$\widehat{f}_R^{(1)}(z, x) = -u_R(x)\psi_{R,\beta}(z, x) \quad (2.79)$$

$$\widehat{f}_R^{(2)}(z, x) = \frac{-u_R(x)\psi_{0,\alpha}(z, x)}{W(\psi_{R,\beta}^{(0)}(z, \cdot), \psi_{0,\alpha}^{(0)}(z, \cdot))}, \quad (2.80)$$

$$x \in [0, R].$$

Introducing

$$U_R(z, x) = \begin{pmatrix} 1 - \int_x^R dx' g_R^{(1)}(z, x') \widehat{f}_R^{(1)}(z, x') & \int_0^x dx' g_R^{(1)}(z, x') \widehat{f}_R^{(2)}(z, x') \\ \int_x^R dx' g_R^{(2)}(z, x') \widehat{f}_R^{(1)}(z, x') & 1 - \int_0^x dx' g_R^{(2)}(z, x') \widehat{f}_R^{(2)}(z, x') \end{pmatrix}, \quad (2.81)$$

then applying [18, Theorem 3.2], one obtains

$$\det_{L^2((0,R);dx)} \left(I_{(0,R)} + u_R \overline{(H_{(0,R),\alpha,\beta}^{(0)} - z)^{-1} v_R} \right) = \det_{\mathbb{C}^2}(U_R(z, 0)) \quad (2.82)$$

$$= \det_{\mathbb{C}^2}(U_R(z, R)). \quad (2.83)$$

Using (2.74) and (2.79)–(2.81), one readily verifies

$$\det_{\mathbb{C}^2}(U_R(z, 0)) = 1 + \int_0^R dx' \frac{V(x')\psi_{0,\alpha}^{(0)}(z, x')\psi_{R,\beta}(z, x')}{W(\psi_{R,\beta}^{(0)}(z, \cdot), \psi_{0,\alpha}^{(0)}(z, \cdot))}, \quad (2.84)$$

$$\det_{\mathbb{C}^2}(U_R(z, R)) = 1 + \int_0^R dx' \frac{V(x')\psi_{R,\beta}^{(0)}(z, x')\psi_{0,\alpha}(z, x')}{W(\psi_{R,\beta}^{(0)}(z, \cdot), \psi_{0,\alpha}^{(0)}(z, \cdot))}. \quad (2.85)$$

By definition,

$$W(\psi_{R,\beta}(z, \cdot), \psi_{0,\alpha}(z, \cdot)) = \psi_{R,\beta}(x)\psi_{0,\alpha}'(x) - \psi_{R,\beta}'(x)\psi_{0,\alpha}(x), \quad x \in [0, R]. \quad (2.86)$$

Substitution of (2.44) and (2.45) into the r.h.s. of (2.86) leads one to the following lengthy expression for the constant $W(\psi_{R,\beta}, \psi_{0,\alpha})$:

$$\begin{aligned} & W(\psi_{R,\beta}(z, \cdot), \psi_{0,\alpha}(z, \cdot)) \\ &= \left[\psi_{R,\beta}^{(0)'}(z, x) - z^{-1/2} \int_x^R dx' \sin(z^{1/2}(x-x'))V(x')\psi_{R,\beta}(z, x') \right] \\ & \quad \times \left[\psi_{0,\alpha}^{(0)'}(z, x) + \int_0^x dx' \cos(z^{1/2}(x-x'))V(x')\psi_{0,\alpha}(z, x') \right] \\ & \quad - \left[\psi_{R,\beta}^{(0)'}(z, x) - \int_x^R dx' \cos(z^{1/2}(x-x'))V(x')\psi_{R,\beta}(z, x') \right] \\ & \quad \times \left[\psi_{0,\alpha}^{(0)}(z, x) + z^{-1/2} \int_0^x dx' \sin(z^{1/2}(x-x'))V(x')\psi_{0,\alpha}(z, x') \right], \quad (2.87) \\ & \quad \quad \quad x \in [0, R]. \end{aligned}$$

Evaluation of the r.h.s. of (2.87) at $x = 0$ and $x = R$ yields (after straightforward manipulation)

$$\begin{aligned} W(\psi_{R,\beta}(z, \cdot), \psi_{0,\alpha}(z, \cdot)) &= W(\psi_{R,\beta}^{(0)}, \psi_{0,\alpha}^{(0)}) + \int_0^R dx' V(x')\psi_{0,\alpha}^{(0)}(z, x')\psi_{R,\beta}(z, x') \\ &= W(\psi_{R,\beta}^{(0)}, \psi_{0,\alpha}^{(0)}) + \int_0^R dx' V(x')\psi_{R,\beta}^{(0)}(z, x')\psi_{0,\alpha}(z, x'), \end{aligned}$$

again making use of the abbreviation (2.75), and it follows from (2.84) and (2.85) that

$$\det_{\mathbb{C}^2}(U_R(z, 0)) = \det_{\mathbb{C}^2}(U_R(z, R)) = \frac{W(\psi_{R,\beta}(z, \cdot), \psi_{0,\alpha}(z, \cdot))}{W(\psi_{R,\beta}^{(0)}(z, \cdot), \psi_{0,\alpha}^{(0)}(z, \cdot))}. \quad (2.88)$$

The identity (2.70) then follows from (2.88) and either (2.82) or (2.83). \square

The corresponding half-line result then reads as follows:

Theorem 2.6. *Assume that $V \in L^1((0, \infty); dx)$ is real-valued and suppose that $z \in \rho(H_{(0,\infty),\alpha}^{(0)})$. Then*

$$\overline{u(H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-1}v} \in \mathcal{B}_1(L^2((0, \infty); dx)), \quad (2.89)$$

and the determinant $\det_{L^2((0,\infty);dx)}(I_{(0,\infty)} + \overline{u(H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-1}v})$ is a quotient of Wronskians,

$$\det_{L^2((0,\infty);dx)}(I_{(0,\infty)} + \overline{u(H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-1}v}) = \frac{W(\psi_+(z, \cdot), \psi_{0,\alpha}(z, \cdot))}{W(\psi_+^{(0)}(z, \cdot), \psi_{0,\alpha}^{(0)}(z, \cdot))} \quad (2.90)$$

$$= \frac{\sin(\alpha)\psi'_+(z, 0) + \cos(\alpha)\psi_+(z, 0)}{\cos(\alpha) + iz^{1/2}\sin(\alpha)}. \quad (2.91)$$

Proof. The equality (2.91) is a trivial consequence of (2.49) and (2.52). Thus, we proceed to verify (2.11). Let $x, x' \in [0, \infty)$ and $z \in \rho(H_{(0,\infty),\alpha}^{(0)})$, $\text{Im}(z^{1/2}) \geq 0$. Define the functions

$$\begin{aligned} f_1(z, x) &= -u(x)\psi_+^{(0)}(z, x), & f_2(z, x) &= \frac{-u(x)\psi_{0,\alpha}^{(0)}(z, x)}{W(\psi_+^{(0)}(z, \cdot), \psi_{0,\alpha}^{(0)}(z, \cdot))}, \\ g_1(z, x) &= \frac{v(x)\psi_{0,\alpha}^{(0)}(z, x)}{W(\psi_+^{(0)}(z, \cdot), \psi_{0,\alpha}^{(0)}(z, \cdot))}, & g_2(z, x) &= v(x)\psi_+^{(0)}(z, x), \end{aligned} \quad (2.92)$$

and introduce the Volterra integral kernel

$$\begin{aligned} H(z, x, x') &= f_1(z, x)g_1(z, x') - f_2(z, x)g_2(z, x') \\ &= u(x) \frac{\sin(z^{1/2}(x - x'))}{z^{1/2}} v(x'). \end{aligned} \quad (2.93)$$

Introducing $\widehat{f}_1, \widehat{f}_2$ as the solutions of the Volterra integral equations

$$\begin{aligned} \widehat{f}_1(z, x) &= f_1(z, x) - \int_x^\infty dx' H(z, x, x') \widehat{f}_1(z, x'), \\ \widehat{f}_2(z, x) &= f_2(z, x) + \int_0^x dx' H(z, x, x') \widehat{f}_2(z, x'), \end{aligned} \quad (2.94)$$

a comparison with (2.50) yields

$$\widehat{f}_1(z, x) = -u(x)\psi_+(z, x). \quad (2.95)$$

Assuming temporarily that V has compact support, one concludes that $\widehat{f}_1 \in L^2((0, \infty); dx)$ and applying [18, Theorem 3.2] results in

$$\det_{L^2((0,\infty);dx)}(I_{(0,\infty)} + \overline{u(H_{(0,\infty),\alpha}^{(0)} - z)^{-1}v}) = \det_{\mathbb{C}^2}(U(z, 0)), \quad (2.96)$$

where

$$U(z, x) = \begin{pmatrix} 1 - \int_x^\infty dx' g_1(z, x') \widehat{f}_1(z, x') & \int_0^x dx' g_1(z, x') \widehat{f}_2(z, x') \\ \int_x^\infty dx' g_2(z, x') \widehat{f}_1(z, x') & 1 - \int_0^x dx' g_2(z, x') \widehat{f}_2(z, x') \end{pmatrix}. \quad (2.97)$$

Using (2.92) and (2.95), equations (2.96) and (2.97) immediately yield

$$\det_{\mathbb{C}^2}(U(z, 0)) = 1 + \int_0^\infty dx' \frac{V(x') \psi_{0,\alpha}^{(0)}(z, x') \psi_+(z, x')}{W(\psi_+^{(0)}(z, \cdot), \psi_{0,\alpha}^{(0)}(z, \cdot))}. \quad (2.98)$$

By definition,

$$W(\psi_+(z, \cdot), \psi_{0,\alpha}(z, \cdot)) = \psi_+(z, x) \psi'_{0,\alpha}(z, x) - \psi'_+(z, x) \psi_{0,\alpha}(z, x), \quad x \in [0, \infty). \quad (2.99)$$

Substitution of (2.50) and (2.51) into the r.h.s. of (2.99) provides

$$\begin{aligned} & W(\psi_+(z, \cdot), \psi_{0,\alpha}(z, \cdot)) \\ &= \left[\psi_+^{(0)}(z, x) - z^{-1/2} \int_x^\infty dx' \sin(z^{1/2}(x-x')) V(x') \psi_+(z, x') \right] \\ & \quad \times \left[\psi_{0,\alpha}^{(0)'}(z, x) + \int_0^x dx' \cos(z^{1/2}(x-x')) V(x') \psi_{0,\alpha}(z, x') \right] \\ & \quad - \left[\psi_{0,\alpha}^{(0)'}(z, x) - \int_x^\infty dx' \cos(z^{1/2}(x-x')) V(x') \psi_+(z, x') \right] \\ & \quad \times \left[\psi_+^{(0)}(z, x) + z^{-1/2} \int_0^x dx' \sin(z^{1/2}(x-x')) V(x') \psi_{0,\alpha}(z, x') \right], \quad (2.100) \\ & \quad \quad \quad x \in [0, \infty). \end{aligned}$$

Evaluation of the r.h.s. of (2.100) at $x = 0$ yields (after straightforward manipulation)

$$W(\psi_+(z, \cdot), \psi_{0,\alpha}(z, \cdot)) = W(\psi_+^{(0)}(z, \cdot), \psi_{0,\alpha}^{(0)}(z, \cdot)) + \int_0^\infty dx' V(x') \psi_{0,\alpha}^{(0)}(z, x') \psi_+(z, x'), \quad (2.101)$$

and it follows from (2.98) and (2.101) that

$$\det_{\mathbb{C}^2}(U(z, 0)) = \frac{W(\psi_+(z, \cdot), \psi_{0,\alpha}(z, \cdot))}{W(\psi_+^{(0)}(z, \cdot), \psi_{0,\alpha}^{(0)}(z, \cdot))}. \quad (2.102)$$

The identities (2.96) and (2.102) yield (2.90) under the additional assumption that V has compact support. Next we now show how to remove this additional assumption.

Let χ_ε denote a smooth cutoff function of the type

$$0 \leq \chi \leq 1, \quad \chi(x) = \begin{cases} 1, & x \in [0, 1], \\ 0 & x \geq 2, \end{cases} \quad \chi_\varepsilon(x) = \chi(\varepsilon x), \quad \varepsilon > 0, \quad (2.103)$$

and

$$u_{(\varepsilon)} = u \chi_\varepsilon, \quad v_{(\varepsilon)} = v \chi_\varepsilon, \quad V_{(\varepsilon)} = V \chi_\varepsilon. \quad (2.104)$$

For convenience, we denote

$$K_{\alpha,(\varepsilon)}^{(0)}(z) = \overline{u_{(\varepsilon)} (H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-1} v_{(\varepsilon)}}, \quad K_\alpha^{(0)}(z) = \overline{u (H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-1} v}. \quad (2.105)$$

The integral kernel $G_{(0,\infty),\alpha}^{(0)}(z, x, x')$ of $(H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-1}$ can be expressed in terms of $G_{(0,\infty),0}^{(0)}(z, x, x')$, the integral kernel of the resolvent of the free half-line Dirichlet operator, that is, $(H_{(0,\infty),0}^{(0)} - zI_{(0,\infty)})^{-1}$ as follows (cf., Lemma 2.2),

$$G_{(0,\infty),\alpha}^{(0)}(z, x, x') = G_{(0,\infty),0}^{(0)}(z, x, x') - \frac{\sin(\alpha)e^{iz^{1/2}x}e^{iz^{1/2}x'}}{\cos(\alpha) + iz^{1/2}\sin(\alpha)}, \quad x, x' \geq 0. \quad (2.106)$$

Therefore,

$$\begin{aligned} K_{\alpha}^{(0)}(z) &= \overline{u(H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-1}v} \\ &= \overline{u(H_{(0,\infty),0}^{(0)} - zI_{(0,\infty)})^{-1}v} \\ &\quad - \frac{\sin(\alpha)}{\cos(\alpha) + iz^{1/2}\sin(\alpha)} \left(\overline{ve^{iz^{1/2}\cdot}}, \cdot \right)_{L^2((0,\infty);dx)} u e^{iz^{1/2}\cdot}, \end{aligned} \quad (2.107)$$

and

$$\begin{aligned} K_{\alpha,(\varepsilon)}^{(0)}(z) &= \overline{u_{(\varepsilon)}(H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-1}v_{(\varepsilon)}} \\ &= \overline{u_{(\varepsilon)}(H_{(0,\infty),0}^{(0)} - zI_{(0,\infty)})^{-1}v_{(\varepsilon)}} \\ &\quad - \frac{\sin(\alpha)}{\cos(\alpha) + iz^{1/2}\sin(\alpha)} \left(v_{(\varepsilon)} \overline{e^{iz^{1/2}\cdot}}, \cdot \right)_{L^2((0,\infty);dx)} u_{(\varepsilon)} e^{iz^{1/2}\cdot}. \end{aligned} \quad (2.108)$$

Thus,

$$\begin{aligned} &\|K_{\alpha}^{(0)}(z) - K_{\alpha,(\varepsilon)}^{(0)}(z)\|_{\mathcal{B}_1(L^2((0,\infty);dx))} \\ &\leq \|K_0^{(0)}(z) - K_{0,\varepsilon}^{(0)}\|_{\mathcal{B}_1(L^2((0,\infty);dx))} \\ &\quad + |\eta(\alpha)| \left\| \left(\overline{ve^{iz^{1/2}\cdot}}, \cdot \right)_{L^2((0,\infty);dx)} u e^{iz^{1/2}\cdot} \right. \\ &\quad \left. - \left(v_{(\varepsilon)} \overline{e^{iz^{1/2}\cdot}}, \cdot \right)_{L^2((0,\infty);dx)} u_{(\varepsilon)} e^{iz^{1/2}\cdot} \right\|_{\mathcal{B}_1(L^2((0,\infty);dx))} \\ &= \left\| \overline{u(H_{(0,\infty),0}^{(0)} - zI_{(0,\infty)})^{-1}v} - \overline{u_{(\varepsilon)}(H_{(0,\infty),0}^{(0)} - zI_{(0,\infty)})^{-1}v} \right. \\ &\quad \left. + \overline{u_{(\varepsilon)}(H_{(0,\infty),0}^{(0)} - zI_{(0,\infty)})^{-1}v} \right. \\ &\quad \left. - \overline{u_{(\varepsilon)}(H_{(0,\infty),0}^{(0)} - zI_{(0,\infty)})^{-1}v_{(\varepsilon)}} \right\|_{\mathcal{B}_1(L^2((0,\infty);dx))} \\ &\quad + |\eta(\alpha)| \left\| \left(\overline{ve^{iz^{1/2}\cdot}}, \cdot \right)_{L^2((0,\infty);dx)} u e^{iz^{1/2}\cdot} \right. \\ &\quad \left. - \left(v_{(\varepsilon)} \overline{e^{iz^{1/2}\cdot}}, \cdot \right)_{L^2((0,\infty);dx)} u_{(\varepsilon)} e^{iz^{1/2}\cdot} \right\|_{\mathcal{B}_1(L^2((0,\infty);dx))} \\ &\leq C(z, \alpha) [\|u - u_{(\varepsilon)}\|_{L^2((0,\infty);dx)} + \|v - v_{(\varepsilon)}\|_{L^2((0,\infty);dx)}] \\ &= 2C(z, \alpha) \|u - u_{(\varepsilon)}\|_{L^2((0,\infty);dx)}, \end{aligned} \quad (2.109)$$

where $C(z, \alpha)$ is an appropriate constant, and we abbreviated

$$\eta(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha) + iz^{1/2}\sin(\alpha)}. \quad (2.110)$$

Since (2.109) converges to zero as $\varepsilon \downarrow 0$, one infers

$$\lim_{\varepsilon \downarrow 0} \det_{L^2((0,\infty);dx)} (I_{(0,\infty)} + K_{\alpha,(\varepsilon)}^{(0)}(z)) = \det_{L^2((0,\infty);dx)} (I_{(0,\infty)} + K_{\alpha}^{(0)}(z)). \quad (2.111)$$

Since $V_{(\varepsilon)}$ has compact support, one has

$$\det_{L^2((0,\infty);dx)}(I_{(0,\infty)} + K_{\alpha,(\varepsilon)}^{(0)}(z)) = \frac{\cos(\alpha)\psi_{+,(\varepsilon)}(z,0) + \sin(\alpha)\psi'_{+,(\varepsilon)}(z,0)}{\cos(\alpha) + iz^{1/2}\sin(\alpha)}, \quad (2.112)$$

where $\psi_{+,(\varepsilon)}$ denotes the Jost solution corresponding to the potential $V_{(\varepsilon)}$. Therefore,

$$\det_{L^2((0,\infty);dx)}(I_{(0,\infty)} + K_{\alpha}^{(0)}(z)) = \lim_{\varepsilon \downarrow 0} \frac{\cos(\alpha)\psi_{+,(\varepsilon)}(z,0) + \sin(\alpha)\psi'_{+,(\varepsilon)}(z,0)}{\cos(\alpha) + iz^{1/2}\sin(\alpha)}. \quad (2.113)$$

Iterating the Volterra integral equations (2.50) and (2.51), with V replaced by $V_{(\varepsilon)}$, and ψ_+ replaced by $\psi_{+,(\varepsilon)}$, using the estimates,

$$|\psi_+(z,x)| + |\psi_{+,(\varepsilon)}(z,x)| \leq Ce^{-\operatorname{Im}(z^{1/2})x}, \quad \operatorname{Im}(z^{1/2}) \geq 0, \quad x \geq 0, \quad (2.114)$$

$$|V_{(\varepsilon)}(x)| \leq |V(x)| \text{ for a.e. } x \geq 0, \quad (2.115)$$

together with $V \in L^1((0,\infty);dx)$ and Lebesgue's dominated convergence theorem readily yield

$$\lim_{\varepsilon \downarrow 0} \psi_{+,(\varepsilon)}(z,x) = \psi_+(z,x), \quad \lim_{\varepsilon \downarrow 0} \psi'_{+,(\varepsilon)}(z,x) = \psi'_+(z,x), \quad x \geq 0. \quad (2.116)$$

Thus, taking the limit $\varepsilon \downarrow 0$ on the right-hand side of (2.113) proves (2.91). \square

In the special Dirichlet case $\alpha = 0$, (2.91) is due to Jost and Pais [30] (see also [9]).

Remark 2.7. Due to our hypotheses (2.1) and (2.8) on V one can view $H_{(0,R),\alpha,\beta}$ as a quadratic form sum of $H_{(0,R),\alpha,\beta}^{(0)}$ and V in $L^2((0,R);dx)$,

$$H_{(0,R),\alpha,\beta} = H_{(0,R),\alpha,\beta}^{(0)} +_q V, \quad (2.117)$$

and similarly, one can view $H_{(0,\infty),\alpha}$ as a quadratic form sum of $H_{(0,\infty),\alpha}^{(0)}$ and V in $L^2((0,\infty);dx)$,

$$H_{(0,\infty),\alpha} = H_{(0,\infty),\alpha}^{(0)} +_q V. \quad (2.118)$$

This represents a special case of the perturbation theory approach used also in Appendix A (cf. (A.7)) and will play a role in our principal Section 4. More precisely, the corresponding closed, symmetric, and densely defined sesquilinear form, denoted by $Q_{H_{(0,R),\alpha,\beta}}$, associated with $H_{(0,R),\alpha,\beta}$ (cf. [31, p. 312, 321, 327–328]), reads as follows:

$$\begin{aligned} Q_{H_{(0,R),\alpha,\beta}}(f,g) &= \int_0^R dx [f'(x)\overline{g'(x)} + V(x)\overline{f(x)}g(x)] \\ &\quad - \cot(\alpha)\overline{f(0)}g(0) - \cot(\beta)\overline{f(R)}g(R), \end{aligned} \quad (2.119)$$

$$f, g \in \operatorname{dom}(Q_{H_{(0,R),\alpha,\beta}}) = H^1((0,R))$$

$$= \{h \in L^2((0,R);dx) \mid h \in AC([0,R]); h' \in L^2((0,R);dx)\}, \quad \alpha, \beta \in (0,\pi),$$

$$Q_{H_{(0,R),0,\beta}}(f,g) = \int_0^R dx [f'(x)\overline{g'(x)} + V(x)\overline{f(x)}g(x)] - \cot(\beta)\overline{f(R)}g(R), \quad (2.120)$$

$$f, g \in \operatorname{dom}(Q_{H_{(0,R),0,\beta}})$$

$$= \{h \in L^2((0,R);dx) \mid h \in AC([0,R]); h(0) = 0; h' \in L^2((0,R);dx)\},$$

$$\beta \in (0, \pi),$$

$$\begin{aligned} Q_{H_{(0,R),\alpha,0}}(f, g) &= \int_0^R dx [f'(x)g'(x) + V(x)\overline{f(x)}g(x)] - \cot(\alpha)\overline{f(0)}g(0), \quad (2.121) \\ f, g &\in \text{dom}(Q_{H_{(0,R),\alpha,0}}) \\ &= \{h \in L^2((0, R); dx) \mid h \in AC([0, R]); h(R) = 0; h' \in L^2((0, R); dx)\}, \\ &\alpha \in (0, \pi), \end{aligned}$$

$$\begin{aligned} Q_{H_{(0,R),0,0}}(f, g) &= \int_0^R dx [f'(x)g'(x) + V(x)\overline{f(x)}g(x)], \quad (2.122) \\ f, g &\in \text{dom}(Q_{H_{(0,R),0,0}}) = \text{dom}(|H_{0,0}|^{1/2}) = H_0^1((0, R)) \\ &= \{h \in L^2((0, R); dx) \mid h \in AC([0, R]); h(0) = 0, h(R) = 0; \\ &\quad h' \in L^2((0, R); dx)\}. \end{aligned}$$

Equations (2.119)–(2.122) follow from the fact that for any $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that for all $h \in H^1((0, R))$,

$$|h(x_0)| \leq \varepsilon \|h'\|_{L^2((0,R);dx)} + \eta(\varepsilon) \|h\|_{L^2((0,R);dx)}, \quad x_0 \in [0, R], \quad (2.123)$$

$$\| |V|^{1/2} h \|_{L^2((0,R);dx)} \leq \varepsilon \|h'\|_{L^2((0,R);dx)} + \eta(\varepsilon) \|h\|_{L^2((0,R);dx)} \quad (2.124)$$

(cf. [31, p. 193, 345–346]). In particular, closely examining the inequalities in [31, p. 193, 345–346] shows that $\eta(\varepsilon)$ can be chosen to be $2/\varepsilon$ and hence independently of $R > 0$. Thus, for fixed $\alpha, \beta \in [0, \pi)$, $H_{(0,R),\alpha,\beta}$ are bounded from below uniformly with respect to $R > 0$,

$$H_{(0,R),\alpha,\beta} \geq c_{\alpha,\beta} I_{(0,R)} \quad (2.125)$$

for some $c_{\alpha,\beta} \in \mathbb{R}$ independent of $R > 0$, a fact to be re-examined in Lemma 3.8 and used in Section 4.

Remark 2.8. We note in passing that in the special case of Dirichlet boundary conditions $\alpha = 0$ at $x = 0$ in (2.3) and (2.5), it is well-known (and discussed in great detail, e.g., in [10, Sects. I.1–I.4]) that one can actually accommodate fairly strongly singular potentials at $x = 0$. More precisely, one can permit a.e. real-valued (singular) potentials of the type

$$V \in L^1((0, R); xdx) \text{ respectively, } V \in L^1((0, \infty); [x/(1+x)]dx), \quad (2.126)$$

and all results in this paper extend to potentials satisfying (2.126) as long as the Dirichlet boundary condition $\alpha = 0$ is chosen at the left endpoint $x = 0$.

3. SOME BASIC CONVERGENCE RESULTS

In this section we prove some basic convergence results involving the Schrödinger operators $H_{(0,R),0,0}$ and $H_{(0,\infty),0}$ with Dirichlet boundary conditions. The case of general separated boundary conditions will subsequently be discussed with the help of Lemma 2.1 and 2.2.

We start by decomposing $L^2((0, \infty); dx)$ into

$$L^2((0, \infty); dx) = L^2((0, R); dx) \oplus L^2((R, \infty); dx), \quad (3.1)$$

and all direct operator sums $A = B \oplus C$ from now on are viewed with respect to the decomposition (3.1).

Lemma 3.1. *Let $z \in \mathbb{C} \setminus [0, \infty)$. Then*

$$\text{s-lim}_{R \rightarrow \infty} \left([H_{(0,R),0,0}^{(0)} \oplus 0] - zI_{(0,\infty)} \right)^{-1} = \left(H_{(0,\infty),0}^{(0)} - zI_{(0,\infty)} \right)^{-1}. \quad (3.2)$$

and

$$\lim_{R \rightarrow \infty} \left[u_R \left(H_{(0,R),0,0}^{(0)} - zI_{(0,R)} \right)^{-1} \oplus 0 \right] = u \left(H_{(0,\infty),0}^{(0)} - zI_{(0,\infty)} \right)^{-1} \quad (3.3)$$

in $\mathcal{B}_2(L^2((0, \infty); dx))$.

as well as

$$\lim_{R \rightarrow \infty} \left[\overline{\left(H_{(0,R),0,0}^{(0)} - zI_{(0,R)} \right)^{-1} v_R \oplus 0} \right] = \overline{\left(H_{(0,\infty),0}^{(0)} - zI_{(0,\infty)} \right)^{-1} v} \quad (3.4)$$

in $\mathcal{B}_2(L^2((0, \infty); dx))$.

Proof. Without loss of generality, it suffices to prove (3.2) and (3.3) for $z < 0$.

Since weak resolvent convergence in the context of self-adjoint operators is equivalent to strong resolvent convergence, in order to prove (3.2) it suffices to prove that

$$\begin{aligned} & \lim_{R \rightarrow \infty} \left(f, \left([H_{(0,R),0,0}^{(0)} \oplus 0] - zI_{(0,\infty)} \right)^{-1} g \right)_{L^2((0,\infty); dx)} \\ &= \left(f, \left(H_{(0,\infty),0}^{(0)} - zI_{(0,\infty)} \right)^{-1} g \right)_{L^2((0,\infty); dx)}, \quad f, g \in L^2((0, \infty); dx). \end{aligned} \quad (3.5)$$

In addition, it suffices to verify (3.5) for all f and g belonging to a dense subset (see [52, Theorem 4.26(b)]), which we choose to be

$$L^1((0, \infty); dx) \cap L^2((0, \infty); dx). \quad (3.6)$$

From now on let f and g belong to the set in (3.6). Then

$$\begin{aligned} & \left| \left(f, \left[\left(H_{(0,R),0,0}^{(0)} - zI_{(0,R)} \right)^{-1} \oplus (-z)^{-1} I_{(R,\infty)} \right] g \right)_{L^2((0,\infty); dx)} \right. \\ & \quad \left. - \left(f, \left(H_{(0,\infty),0}^{(0)} - zI_{(0,\infty)} \right)^{-1} g \right)_{L^2((0,\infty); dx)} \right| \\ & \leq \int_0^\infty \int_0^\infty dx' dx \left| \overline{f(x)} [\chi_{(0,R)}(x) \chi_{(0,R)}(x') G_{(0,R),0,0}^{(0)}(z, x, x') \right. \\ & \quad \left. - G_{(0,\infty),0}^{(0)}(z, x, x')] g(x') \right| + \left| \frac{1}{z} \int_R^\infty dx \overline{f(x)} g(x) \right|, \end{aligned} \quad (3.7)$$

where $G_{(0,R),0,0}^{(0)}(z, \cdot, \cdot)$ and $G_{(0,\infty),0}^{(0)}(z, \cdot, \cdot)$ are the integral kernels defined in (2.6) and (2.13), respectively.

It is clear that the second term on the right-hand side of (3.7) converges to zero as $R \rightarrow \infty$.

In view of the identity

$$\psi_{R,0}^{(0)}(z, x) = \frac{\sin(z^{1/2}(R-x))}{\sin(z^{1/2}R)} = e^{iz^{1/2}x} + 2i \frac{e^{iz^{1/2}2R}}{e^{iz^{1/2}2R} - 1} \sin(z^{1/2}x), \quad \text{Im}(z^{1/2}) \geq 0, \quad (3.8)$$

and noting that the second term on the right-hand side of (3.8) converges to zero for all $x > 0$ since by hypothesis $\text{Im}(z^{1/2}) > 0$, it is clear from (2.6) that the integrand in

$$\int_0^\infty \int_0^\infty dx' dx \left| \overline{f(x)} [\chi_{(0,R)}(x) \chi_{(0,R)}(x') G_{(0,\infty),0,0}^{(0)}(z, x, x') - G_{(0,\infty),0}^{(0)}(z, x, x')] g(x') \right| \quad (3.9)$$

converges pointwise to zero as $R \rightarrow \infty$.

Due to the explicit structure of $G_{(0,R),0,0}^{(0)}(z, \cdot, \cdot)$ and $G_{(0,\infty),0}^{(0)}(z, \cdot, \cdot)$ (cf. (2.6), (2.13), (2.53), and (2.54)), there is a constant $C(z) > 0$, depending only on z , such that

$$|\chi_{(0,R)}(x)\chi_{(0,R)}(x')G_{(0,\infty),0,0}^{(0)}(z, x, x') - G_{(0,\infty),0}^{(0)}(z, x, x')| \leq C(z) \quad (3.10)$$

uniformly for a.e. $x, x' \in (0, \infty)$ and all $R > 0$.

Thus, the integrand in (3.9) is majorized by the integrable function

$$C(z)|f(x)g(x')|. \quad (3.11)$$

for R sufficiently large. (One recalls we assumed $f, g \in L^1((0, \infty); dx)$). Consequently, convergence of (3.9) to zero as $R \rightarrow \infty$ follows from an application of Lebesgue's dominated convergence theorem, implying (3.2).

To prove (3.3), we first note that by (2.69) and (2.89),

$$u_R(H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1/2} \in \mathcal{B}_2(L^2((0, R); dx)), \quad (3.12)$$

$$u(H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-1/2} \in \mathcal{B}_2(L^2((0, \infty); dx)), \quad (3.13)$$

in particular,

$$u_R(H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1} \in \mathcal{B}_2(L^2((0, R); dx)), \quad (3.14)$$

$$u(H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-1} \in \mathcal{B}_2(L^2((0, \infty); dx)). \quad (3.15)$$

Thus, it suffices to verify that

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{(0,\infty)} \int_{(0,\infty)} dx dx' & |u(x)\chi_{(0,R)}(x)\chi_{(0,R)}(x')G_{(0,R),0,0}^{(0)}(z, x, x') \\ & - u(x)G_{(0,\infty),0}^{(0)}(z, x, x')|^2 = 0. \end{aligned} \quad (3.16)$$

As in the case of (3.2), the integrand in (3.16) converges pointwise to zero as $R \rightarrow \infty$.

Employing the domain monotonicity (2.65) for Dirichlet Green's functions one obtains the desired R -independent integrable majorant (cf. (3.15))

$$\begin{aligned} & |u(x)\chi_{(0,R)}(x)\chi_{(0,R)}(x')G_{(0,R),0,0}^{(0)}(z, x, x') - u(x)G_{(0,\infty),0}^{(0)}(z, x, x')|^2 \\ & \leq 2|u(x)|^2 |G_{(0,\infty),0}^{(0)}(z, x, x')|^2 \quad \text{for a.e. } x, x' \in (0, \infty). \end{aligned} \quad (3.17)$$

An application of Lebesgue's dominated convergence theorem then completes the proof of (3.3).

Relation (3.4) follows after taking adjoints in (3.3) (replacing u_R, u by v_R, v , resp.). \square

Next, recalling our notation u_R, v_R, V_R in (2.68) and the trace class properties proved in (2.69) and (2.89), we now state the following result:

Lemma 3.2. *Assume (2.8) and let $z \in \mathbb{C} \setminus [0, \infty)$. Then*

$$\begin{aligned} \lim_{R \rightarrow \infty} \left\| \left[u_R(H_{(0,R),0,0}^{(0)} - zI_{(0,R)})^{-1} v_R \oplus 0 \right] \right. \\ \left. - \overline{u(H_{(0,\infty),0}^{(0)} - zI_{(0,\infty)})^{-1} v} \right\|_{\mathcal{B}_1(L^2((0,\infty); dx))} = 0 \end{aligned} \quad (3.18)$$

and

$$\begin{aligned} & \lim_{R \rightarrow \infty} \det_{L^2((0,R);dx)} \left(I_{(0,R)} + \overline{u_R(H_{(0,R),0,0} - zI_{(0,R)})^{-1} v_R} \right) \\ &= \det_{L^2((0,\infty);dx)} \left(I_{(0,\infty)} + \overline{u(H_{(0,\infty),0} - zI_{(0,\infty)})^{-1} v} \right). \end{aligned} \quad (3.19)$$

Proof. It suffices to verify (3.18) for real energies $z < 0$. Convergence for $z \in \mathbb{C} \setminus \mathbb{R}$ then follows from convergence for real energies via the first resolvent identity. Therefore, let $z = -E < 0$ be fixed for the remainder of this proof. In order to establish (3.18) it will be sufficient to show that

$$\begin{aligned} & \lim_{R \rightarrow \infty} \left\| \left[u_R(H_{(0,R),0,0}^{(0)} + EI_{(0,R)})^{-1/2} \oplus 0 \right] \right. \\ & \quad \left. - u(H_{(0,\infty),0}^{(0)} + EI_{(0,\infty)})^{-1/2} \right\|_{\mathcal{B}_2(L^2((0,\infty);dx))}^2 \\ &= \lim_{R \rightarrow \infty} \int_0^\infty \int_0^\infty dx dx' |\mathcal{K}_R(-E, x, x') - \mathcal{K}(-E, x, x')|^2 = 0, \end{aligned} \quad (3.20)$$

where $\mathcal{K}_R(-E, \cdot, \cdot)$ and $\mathcal{K}(-E, \cdot, \cdot)$ denote the integral kernels for the Hilbert-Schmidt operators $u_R[(H_{(0,R),0,0}^{(0)} + EI_{(0,R)})^{-1/2} \oplus 0]$ and $u(H_{(0,\infty),0}^{(0)} + EI_{(0,\infty)})^{-1/2}$, respectively. Specifically,

$$\mathcal{K}_R(-E, x, x') = \begin{cases} u_R(x) R_{(0,R),0,0}^{(0),1/2}(-E, x, x') & \text{for a.e. } x, x' \in (0, R), \\ 0, & \text{for a.e. } x \geq R \text{ or } x' \geq R, \end{cases} \quad (3.21)$$

$$\mathcal{K}(-E, x, x') = u(x) R_{(0,\infty),0}^{(0),1/2}(-E, x, x') \quad \text{for a.e. } x, x' \in (0, \infty). \quad (3.22)$$

An application of (2.56) and (2.64) then yields the R -independent majorant

$$\begin{aligned} & |\mathcal{K}_R(-E, x, x') - \mathcal{K}(-E, x, x')|^2 \leq 4|\mathcal{K}(-E, x, x')|^2 \\ & \leq 4|u(x)|^2 |R_{(0,R),0,0}^{(0),1/2}(-E, x, x')|^2 \\ & \leq 4\pi^{-2}|u(x)|^2 [|K_0(E^{1/2}|x-x'|)|^2 + |K_0(E^{1/2}(x+x'))|^2] \end{aligned} \quad (3.23)$$

for a.e. $x, x' \in (0, \infty)$. The zeroth order irregular modified Bessel function $K_0(\cdot)$ satisfies the estimate

$$0 \leq K_0(x) \leq C \ln \left(\frac{ex}{1+x} \right) \frac{e^{-x}}{2\pi x^{1/2} + 1}, \quad x > 0, \quad (3.24)$$

for a suitable constant $C > 0$ (cf. [1, Sect. 9.6] for the proper asymptotic relations as $x \downarrow 0$ and $x \rightarrow \infty$, implying (3.24)). Thus, the right-hand side of (3.23) represents an R -independent integrable majorant of $|\mathcal{K}_R(-E, \cdot, \cdot) - \mathcal{K}(-E, \cdot, \cdot)|^2$ a.e. on $(0, \infty)^2$.

Applying Lebesgue's dominated convergence theorem to (2.59), employing the integrable majorant (2.63) (equivalently, applying the domain monotonicity (2.65)) one infers for a.e. $x, x' \in (0, \infty)$ that

$$\begin{aligned} \lim_{R \rightarrow \infty} \mathcal{K}_R(z, x, x') &= \lim_{\substack{R \rightarrow \infty \\ R \geq \max\{x, x'\}}} u_R(x) R_{(0,R),0,0}^{(0),1/2}(z, x, x') \\ &= u(x) R_{(0,\infty),0}^{(0),1/2}(z, x, x') \\ &= \mathcal{K}(z, x, x'), \end{aligned} \quad (3.25)$$

that is, $\mathcal{K}_R(z, \cdot, \cdot)$ converges pointwise almost everywhere on $(0, \infty)^2$ to $\mathcal{K}(z, \cdot, \cdot)$.

An application of Lebesgue's dominated convergence theorem to (3.20), using the R -independent integrable majorant in (3.23) then completes the proof of (3.18).

In view of the identities,

$$\begin{aligned} & \det_{L^2((0,\infty);dx)} \left(I_{(0,\infty)} + \overline{u \left[\left(H_{(0,R),0,0}^{(0)} - zI_{(0,R)} \right)^{-1} \oplus 0 \right] v} \right) \\ &= \det_{L^2((0,\infty);dx)} \left(I_{(0,R)} \oplus I_{(R,\infty)} + \overline{u_R \left(H_{(0,R),0,0}^{(0)} - zI_{(0,R)} \right)^{-1} v_R \oplus 0} \right) \\ &= \det_{L^2((0,R);dx)} \left(I_{(0,R)} + \overline{u_R \left(H_{(0,R),0,0}^{(0)} - zI_{(0,R)} \right)^{-1} v_R} \right). \end{aligned} \quad (3.26)$$

and the fact that the mapping

$$\begin{cases} \mathcal{B}_1(\mathcal{H}) \rightarrow \mathbb{C}, \\ A \mapsto \det_{\mathcal{H}}(I_{\mathcal{H}} + A), \end{cases} \quad (3.27)$$

is continuous in A with respect to the trace norm on $\mathcal{B}_1(\mathcal{H})$, (3.19) follows from (3.18). \square

Lemma 3.3. *Assume (2.8) and let $z \in \mathbb{C} \setminus [0, \infty)$. Then*

$$\text{s-lim}_{R \rightarrow \infty} \left([H_{(0,R),0,0} \oplus 0] - zI_{(0,\infty)} \right)^{-1} = (H_{(0,\infty),0} - zI_{(0,\infty)})^{-1}. \quad (3.28)$$

Proof. Since

$$\begin{aligned} & \begin{pmatrix} H_{(0,R),0,0} - zI_{(0,R)} & 0 \\ 0 & -zI_{(R,\infty)} \end{pmatrix}^{-1} = \begin{pmatrix} (H_{(0,R),0,0} - zI_{(0,R)})^{-1} & 0 \\ 0 & -z^{-1}I_{(R,\infty)} \end{pmatrix} \\ &= [(H_{(0,R),0,0} - zI_{(0,R)})^{-1} \oplus 0] - [0 \oplus z^{-1}I_{(R,\infty)}] \end{aligned} \quad (3.29)$$

with respect to the decomposition (3.1), and since $\text{s-lim}_{R \rightarrow \infty} [0 \oplus z^{-1}I_{(R,\infty)}] = 0$, it suffices to show that

$$\text{s-lim}_{R \rightarrow \infty} [(H_{(0,R),0,0} - zI_{(0,R)})^{-1} \oplus 0] = (H_{(0,\infty),0} - zI_{(0,\infty)})^{-1}. \quad (3.30)$$

To verify (3.30), we use Lemma 3.1 (i), (ii) and (3.18) together with the resolvent identities

$$\begin{aligned} & (H_{(0,R),0,0} - zI_{(0,R)})^{-1} = (H_{(0,R),0,0}^{(0)} - zI_{(0,R)})^{-1} \\ & \quad - \overline{(H_{(0,R),0,0}^{(0)} - zI_{(0,R)})^{-1} v_R \left[I_{(0,R)} + u_R (H_{(0,R),0,0}^{(0)} - zI_{(0,R)})^{-1} v_R \right]^{-1}} \\ & \quad \times u_R (H_{(0,R),0,0}^{(0)} - zI_{(0,R)})^{-1}, \end{aligned} \quad (3.31)$$

$$\begin{aligned} & (H_{(0,\infty),0} - zI_{(0,\infty)})^{-1} = (H_{(0,\infty),0}^{(0)} - zI_{(0,\infty)})^{-1} \\ & \quad - \overline{(H_{(0,\infty),0}^{(0)} - zI_{(0,\infty)})^{-1} v \left[I_{(0,\infty)} + u (H_{(0,\infty),0}^{(0)} - zI_{(0,\infty)})^{-1} v \right]^{-1}} \\ & \quad \times u (H_{(0,\infty),0}^{(0)} - zI_{(0,\infty)})^{-1}. \end{aligned} \quad (3.32)$$

Using

$$\begin{aligned} & \left[I_{(0,R)} + \overline{u_R (H_{(0,R),0,0}^{(0)} - zI_{(0,R)})^{-1} v_R} \right]^{-1} \oplus 0 \\ &= \left[I_{(0,\infty)} + \overline{u_R (H_{(0,R),0,0}^{(0)} - zI_{(0,R)})^{-1} v_R \oplus 0} \right]^{-1} - (0 \oplus I_{(R,\infty)}) \end{aligned}$$

together with (3.31) yields

$$\begin{aligned}
(H_{(0,R),0,0} - zI_{(0,R)})^{-1} \oplus 0 &= \left[(H_{(0,R),0,0}^{(0)} - zI_{(0,R)})^{-1} \oplus 0 \right] \\
&\quad - \overline{\left((H_{(0,R),0,0}^{(0)} - zI_{(0,R)})^{-1} v_R \oplus 0 \right)} \\
&\quad \times \left(\left[I_{(0,\infty)} + \overline{(u_R(H_{(0,R),0,0}^{(0)} - zI_{(0,R)})^{-1} v_R \oplus 0)} \right]^{-1} - (0 \oplus I_{(R,\infty)}) \right) \\
&\quad \times \left(u_R(H_{(0,R),0,0}^{(0)} - zI_{(0,R)})^{-1} \oplus 0 \right). \tag{3.33}
\end{aligned}$$

Based on the following convergence results,

$$\begin{aligned}
\text{n-}\lim_{R \rightarrow \infty} \left[I_{(0,\infty)} + \overline{u_R(H_{(0,R),0,0}^{(0)} - zI_{(0,R)})^{-1} v_R \oplus 0} \right]^{-1} \\
= \left[I_{(0,\infty)} + \overline{u(H_{(0,\infty),0}^{(0)} - zI_{(0,\infty)})^{-1} v} \right]^{-1}, \tag{3.34}
\end{aligned}$$

$$\text{s-}\lim_{R \rightarrow \infty} (0 \oplus I_{(R,\infty)}) = 0, \tag{3.35}$$

$$\text{s-}\lim_{R \rightarrow \infty} \left((H_{(0,R),0,0}^{(0)} - zI_{(0,R)})^{-1} \oplus 0 \right) = (H_{(0,\infty),0}^{(0)} - zI_{(0,\infty)})^{-1}, \tag{3.36}$$

$$\begin{aligned}
\lim_{R \rightarrow \infty} \left\| \left[u_R(H_{(0,R),0,0}^{(0)} - zI_{(0,R)})^{-1} \oplus 0 \right] \right. \\
\left. - u(H_{(0,\infty),0}^{(0)} - zI_{(0,\infty)})^{-1} \right\|_{\mathcal{B}_2(L^2((0,\infty); dx))} = 0, \tag{3.37}
\end{aligned}$$

$$\begin{aligned}
\lim_{R \rightarrow \infty} \left\| \overline{\left[(H_{(0,R),0,0}^{(0)} - zI_{(0,R)})^{-1} v_R \oplus 0 \right]} \right. \\
\left. - \overline{(H_{(0,\infty),0}^{(0)} - zI_{(0,\infty)})^{-1} v} \right\|_{\mathcal{B}_2(L^2((0,\infty); dx))} = 0, \tag{3.38}
\end{aligned}$$

the right-hand side of (3.33) converges strongly to the right-hand side of (3.32). We note that (3.34) relies on (3.18), while (3.36), (3.37), and (3.38) follow from Lemma 3.1. \square

Next we extend Lemmas 3.1–3.3 to the case of general separated boundary conditions applying Lemmas 2.1 and 2.2.

Lemma 3.4. *Assume (2.8) and let $z \in \mathbb{C} \setminus [0, \infty)$ and $\alpha, \beta \in [0, \pi)$. Then*

$$\text{s-}\lim_{R \rightarrow \infty} \left([H_{(0,R),\alpha,\beta}^{(0)} \oplus 0] - zI_{(0,\infty)} \right)^{-1} = (H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-1}. \tag{3.39}$$

and

$$\begin{aligned}
\lim_{R \rightarrow \infty} \left[u_R(H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1} \oplus 0 \right] &= u(H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-1} \\
&\quad \text{in } \mathcal{B}_2(L^2((0, \infty); dx)). \tag{3.40}
\end{aligned}$$

as well as

$$\begin{aligned}
\lim_{R \rightarrow \infty} \left[\overline{(H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1} v_R \oplus 0} \right] &= \overline{(H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-1} v} \\
&\quad \text{in } \mathcal{B}_2(L^2((0, \infty); dx)). \tag{3.41}
\end{aligned}$$

Moreover,

$$\lim_{R \rightarrow \infty} \left\| \left[u_R(H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1} v_R \oplus 0 \right] \right.$$

$$-u \left(\overline{H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)}} \right)^{-1} v \Big\|_{\mathcal{B}_1(L^2((0,\infty);dx))} = 0 \quad (3.42)$$

and

$$\begin{aligned} & \lim_{R \rightarrow \infty} \det_{L^2((0,R);dx)} \left(I_{(0,R)} + \overline{u_R \left(H_{(0,R),\alpha,\beta} - zI_{(0,R)} \right)^{-1} v_R} \right) \\ &= \det_{L^2((0,\infty);dx)} \left(I_{(0,\infty)} + \overline{u \left(H_{(0,\infty),\alpha} - zI_{(0,\infty)} \right)^{-1} v} \right). \end{aligned} \quad (3.43)$$

In addition, assume (2.8). Then

$$\text{s-lim}_{R \rightarrow \infty} \left([H_{(0,R),\alpha,\beta} \oplus 0] - zI_{(0,\infty)} \right)^{-1} = \left(H_{(0,\infty),\alpha} - zI_{(0,\infty)} \right)^{-1}. \quad (3.44)$$

Proof. Consecutively employing Lemmas 2.1 and 2.2, we will reduce the proofs for Lemma 3.4 to those of Lemmas 3.1–14.3 as follows:

$$\begin{aligned} G_{(0,R),\alpha,\beta}(z, x, x') &= G_{(0,\infty),\alpha}^{(0)}(z, x, x') \\ &+ \frac{\sin(\beta)\psi'_+(z, R) + \cos(\beta)\psi_+(z, R)}{\sin(\beta)\psi'_{0,\alpha}(z, R) + \cos(\beta)\psi_{0,\alpha}(z, R)} \psi_{0,\alpha}(z, x)\psi_{0,\alpha}(z, x') \\ &= G_{(0,\infty),0}^{(0)}(z, x, x') \\ &+ \frac{\sin(\alpha)\psi'_{0,0}(z, 0) + \cos(\alpha)\psi_{0,0}(z, 0)}{W(\psi_{0,0}(z, \cdot), \psi_+(z, \cdot))[\sin(\alpha)\psi'_+(z, 0) + \cos(\beta)\psi_+(z, 0)]} \psi_+(z, x)\psi_+(z, x') \\ &+ \frac{\sin(\beta)\psi'_+(z, R) + \cos(\beta)\psi_+(z, R)}{\sin(\beta)\psi'_{0,\alpha}(z, R) + \cos(\beta)\psi_{0,\alpha}(z, R)} \psi_{0,\alpha}(z, x)\psi_{0,\alpha}(z, x') \\ &= G_{(0,R),0,0}(z, x, x') \\ &+ \frac{\sin(\alpha)\psi'_{0,0}(z, 0) + \cos(\alpha)\psi_{0,0}(z, 0)}{W(\psi_{0,0}(z, \cdot), \psi_+(z, \cdot))[\sin(\alpha)\psi'_+(z, 0) + \cos(\beta)\psi_+(z, 0)]} \psi_+(z, x)\psi_+(z, x') \\ &- \frac{\psi_+(z, R)}{W(\psi_{0,0}(z, \cdot), \psi_+(z, \cdot))\psi_{0,0}(z, R)} \psi_{0,0}(z, x)\psi_{0,0}(z, x') \\ &+ \frac{\sin(\beta)\psi'_+(z, R) + \cos(\beta)\psi_+(z, R)}{\sin(\beta)\psi'_{0,\alpha}(z, R) + \cos(\beta)\psi_{0,\alpha}(z, R)} \psi_{0,\alpha}(z, x)\psi_{0,\alpha}(z, x'), \quad x, x' \in (0, R). \end{aligned} \quad (3.45)$$

Similarly, one has

$$\begin{aligned} G_{(0,\infty),\alpha}^{(0)}(z, x, x') &= G_{(0,\infty),0}^{(0)}(z, x, x') \\ &+ \frac{\sin(\alpha)\psi'_{0,0}(z, 0) + \cos(\alpha)\psi_{0,0}(z, 0)}{W(\psi_{0,0}(z, \cdot), \psi_+(z, \cdot))[\sin(\alpha)\psi'_+(z, 0) + \cos(\beta)\psi_+(z, 0)]} \psi_+(z, x)\psi_+(z, x'), \\ & \quad x, x' \in (0, \infty). \end{aligned} \quad (3.46)$$

One notes that the second terms on the right-hand sides in (3.45) and (3.46) are identical except for the underlying interval $(0, R)$ versus $(0, \infty)$. In this context we refer to the subsequent, elementary result (3.61) in Lemma 3.6, which deals with precisely such situations abstractly.

Next, one observes that iterating the Volterra integral equations (2.45) and (2.50) yields the bounds (cf. also [10, Sects. I.2, I.3]),

$$|\psi_{0,0}(z, R)| \leq C \frac{R}{1 + |z^{1/2}|R} e^{\text{Im}(z^{1/2})R}, \quad (3.47)$$

$$|\psi_{0,0}(z, R) - \psi_{(0,0)}^{(0)}(z, R)| \leq C \frac{R e^{\operatorname{Im}(z^{1/2})R}}{1 + |z^{1/2}|R} \int_0^R dx \frac{x}{1 + |z^{1/2}|x} |V(x)|, \quad (3.48)$$

$$|\psi_+(z, R)| \leq C e^{-\operatorname{Im}(z^{1/2})R}, \quad (3.49)$$

$$|\psi'_+(z, R)| \leq C |z|^{1/2} e^{-\operatorname{Im}(z^{1/2})R}, \quad (3.50)$$

$$|\psi_+(z, R) - \psi_+^{(0)}(z, R)| \leq C \frac{e^{\operatorname{Im}(z^{1/2})R}}{|z^{1/2}|} \int_R^\infty dx |V(x)|, \quad z \in \mathbb{C} \setminus \{0\}, \quad (3.51)$$

$$|\psi'_+(z, R) - \psi_+^{(0)'}(z, R)| \leq C e^{\operatorname{Im}(z^{1/2})R} \int_R^\infty dx |V(x)|, \quad (3.52)$$

where we recall our convention of $\operatorname{Im}(z^{1/2}) \geq 0$, $z \in \mathbb{C}$. In addition one notes that $W(\psi_{0,0}(z, \cdot), \psi_+(z, \cdot))$ is x -independent and hence just a function of z .

Due to the estimates (3.47)–(3.52), the third and fourth terms on the right-hand side of (3.45) generate rank-one operators whose norms exhibit a decay of the order $e^{-2\operatorname{Im}(z^{1/2})R}$ as $R \rightarrow \infty$.

At this point one can multiply (3.45) and (3.46) on the left with $u_R(x)$ and $u(x)$ and reduce the proof of (3.39) and (3.40) to (3.2) and (3.3) in Lemma 3.1 together with the exponential decay of the two rank-one operators generated by the third and fourth term on the right-hand side of (3.45). Analogous observations apply to (3.41).

Similarly, multiplying (3.45) and (3.46) on the left with $u_R(x)$ and $u(x)$ and on the right with $v_R(x')$ and $v(x')$, respectively, one can prove (3.42) and (3.43) by reducing them to (3.18) and (3.19) in Lemma 3.2.

Finally, (3.44) follows in the same vein from (3.28) in Lemma 3.3. \square

Remark 3.5. For subsequent purposes, we decided to provide a purely operator theoretic proof of the convergence of Fredholm determinants in (3.43). An elementary alternative proof based on Jost functions and the Wronski relations (2.70) and (2.90) is also possible, see [39].

In order to deal with the the second terms on the right-hand sides in (3.45) and (3.46) we formulate the following elementary abstract result: Let \mathcal{H} be a complex, separable Hilbert space and $\{P_n\}_{n \in \mathbb{N}}$ a sequence of orthogonal projections in \mathcal{H} converging strongly to the identity as $n \rightarrow \infty$,

$$P_n = P_n^2 = P_n^*, \quad n \in \mathbb{N}, \quad \text{s-lim}_{n \rightarrow \infty} P_n = I_{\mathcal{H}}. \quad (3.53)$$

We also introduce

$$P_n^\perp = I_{\mathcal{H}} - P_n, \quad \mathcal{H}_n = P_n \mathcal{H}, \quad \mathcal{H}_n^\perp = P_n^\perp \mathcal{H}, \quad n \in \mathbb{N}, \quad (3.54)$$

and write for any $h \in \mathcal{H}$, and $n \in \mathbb{N}$,

$$h = \langle h_n, h_n^\perp \rangle = h_n \oplus h_n^\perp \quad \text{with respect to } \mathcal{H} = \mathcal{H}_n \oplus \mathcal{H}_n^\perp, \quad (3.55)$$

with the standard convention for the inner product,

$$\langle h, k \rangle_{\mathcal{H}} = (\langle h_n, h_n^\perp \rangle, \langle k_n, k_n^\perp \rangle)_{\mathcal{H}} = \langle h_n, k_n \rangle_{\mathcal{H}_n} + \langle h_n^\perp, k_n^\perp \rangle_{\mathcal{H}_n^\perp}, \quad h, k \in \mathcal{H}. \quad (3.56)$$

Finally, for $f = \langle f_n, f_n^\perp \rangle, g = \langle g_n, g_n^\perp \rangle \in \mathcal{H}$, we introduce the rank-one operator T in \mathcal{H} by

$$Th = \langle f, h \rangle_{\mathcal{H}} g, \quad h = \langle h_n, h_n^\perp \rangle \in \mathcal{H}, \quad (3.57)$$

and for each $n \in \mathbb{N}$, the rank-one operator T_n in \mathcal{H}_n by

$$T_n h_n = (f_n, h_n) g_n, \quad n \in \mathbb{N}. \quad (3.58)$$

Lemma 3.6. *With the notation introduced in (3.53)–(3.58), one has*

$$[T - (T_n \oplus 0)]h = ((I - P_n)g, h)_{\mathcal{H}}g + (P_n f, h)_{\mathcal{H}}(I - P_n)g, \quad h \in \mathcal{H}, \quad (3.59)$$

and

$$\begin{aligned} \|T - (T_n \oplus 0)\|_{\mathcal{B}_1(\mathcal{H})} &\leq \|(I - P_n)f\|_{\mathcal{H}}\|g\|_{\mathcal{H}} + \|P_n f\|_{\mathcal{H}}\|(I - P_n)g\|_{\mathcal{H}} \\ &\leq 2\|f\|_{\mathcal{H}}\|g\|_{\mathcal{H}}. \end{aligned} \quad (3.60)$$

In particular,

$$\lim_{n \rightarrow \infty} \|T_n \oplus 0 - T\|_{\mathcal{B}_1(\mathcal{H})} = 0. \quad (3.61)$$

Proof. One computes for $h = h_n \oplus h_n^\perp \in \mathcal{H}$

$$\begin{aligned} [T - (T_n \oplus 0)]h &= Th - (T_n \oplus 0)h \\ &= (f, h)_{\mathcal{H}}g - [(f_n, h_n)_{\mathcal{H}_n}g_n \oplus 0] \\ &= [(f_n, h_n)_{\mathcal{H}_n} + (f_n^\perp, h_n^\perp)_{\mathcal{H}_n^\perp}]g - [(f_n, h_n)_{\mathcal{H}_n} \oplus 0] \\ &= (f_n^\perp, h_n^\perp)_{\mathcal{H}_n^\perp}g_n \oplus [(f_n, h_n)_{\mathcal{H}_n} + (f_n^\perp, h_n^\perp)_{\mathcal{H}_n^\perp}]g_n^\perp \\ &= (f_n^\perp, h_n^\perp)_{\mathcal{H}_n^\perp}g + [0 \oplus (f_n, h_n)_{\mathcal{H}_n}g_n^\perp] \\ &= ((I - P_n)g, h)_{\mathcal{H}}g + (P_n f, h)_{\mathcal{H}}(I - P_n)g. \end{aligned} \quad (3.62)$$

Thus,

$$T - (T_n \oplus 0) = ((I - P_n)f, \cdot)_{\mathcal{H}}g + (P_n f, \cdot)_{\mathcal{H}}(I - P_n)g, \quad (3.63)$$

implying

$$\begin{aligned} \|T - (T_n \oplus 0)\|_{\mathcal{B}_1(\mathcal{H})} &\leq \|((I - P_n)f, \cdot)_{\mathcal{H}}g\|_{\mathcal{B}_1(\mathcal{H})} + \\ &\quad + \|(P_n f, \cdot)_{\mathcal{H}}(I - P_n)g\|_{\mathcal{B}_1(\mathcal{H})} \\ &\leq \|(I - P_n)f\|_{\mathcal{H}}\|g\|_{\mathcal{H}} + \|P_n f\|_{\mathcal{H}}\|(I - P_n)g\|_{\mathcal{H}}. \end{aligned} \quad (3.64)$$

The two terms in (3.64) go to zero since $s\text{-}\lim_{n \rightarrow \infty} P_n = I_{\mathcal{H}}$ and $\|P_n f\|_{\mathcal{B}(\mathcal{H})} = 1$, $n \in \mathbb{N}$. \square

Remark 3.7. The convergence results of Lemma 3.1 (i) and Lemma 3.3 and similarly those of (3.39) and (3.44) in Lemma 3.4 are similar in spirit to [49, Theorem 6] (c.f. also [4], [49, Theorem 5], [50, Theorem 4], and [51, Lemma 4]). More specifically, the authors show that $A_n \rightarrow A$ in the sense of *generalized strong convergence*, that is,

$$s\text{-}\lim_{n \rightarrow \infty} (A_n - zI_{(a_n, b_n)})^{-1}P_n = (A - zI_{(a, b)})^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (3.65)$$

where A_n (resp., A) is the self-adjoint realization of a (rather general) Sturm–Liouville differential expression in $L^2((a_n, b_n); r(x)dx)$ (resp., $L^2((a, b); r(x)dx)$), with $a_n \downarrow a$ (or $a_n = a$ identically), $b_n \uparrow b$, and P_n denotes the orthogonal projection in $L^2((a, b); r(x)dx)$ onto $L^2((a_n, b_n); r(x)dx)$, with the latter identified with a closed subspace of $L^2((a, b); r(x)dx)$.

Next, we briefly re-examine the uniform boundedness from below of $H_{(0, R), \alpha, \beta}$ with respect to R (cf. (2.125)).

Lemma 3.8. *Assume (2.8).*

(i) *There exists $E(\alpha, \beta) < 0$, such that for all $z \leq E(\alpha, \beta)$ and all $R > 0$,*

$$\left\| \overline{u_R(H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1}v_R \oplus 0} \right\|_{\mathcal{B}_1(L^2((0,\infty);dx))} \leq C_{\alpha,\beta}|z|^{-1/2}. \quad (3.66)$$

for some constant $C_{\alpha,\beta} > 0$, independent of $R > 0$. In particular,

$$H_{(0,R),\alpha,\beta} \geq c_{\alpha,\beta}I_{(0,R)} \quad (3.67)$$

for some $c_{\alpha,\beta} \leq 0$ independent of $R > 0$.

(ii) *There exists $E(\alpha) < 0$, such that for all $z \leq E(\alpha)$,*

$$\left\| \overline{u(H_{(0,\infty),\alpha}^{(0)} - zI_{(0,R)})^{-1}v} \right\|_{\mathcal{B}_1(L^2((0,\infty);dx))} \leq C_\alpha|z|^{-1/2}. \quad (3.68)$$

for some constant $C(\alpha) > 0$. In particular,

$$H_{(0,\infty),\alpha} \geq c_\alpha I_{(0,\infty)} \quad (3.69)$$

for some $c_\alpha \leq 0$.

Proof. Combining (2.6), (2.13), (2.42), (2.43), (2.48), and (2.49), one infers that

$$\left| G_{(0,R),\alpha,\beta}^{(0)}(z, x, x) \right| \leq C(\alpha, \beta)|z|^{-1/2}, \quad x \in [0, R], \quad R > 0, \quad z < E(\alpha, \beta), \quad (3.70)$$

$$\left| G_{(0,\infty),\alpha}^{(0)}(z, x, x) \right| \leq C(\alpha)|z|^{-1/2}, \quad x \in [0, \infty), \quad z < E(\alpha) \quad (3.71)$$

for some constants $C(\alpha, \beta) > 0$, $C(\alpha) > 0$.

Thus, using $u = \text{sgn}(V)v$, with $\text{sgn}(V(x)) = 1$ if $V(x) \geq 0$ a.e., and $\text{sgn}(V(x)) = -1$ if $V(x) < 0$ a.e., rendering $\text{sgn}(V)$ a unitary operator of multiplication in $L^2((0, \infty); dx)$, and analogously for $u_R = \text{sgn}(V_R)v_R$, one infers for $z < 0$ sufficiently negative, that

$$\begin{aligned} & \left\| u_R(H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1}v_R \right\|_{\mathcal{B}_1(L^2((0,R);dx))} \\ &= \left\| v_R(H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1}v_R \right\|_{\mathcal{B}_1(L^2((0,R);dx))} \\ &= \text{tr}_{L^2((0,R);dx)} \left(v_R(H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1}v_R \right) \\ &= \int_0^R dx |V(x)| G_{(0,R),\alpha,\beta}^{(0)}(z, x, x) \\ &\leq C(\alpha, \beta) \int_0^R dx |V(x)| |z|^{-1/2}, \end{aligned} \quad (3.72)$$

and analogously, for $z < 0$ sufficiently negative,

$$\left\| u(H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-1}v \right\|_{\mathcal{B}_1(L^2((0,\infty);dx))} \leq C(\alpha) \int_0^\infty dx |V(x)| |z|^{-1/2}, \quad (3.73)$$

proving (3.66) and (3.68).

Thus, for $z \leq c_{\alpha,\beta}$, for some $c_{\alpha,\beta} \leq 0$ independent of $R > 0$,

$$\left[I_{(0,\infty)} + \overline{u_R(H_{(0,R),0,0}^{(0)} - zI_{(0,R)})^{-1}v_R \oplus 0} \right]^{-1} \in \mathcal{B}(L^2((0, \infty); dx)), \quad (3.74)$$

proving (3.67), employing the resolvent equation (3.31). The lower bound (3.69) is proved analogously (but also follows from the estimate (2.124) with $(0, R)$ replaced by $(0, \infty)$, cf., e.g., [44, eq. (2.7.8)]). \square

We conclude this section with the following observation that will play an important role in Section 4 later on:

Remark 3.9. We emphasize that our choice of $H_{(0,R),0,0}^{(0)}$ and $H_{(0,\infty),0}^{(0)}$ in Lemmas 3.1 and 3.2 is of no importance. In fact, using the general inequalities (2.65) and (2.66) (for $q = 1/2$), and iterating the corresponding Volterra integral equations (2.45) and (2.50) to obtain standard bounds on $\psi_{0,0}(z, \cdot)$, $\psi_{R,0}(z, \cdot)$, and $\psi_+(z, \cdot)$ in terms of those of $\psi_{0,0}^{(0)}(z, \cdot)$, $\psi_{R,0}^{(0)}(z, \cdot)$, and $\psi_+^{(0)}(z, \cdot)$ with certain z -dependent constants shows that Lemmas 3.1 and 3.2 extend to the case where $H_{(0,R),0,0}^{(0)}$ and $H_{(0,\infty),0}^{(0)}$ are replaced by general Schrödinger operators $H_{(0,R),0,0}$ and $H_{(0,\infty),0}$ as long as their underlying potentials W satisfy $W \in L^1((0, R); dx)$ and $W \in L^1((0, \infty); dx)$, respectively.

Using again (3.45) and (3.46) this observation extends to $H_{(0,R),\alpha,\beta}^{(0)}$ and $H_{(0,\infty),\alpha}^{(0)}$ in Lemma 3.4. Hence adding a potential W satisfying $W \in L^1((0, R); dx)$ and $W \in L^1((0, \infty); dx)$, respectively, to $H_{(0,R),\alpha,\beta}^{(0)}$ and $H_{(0,\infty),\alpha}^{(0)}$ will not alter the conclusions in Lemma 3.4. Put differently, our methods immediately apply to the case where the “unperturbed” operator contains a background potential W (with $W \in L^1((0, \infty); dx)$).

4. WEAK CONVERGENCE OF SPECTRAL SHIFT FUNCTIONS

In this section we prove our principal new results, the weak convergence of the absolutely continuous measures $\xi(\lambda; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)})d\lambda$, associated with the pair of self-adjoint operators $(H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)})$, to the absolutely continuous measure $\xi(\lambda; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)})d\lambda$, associated with the pair $(H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)})$. Our approach will be based on convergence of underlying Fredholm determinants combined with certain measure theoretic facts.

We will freely use the facts recorded (and notations employed) on spectral shift functions in Appendix A and, in particular, note that (A.8), (A.9) imply

$$\begin{aligned} & -\frac{d}{dz} \det_{L^2((0,\infty); dx)} \left(I_{(0,\infty)} + \overline{u(H_{(0,\infty),\alpha} - zI_{(0,\infty)})^{-1}v} \right) \\ & = \operatorname{tr}_{L^2((0,\infty); dx)} \left((H_{(0,\infty),\alpha} - z)^{-1} - (H_{(0,\infty),\alpha}^{(0)} - z)^{-1} \right) \\ & = - \int_{\mathbb{R}} \frac{\xi(\lambda; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)})d\lambda}{(\lambda - z)^2}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \end{aligned} \quad (4.1)$$

Combined with (2.91), this represents a result due to Buslaev and Faddeev [9] in the Dirichlet case $\alpha = 0$. We will also use the analog of (4.1) with $(0, \infty)$ replaced by $(0, R)$ and u, v replaced by u_R, v_R , etc.

We start with the following basic result:

Lemma 4.1. *Assume (2.8) and let $a, z \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$\lim_{R \rightarrow \infty} \ln \left(\frac{\det_{L^2((0,R); dx)} \left(I_{(0,R)} + \overline{u_R(H_{(0,R),\alpha,\beta} - zI_{(0,R)})^{-1}v_R} \right)}{\det_{L^2((0,R); dx)} \left(I_{(0,R)} + \overline{u_R(H_{(0,R),\alpha,\beta} - aI_{(0,R)})^{-1}v_R} \right)} \right)$$

$$= \ln \left(\frac{\det_{L^2((0,\infty);dx)} \left(I_{(0,\infty)} + u \overline{(H_{(0,\infty),\alpha} - zI_{(0,\infty)})^{-1}v} \right)}{\det_{L^2((0,\infty);dx)} \left(I_{(0,\infty)} + u \overline{(H_{(0,\infty),\alpha} - aI_{(0,\infty)})^{-1}v} \right)} \right), \quad (4.2)$$

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}} \frac{\xi(\lambda; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)}) d\lambda}{(\lambda - a)(\lambda - z)^n} = \int_{\mathbb{R}} \frac{\xi(\lambda; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)}) d\lambda}{(\lambda - a)(\lambda - z)^n}, \quad n \in \mathbb{N}. \quad (4.3)$$

Proof. Convergence in (4.2) is a direct consequence of (3.43). Moreover, since

$$\begin{aligned} & \ln \left(\frac{\det_{L^2((0,R);dx)} \left(I_{(0,R)} + u_R \overline{(H_{(0,R),\alpha,\beta} - zI_{(0,R)})^{-1}v_R} \right)}{\det_{L^2((0,R);dx)} \left(I_{(0,R)} + u_R \overline{(H_{(0,R),\alpha,\beta} - aI_{(0,R)})^{-1}v_R} \right)} \right) \\ &= (z - a) \int_{\mathbb{R}} \frac{\xi(\lambda; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)}) d\lambda}{(\lambda - a)(\lambda - z)}, \end{aligned} \quad (4.4)$$

$$\begin{aligned} & \ln \left(\frac{\det_{L^2((0,\infty);dx)} \left(I_{(0,\infty)} + u \overline{(H_{(0,\infty),\alpha} - zI_{(0,\infty)})^{-1}v} \right)}{\det_{L^2((0,\infty);dx)} \left(I_{(0,\infty)} + u \overline{(H_{(0,\infty),\alpha} - aI_{(0,\infty)})^{-1}v} \right)} \right) \\ &= (z - a) \int_{\mathbb{R}} \frac{\xi(\lambda; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)}) d\lambda}{(\lambda - a)(\lambda - z)} \end{aligned} \quad (4.5)$$

(cf. (4.1) and its half-line analog), one immediately gets

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}} \frac{\xi(\lambda; H_{(0,R),\alpha}, H_{(0,R),\alpha}^{(0)}) d\lambda}{(\lambda - a)(\lambda - z)} = \int_{\mathbb{R}} \frac{\xi(\lambda; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)}) d\lambda}{(\lambda - a)(\lambda - z)}, \quad (4.6)$$

$z \neq a, \quad z, a \in \mathbb{C} \setminus \mathbb{R}.$

To verify (4.3), we start with the basic identities (see, e.g., [53, Ch. 8])

$$\begin{aligned} & \operatorname{tr}_{L^2((0,R);dx)} \left((H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-n} - (H_{(0,R),\alpha,\beta} - zI_{(0,R)})^{-n} \right) \\ &= n \int_{\mathbb{R}} \frac{\xi(\lambda; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)}) d\lambda}{(\lambda - z)^{n+1}}, \end{aligned} \quad (4.7)$$

$$\begin{aligned} & \operatorname{tr}_{L^2((0,\infty);dx)} \left((H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-n} - (H_{(0,\infty),\alpha} - zI_{(0,\infty)})^{-n} \right) \\ &= n \int_{\mathbb{R}} \frac{\xi(\lambda; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)}) d\lambda}{(\lambda - z)^{n+1}}, \end{aligned} \quad (4.8)$$

$n \in \mathbb{N}, \quad z \in \mathbb{C} \setminus \mathbb{R}.$

Next we claim that

$$\begin{aligned} & \lim_{R \rightarrow \infty} \operatorname{tr}_{L^2((0,R);dx)} \left((H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-n} - (H_{(0,R),\alpha,\beta} - zI_{(0,R)})^{-n} \right) \\ &= \operatorname{tr}_{L^2((0,\infty);dx)} \left((H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-n} - (H_{(0,\infty),\alpha} - zI_{(0,\infty)})^{-n} \right), \end{aligned} \quad (4.9)$$

$n \in \mathbb{N}, \quad z \in \mathbb{C} \setminus \mathbb{R}.$

To see this, one notes that

$$\operatorname{tr}_{L^2((0,R);dx)} \left((H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-n} - (H_{(0,R),\alpha,\beta} - zI_{(0,R)})^{-n} \right)$$

$$\begin{aligned}
&= \operatorname{tr}_{L^2((0,\infty);dx)} \left((H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-n} - (H_{(0,R),\alpha,\beta} - zI_{(0,R)})^{-n} \oplus 0 \right) \\
&= \operatorname{tr}_{L^2((0,\infty);dx)} \left[\left((H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1} \oplus \frac{-1}{z} I_{(R,\infty)} \right)^n \right. \\
&\quad \left. - \left((H_{(0,R),\alpha,\beta} - zI_{(0,R)})^{-1} \oplus \frac{-1}{z} I_{(R,\infty)} \right)^n \right], \quad n \in \mathbb{N}, z \in \mathbb{C} \setminus \mathbb{R}. \quad (4.10)
\end{aligned}$$

Since the trace functional is continuous with respect to the $\mathcal{B}_1(L^2((0,\infty);dx))$ -norm, to verify (4.9), it suffices to prove that

$$\begin{aligned}
&\sum_{k=1}^n \left((H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1} \oplus \frac{-1}{z} I_{(R,\infty)} \right)^{n-k} \left[(H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1} \right. \\
&\quad \left. - (H_{(0,R),\alpha,\beta} - zI_{(0,R)})^{-1} \oplus 0 \right] \left((H_{(0,R),\alpha,\beta} - zI_{(0,R)})^{-1} \oplus \frac{-1}{z} I_{(R,\infty)} \right)^{k-1} \\
&\hspace{15em} (4.11)
\end{aligned}$$

converges to

$$\begin{aligned}
&\sum_{k=1}^n (H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{k-n} [(H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-1} - (H_{(0,\infty),\alpha} - zI_{(0,\infty)})^{-1}] \\
&\quad \times (H_{(0,\infty),\alpha} - zI_{(0,\infty)})^{1-k} \\
&\hspace{15em} (4.12)
\end{aligned}$$

in $\mathcal{B}_1(L^2((0,\infty);dx))$ as $R \rightarrow \infty$, since (4.11) is the operator under the trace on the right-hand side of (4.10) and (4.12) is the operator under the trace on the right-hand side of (4.9).¹

By (3.39), one concludes that

$$\begin{aligned}
\operatorname{s-lim}_{R \rightarrow \infty} \left((H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1} \oplus \frac{-1}{z} I_{(R,\infty)} \right)^{n-k} &= (H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{k-n}, \\
&\hspace{15em} (4.13)
\end{aligned}$$

$$\begin{aligned}
\operatorname{s-lim}_{R \rightarrow \infty} \left((H_{(0,R),\alpha,\beta} - zI_{(0,R)})^{-1} \oplus \frac{-1}{z} I_{(R,\infty)} \right)^{k-1} &= (H_{(0,\infty),\alpha} - zI_{(0,\infty)})^{1-k}. \\
&\hspace{15em} (4.14)
\end{aligned}$$

Thus, convergence of (4.11) to (4.12), will follow from Grumm's Theorem [23] (see also the discussion in [47, Ch. 2]) if we can show that

$$\begin{aligned}
&\lim_{R \rightarrow \infty} \left[((H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1} - (H_{(0,R),\alpha,\beta} - zI_{(0,R)})^{-1}) \oplus 0 \right] \\
&= (H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-1} - (H_{(0,\infty),\alpha} - zI_{(0,\infty)})^{-1} \text{ in } \mathcal{B}_1(L^2((0,\infty);dx)). \\
&\hspace{15em} (4.15)
\end{aligned}$$

The $\mathcal{B}_1(L^2((0,\infty);dx))$ -convergence in (4.15) follows readily from the identities

$$\begin{aligned}
&((H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1} - (H_{(0,R),\alpha,\beta} - zI_{(0,R)})^{-1}) \oplus 0 \\
&= \overline{(H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1} v_R \oplus 0} \\
&\quad \times \left(\left[I_{(0,\infty)} + \overline{(u_R (H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1} v_R \oplus 0)} \right]^{-1} - (0 \oplus I_{(R,\infty)}) \right)
\end{aligned}$$

¹Here we have made use of the identity $A^n - B^n = \sum_{k=1}^n A^{n-k}(A-B)B^{k-1}$ for $A, B \in \mathcal{B}(\mathcal{H})$.

$$\times (u_R(H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1} \oplus 0), \quad (4.16)$$

$$\begin{aligned} & (H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-1} - (H_{(0,\infty),\alpha} - zI_{(0,\infty)})^{-1} \\ &= \overline{(H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-1} v \left[I_{(0,\infty)} + \overline{u(H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-1} v} \right]^{-1}} \\ & \times u(H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-1}. \end{aligned} \quad (4.17)$$

Applying (3.34) and (3.35) yields the strong convergence

$$\begin{aligned} & \text{s-lim}_{R \rightarrow \infty} \left(\left[I_{(0,\infty)} + \overline{(u_R(H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1} v_R \oplus 0)} \right]^{-1} - (0 \oplus I_{(R,\infty)}) \right) \\ &= \left[I_{(0,\infty)} + \overline{u(H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-1} v} \right]^{-1}. \end{aligned} \quad (4.18)$$

Therefore, (3.40) and (4.18) together with Gr\"umm's Theorem [23] yield

$$\begin{aligned} & \lim_{R \rightarrow \infty} \left(\left[I_{(0,\infty)} + \overline{(u_R(H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1} v_R \oplus 0)} \right]^{-1} - (0 \oplus I_{(R,\infty)}) \right) \\ & \times (u_R(H_{(0,R),\alpha,\beta}^{(0)} - zI_{(0,R)})^{-1} \oplus 0) \\ &= \left[I_{(0,\infty)} + \overline{v(H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-1} v} \right]^{-1} u(H_{(0,\infty),\alpha}^{(0)} - zI_{(0,\infty)})^{-1} \end{aligned} \quad (4.19)$$

in $\mathcal{B}_2(L^2((0, \infty); dx))$. The convergence in (4.19) and (3.41) yields convergence of the right-hand side of (4.16) to (4.17) in $\mathcal{B}_1(L^2((0, \infty); dx))$, implying (4.15).

Employing (4.7), (4.8), and (4.15), we have shown that

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}} \frac{\xi(\lambda; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)}) d\lambda}{(\lambda - z)^{n+1}} = \int_{\mathbb{R}} \frac{\xi(\lambda; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)}) d\lambda}{(\lambda - z)^{n+1}}, \quad n \in \mathbb{N}, \quad (4.20)$$

so that (4.3) holds in the special case $a = z$. Thus, it remains to settle the case $z \neq a$.

For $z \neq a$ one notes that

$$\begin{aligned} \int_{\mathbb{R}} \frac{\xi(\lambda; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)}) d\lambda}{(\lambda - a)(\lambda - z)^{n+1}} &= \frac{1}{z - a} \left[\int_{\mathbb{R}} \frac{\xi(\lambda; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)}) d\lambda}{(\lambda - z)^{n+1}} \right. \\ & \left. - \int_{\mathbb{R}} \frac{\xi(\lambda; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)}) d\lambda}{(\lambda - a)(\lambda - z)^n} \right], \end{aligned} \quad (4.21)$$

$$\begin{aligned} \int_{\mathbb{R}} \frac{\xi(\lambda; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)}) d\lambda}{(\lambda - a)(\lambda - z)^{n+1}} &= \frac{1}{z - a} \left[\int_{\mathbb{R}} \frac{\xi(\lambda; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)}) d\lambda}{(\lambda - z)^{n+1}} \right. \\ & \left. - \int_{\mathbb{R}} \frac{\xi(\lambda; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)}) d\lambda}{(\lambda - a)(\lambda - z)^n} \right]. \end{aligned} \quad (4.22)$$

Convergence in (4.3) now follows from (4.20) and the two identities (4.21) and (4.22) via a simple induction on $n \in \mathbb{N}$. We emphasize that (4.6) yields the crucial first induction step, $n = 1$, since (4.6) implies, via (4.21) and (4.22), that

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}} \frac{\xi(\lambda; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)}) d\lambda}{(\lambda - a)(\lambda - z)} = \int_{\mathbb{R}} \frac{\xi(\lambda; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)}) d\lambda}{(\lambda - a)(\lambda - z)} \quad (4.23)$$

for $z \neq a$. □

In the following we denote by $C_\infty(\mathbb{R})$ the space of continuous functions on \mathbb{R} vanishing at infinity.

Lemma 4.2. *Let $f, f_n \in L^1(\mathbb{R}; d\lambda)$ and suppose that for some fixed $M > 0$, $\|f_n\|_{L^1(\mathbb{R}; d\lambda)} \leq M$, $n \in \mathbb{N}$. If*

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f_n(\lambda) d\lambda P((\lambda+i)^{-1}, (\lambda-i)^{-1}) = \int_{\mathbb{R}} f(\lambda) d\lambda P((\lambda+i)^{-1}, (\lambda-i)^{-1}) \quad (4.24)$$

for all polynomials $P(\cdot, \cdot)$ in two variables, then

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f_n(\lambda) d\lambda g(\lambda) = \int_{\mathbb{R}} f(\lambda) d\lambda g(\lambda), \quad g \in C_\infty(\mathbb{R}). \quad (4.25)$$

Proof. Let $\varepsilon > 0$ and $g \in C_\infty(\mathbb{R})$. Since by a Stone–Weierstrass argument, polynomials in $(\lambda \pm i)^{-1}$ are dense in $C_\infty(\mathbb{R})$, there is a polynomial $P(\cdot, \cdot)$ in two variables such that writing

$$\mathcal{P}(\lambda) = P((\lambda+i)^{-1}, (\lambda-i)^{-1}), \quad \lambda \in \mathbb{R}, \quad (4.26)$$

one concludes that

$$\|g - \mathcal{P}\|_{L^\infty(\mathbb{R}; d\lambda)} \leq \frac{\varepsilon}{2[M + \|f\|_{L^1(\mathbb{R}; d\lambda)}]}. \quad (4.27)$$

By (4.24), there exists an $N(\varepsilon) \in \mathbb{N}$ such that

$$\left| \int_{\mathbb{R}} f_n(\lambda) d\lambda \mathcal{P}(\lambda) - \int_{\mathbb{R}} f(\lambda) d\lambda \mathcal{P}(\lambda) \right| \leq \frac{\varepsilon}{2} \text{ for all } n \geq N(\varepsilon). \quad (4.28)$$

Therefore, if $n \geq N(\varepsilon)$,

$$\begin{aligned} & \left| \int_{\mathbb{R}} f_n(\lambda) d\lambda g(\lambda) - \int_{\mathbb{R}} f(\lambda) d\lambda g(\lambda) \right| \leq [\|f_n\|_{L^1(\mathbb{R}; d\lambda)} + \|f\|_{L^1(\mathbb{R}; d\lambda)}] \|g - \mathcal{P}\|_{L^\infty(\mathbb{R}; d\lambda)} \\ & + \left| \int_{\mathbb{R}} f_n(\lambda) d\lambda \mathcal{P}(\lambda) - \int_{\mathbb{R}} f(\lambda) d\lambda \mathcal{P}(\lambda) \right| \leq \varepsilon. \end{aligned} \quad (4.29)$$

□

Next, we continue with some preparations needed to prove the principal results of this section. We start by recalling some basic notions regarding the convergence of positive measures (essentially following Bauer [5, § 30]). Denoting by $\mathcal{M}_+(E)$ the set of all *positive Radon measures* on a nonvoid locally compact space E , and by

$$\mathcal{M}_+^b(E) = \{\mu \in \mathcal{M}_+(E) \mid \mu(E) < +\infty\}, \quad (4.30)$$

the set of all *finite positive Radon measures* on E , we note that in the special case $E = \mathbb{R}^n$, $n \in \mathbb{N}$, $\mathcal{M}_+^b(\mathbb{R}^n)$ represents the set of all finite positive Borel measures on \mathbb{R}^n .

If μ is a Radon measure, a point $x \in E$ is called an *atom* of μ if $\mu(\{x\}) > 0$.

In the following, $C_0(E)$ denotes the continuous functions on E with compact support, and $C_b(E)$ represents the bounded continuous functions on E .

Definition 4.3. Let E be a locally compact space.

(i) A sequence $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_+(E)$ is said to be *vaguely convergent* to a Radon measure $\mu \in \mathcal{M}_+(E)$ if

$$\lim_{n \rightarrow \infty} \int_E d\mu_n g = \int_E d\mu g, \quad g \in C_0(E). \quad (4.31)$$

(ii) A sequence $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_+^b(E)$ is said to be *weakly convergent* to $\mu \in \mathcal{M}_+^b(E)$ if

$$\lim_{n \rightarrow \infty} \int_E d\mu_n f = \int_E d\mu f, \quad f \in C_b(E). \quad (4.32)$$

(iii) A Borel set $B \subset E$ is called *boundaryless* with respect to the measure $\mu \in \mathcal{M}_+^b(E)$ (in short, μ -*boundaryless*), if the boundary ∂B of B has μ -measure equal to zero, $\mu(\partial B) = 0$.

Theorem 4.4 ([29], Proposition 4.3). *Let $\mu, \mu_n, n \in \mathbb{N}$, be positive (not necessarily finite) Borel measures on \mathbb{R} . Then the following two items are equivalent:*

(i) *The finiteness $\mu((-\infty, \lambda)) < \infty$ holds for all $\lambda \in \mathbb{R}$ and the relation*

$$\lim_{n \rightarrow \infty} \mu_n((-\infty, \lambda)) = \mu((-\infty, \lambda)) \quad (4.33)$$

holds for all $\lambda \in \mathbb{R}$ except for at most countably many $\lambda \in \mathbb{R}$ with $\mu(\{\lambda\}) \neq 0$.

(ii) *The sequence $\{\mu_n\}_{n \in \mathbb{N}}$ converges vaguely to μ as $n \rightarrow \infty$ and the relation*

$$\lim_{\lambda \downarrow -\infty} \left(\limsup_{n \rightarrow \infty} (\mu_n((-\infty, \lambda))) \right) = 0 \quad (4.34)$$

holds.

Theorem 4.5 ([5], Theorem 30.8). *Suppose that the sequence $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_+^b(E)$ converges vaguely to the measure $\mu \in \mathcal{M}_+^b(E)$. Then the following statements are equivalent:*

(i) *The sequence μ_n converges weakly to μ as $n \rightarrow \infty$.*

(ii) $\lim_{n \rightarrow \infty} \mu_n(E) = \mu(E)$.

(iii) *For every $\varepsilon > 0$ there exists a compact set K_ε of E such that*

$$\mu_n(E \setminus K_\varepsilon) \leq \varepsilon, \quad n \in \mathbb{N}. \quad (4.35)$$

Theorem 4.6 ([5], Theorem 30.12). *Suppose that the sequence $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_+^b(E)$ converges weakly to $\mu \in \mathcal{M}_+^b(E)$. Then*

$$\lim_{n \rightarrow \infty} \int_E d\mu_n f = \int_E d\mu f \quad (4.36)$$

holds for every bounded Borel measurable function f that is μ -almost everywhere continuous on E . In particular,

$$\lim_{n \rightarrow \infty} \mu_n(B) = \mu(B) \quad (4.37)$$

holds for every μ -boundaryless Borel set B .

As usual, finite signed Radon measures are viewed as differences of finite positive Radon measures in the following.

Given these preparations, we now return to our concrete case at hand: Decomposing V as

$$V = V_+ - V_-, \quad V_\pm = [|V| \pm V]/2, \quad (4.38)$$

and analogously for V_R , that is, $V_R = V_{R,+} - V_{R,-}$, we now decompose the spectral shift functions $\xi(\cdot; H_{(0,R);\alpha,\beta}, H_{(0,R);\alpha,\beta}^{(0)})$ and $\xi(\cdot; H_{(0,\infty);\alpha}, H_{(0,\infty);\alpha}^{(0)})$ according to (A.13), (A.15) into a positive and negative contribution as follows,

$$\begin{aligned} \xi(\cdot; H_{(0,R);\alpha,\beta}, H_{(0,R);\alpha,\beta}^{(0)}) &= \xi(\cdot; H_{(0,R);\alpha,\beta}^{(0)} +_q V_{R,+} -_q V_{R,-}, H_{(0,R);\alpha,\beta}^{(0)} +_q V_{R,+}) \\ &\quad + \xi(\cdot; H_{(0,R);\alpha,\beta}^{(0)} +_q V_{R,+}, H_{(0,R);\alpha,\beta}^{(0)}), \end{aligned} \quad (4.39)$$

$$\begin{aligned} \xi(\cdot; H_{(0,\infty);\alpha}, H_{(0,\infty);\alpha}^{(0)}) &= \xi(\cdot; H_{(0,\infty);\alpha}^{(0)} +_q V_+ -_q V_-, H_{(0,\infty);\alpha}^{(0)} +_q V_+) \\ &\quad + \xi(\cdot; H_{(0,\infty);\alpha}^{(0)} +_q V_+, H_{(0,\infty);\alpha}^{(0)}) \end{aligned} \quad (4.40)$$

where

$$\xi(\cdot; H_{(0,R);\alpha,\beta}^{(0)} +_q V_{R,+}, H_{(0,R);\alpha,\beta}^{(0)}) \geq 0, \quad (4.41)$$

$$\xi(\cdot; H_{(0,R);\alpha,\beta}^{(0)} +_q V_{R,+} -_q V_{R,-}, H_{(0,R);\alpha,\beta}^{(0)} +_q V_{R,+}) \leq 0, \quad (4.42)$$

$$\xi(\cdot; H_{(0,\infty);\alpha}^{(0)} +_q V_+, H_{(0,\infty);\alpha}^{(0)}) \geq 0, \quad (4.43)$$

$$\xi(\cdot; H_{(0,\infty);\alpha}^{(0)} +_q V_+ -_q V_-, H_{(0,\infty);\alpha}^{(0)} +_q V_+) \leq 0. \quad (4.44)$$

Theorem 4.7. *Assume (2.8) and let $g \in C_\infty(\mathbb{R})$. Then*

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}} \frac{\xi(\lambda; H_{(0,R);\alpha,\beta}, H_{(0,R);\alpha,\beta}^{(0)}) d\lambda}{\lambda^2 + 1} g(\lambda) = \int_{\mathbb{R}} \frac{\xi(\lambda; H_{(0,\infty);\alpha}, H_{(0,\infty);\alpha}^{(0)}) d\lambda}{\lambda^2 + 1} g(\lambda). \quad (4.45)$$

Proof. Abbreviating temporarily

$$\xi(\cdot; H_{(0,R);\alpha,\beta}, H_{(0,R);\alpha,\beta}^{(0)}) = \xi_{R,+}(\cdot) - \xi_{R,-}(\cdot), \quad (4.46)$$

$$\xi(\cdot; H_{(0,\infty);\alpha}, H_{(0,\infty);\alpha}^{(0)}) = \xi_+(\cdot) - \xi_-(\cdot), \quad (4.47)$$

for ease of notation, where

$$\xi_{R,+}(\cdot) = \xi(\cdot; H_{(0,R);\alpha,\beta}^{(0)} +_q V_{R,+}, H_{(0,R);\alpha,\beta}^{(0)}) \geq 0, \quad (4.48)$$

$$\xi_{R,-}(\cdot) = -\xi(\cdot; H_{(0,R);\alpha,\beta}^{(0)} +_q V_{R,+} -_q V_{R,-}, H_{(0,R);\alpha,\beta}^{(0)} +_q V_{R,+}) \geq 0, \quad (4.49)$$

$$\xi_+(\cdot) = \xi(\cdot; H_{(0,\infty);\alpha}^{(0)} +_q V_+, H_{(0,\infty);\alpha}^{(0)}) \geq 0, \quad (4.50)$$

$$\xi_-(\cdot) = -\xi(\cdot; H_{(0,\infty);\alpha}^{(0)} +_q V_+ -_q V_-, H_{(0,\infty);\alpha}^{(0)} +_q V_+) \geq 0, \quad (4.51)$$

the basic idea is to verify that

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}} \frac{\xi_{R,\pm}(\lambda) d\lambda}{\lambda^2 + 1} P((\lambda + i)^{-1}, (\lambda - i)^{-1}) = \int_{\mathbb{R}} \frac{\xi_{\pm}(\lambda) d\lambda}{\lambda^2 + 1} P((\lambda + i)^{-1}, (\lambda - i)^{-1}) \quad (4.52)$$

for all polynomials $P(\cdot, \cdot)$ in two variables, and then rely on the Stone–Weierstrass approximation in Lemma 4.2 to get

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}} \frac{\xi_{R,\pm}(\lambda) d\lambda}{\lambda^2 + 1} g(\lambda) = \int_{\mathbb{R}} \frac{\xi_{\pm}(\lambda) d\lambda}{\lambda^2 + 1} g(\lambda), \quad g \in C_\infty(\mathbb{R}), \quad (4.53)$$

and hence (4.45). (For another application of this technique in spectral theory, see, e.g., [42, Theorem VIII.20].) To prove (4.52), it suffices to verify that

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}} \frac{\xi_{R,\pm}(\lambda) d\lambda}{\lambda^2 + 1} \frac{1}{(\lambda + i)^m (\lambda - i)^n} = \int_{\mathbb{R}} \frac{\xi_{\pm}(\lambda) d\lambda}{\lambda^2 + 1} \frac{1}{(\lambda + i)^m (\lambda - i)^n}, \quad (4.54)$$

$m, n \in \mathbb{N} \cup \{0\}$,

which, in turn, follows once one proves

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}} \frac{\xi_{R,\pm}(\lambda) d\lambda}{\lambda^2 + 1} \frac{1}{(\lambda \pm i)^n} = \int_{\mathbb{R}} \frac{\xi_{\pm}(\lambda) d\lambda}{\lambda^2 + 1} \frac{1}{(\lambda \pm i)^n}, \quad n \in \mathbb{N} \cup \{0\}, \quad (4.55)$$

since

$$\frac{1}{(\lambda + i)^m (\lambda - i)^n} = \sum_{j=1}^m \frac{c_j}{(\lambda + i)^j} + \sum_{j=1}^n \frac{\widehat{c}_j}{(\lambda - i)^j} \quad (4.56)$$

for appropriate constants c_j and \widehat{c}_j . Choosing $z = \pm i$ and $a = \mp i$ in (4.3) yields (4.55), and therefore (4.52) for all polynomials P . At this point, (4.45) follows from Lemma 4.2 once one shows the existence of an $M > 0$ for which

$$\left| \int_{\mathbb{R}} \frac{\xi_{R,\pm}(\lambda) d\lambda}{\lambda^2 + 1} \right| \leq M \quad (4.57)$$

for R sufficiently large. Taking (4.55) with $n = 0$, yields the convergence

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}} \frac{\xi_{R,\pm}(\lambda) d\lambda}{\lambda^2 + 1} = \int_{\mathbb{R}} \frac{\xi_{\pm}(\lambda) d\lambda}{\lambda^2 + 1}. \quad (4.58)$$

As a result, $\int_{\mathbb{R}} \xi_{R,\pm}(\lambda) d\lambda (\lambda^2 + 1)^{-1}$ is uniformly bounded with respect to R . \square

An immediate consequence of Theorem 4.7 is the following vague convergence result:

Corollary 4.8. *Assume (2.8) and let $g \in C_0(\mathbb{R})$. Then*

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}} \xi(\lambda; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)}) d\lambda g(\lambda) = \int_{\mathbb{R}} \xi(\lambda; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)}) d\lambda g(\lambda). \quad (4.59)$$

Remark 4.9. Since by (3.67) and (3.69), $H_{(0,R),\alpha,\beta}$ and $H_{(0,\infty),\alpha}$ are uniformly bounded from below and hence we use our convention (A.10) of vanishing spectral shift functions in a fixed (i.e., R -independent) neighborhood of $-\infty$, no restrictions on g near $-\infty$ need to be imposed (apart from measurability of g , of course).

Remark 4.10. In the finite interval context, the corresponding spectral shift function $\xi(\cdot; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)})$ is actually the difference of two eigenvalue counting functions (c.f. also [25], [26], [32]), that is,

$$\xi(\lambda; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)}) = N(\lambda; H_{(0,R),\alpha,\beta}^{(0)}) - N(\lambda; H_{(0,R),\alpha,\beta}), \quad \lambda \in \mathbb{R}, \quad (4.60)$$

where we have written $N(\lambda; S)$ to denote the number of eigenvalues (counted according to multiplicity) of the self-adjoint operator S that are less than or equal to λ . Consequently, $\xi(\cdot; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)})$ is an integer-valued function. Thus, one cannot expect that the integer-valued functions $\xi(\cdot; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)})$ converge pointwise to the spectral shift function $\xi(\cdot; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)})$, since the latter may, under the assumption $V \in L^1((0, \infty); dx)$ (resp., (2.126)), V real-valued, be taken continuous (and non-constant) on the half-line $(0, \infty)$, and the pointwise

limit of integer-valued functions (if it exists) is necessarily integer-valued. This line of reasoning also rules out convergence of $\xi(\lambda; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)})(1 + \lambda^2)^{-1}$ to $\xi(\lambda; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)})(1 + \lambda^2)^{-1}$ in the topology of $L^1((0, \infty); d\lambda)$ as $R \rightarrow \infty$, since the latter would imply the existence of a pointwise a.e. convergent subsequence, $\xi(\cdot; H_{(0,R_k),\alpha,\beta}, H_{(0,R_k),\alpha,\beta}^{(0)}) \xrightarrow[k \rightarrow \infty]{} \xi(\cdot; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)})$.

A weaker notion of convergence is that of *convergence in measure*. However, since convergence in measure implies the existence of a pointwise a.e. convergent subsequence (cf., e.g., [15, Corollary 2.32]), again one cannot expect that $\xi(\cdot; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)}) \xrightarrow[R \rightarrow \infty]{} \xi(\cdot; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)})$ in measure.

This argument (and its analog in higher dimensions) shows that the hope of a.e. convergence of spectral shift functions expressed in [25, Remark 1.5 (ii)] for appropriate subsequences of $\{R_k\}_{k \in \mathbb{N}}$ cannot possibly materialize.

However, while Theorem 4.7 already improves on known results in the literature, it is not the best possible result in this direction and in the remainder of this section we will show how to improve on it.

First, however, we note the following consequence of Theorem 4.4:

Lemma 4.11. *Assume (2.8) and $E, E_1, E_2 \in \mathbb{R}$, $E_1 < E_2$. Then*

$$\xi(\cdot; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)}), \xi(\cdot; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)}) \in L^1((-\infty, E); d\lambda) \quad (4.61)$$

and

$$\lim_{R \rightarrow \infty} \int_{-\infty}^E \xi(\lambda; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)}) d\lambda = \int_{-\infty}^E \xi(\lambda; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)}) d\lambda, \quad (4.62)$$

and hence,

$$\lim_{R \rightarrow \infty} \int_{E_1}^{E_2} \xi(\lambda; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)}) d\lambda = \int_{E_1}^{E_2} \xi(\lambda; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)}) d\lambda. \quad (4.63)$$

Proof. Since $\xi(\cdot; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)})$, $\xi(\cdot; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)})$ both vanish in a neighborhood of $-\infty$, the integrability assertion (4.61) holds.

Next we decompose $\xi(\cdot; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)})$ and $\xi(\cdot; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)})$ as in (4.46)–(4.51).

Since by (3.67) and (3.69), $H_{(0,R),\alpha,\beta}$ and $H_{(0,\infty),\alpha}$, and similarly, these operators with V replaced by $V_{R,\pm}$, V_{\pm} are all uniformly bounded from below with respect to $R > 0$, (4.34) applies with μ_n replaced by $\xi_{R,\pm}(\lambda)d\lambda$ and $\xi_{\pm}(\lambda)d\lambda$. Moreover, since these measures are all absolutely continuous and hence have no atoms, (4.62) holds for all $E \in \mathbb{R}$. The result (4.63) now follows from (4.62) by taking differences. \square

Given the decomposition (4.46)–(4.51), and introducing the measures

$$\eta_{\xi_{R,\pm}}(A) = \int_A \frac{\xi_{R,\pm}(\lambda)d\lambda}{\lambda^2 + 1}, \quad R > 0, \quad \eta_{\xi_{\pm}}(A) = \int_A \frac{\xi_{\pm}(\lambda)d\lambda}{\lambda^2 + 1}, \quad (4.64)$$

$A \subseteq \mathbb{R}$ Lebesgue measurable,

we are now ready for the principal result of this paper.

Theorem 4.12. *Assume (2.8). Then*

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}} \frac{\xi(\lambda; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)})}{\lambda^2 + 1} f(\lambda) = \int_{\mathbb{R}} \frac{\xi(\lambda; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)})}{\lambda^2 + 1} f(\lambda),$$

$$f \in C_b(\mathbb{R}). \quad (4.65)$$

Moreover, (4.65) holds for every bounded Borel measurable function f that is η_{ξ_+} and η_{ξ_-} -almost everywhere continuous on \mathbb{R} . In particular,

$$\lim_{R \rightarrow \infty} \int_S \frac{\xi(\lambda; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)}) d\lambda}{\lambda^2 + 1} = \int_S \frac{\xi(\lambda; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)}) d\lambda}{\lambda^2 + 1} \quad (4.66)$$

for every set S whose boundary has η_{ξ_+} and η_{ξ_-} -measure equal to zero.

Proof. Again we decompose $\xi(\cdot; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)})$ and $\xi(\cdot; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)})$ as in (4.46)–(4.51). By (4.53) and Corollary 4.8, the measure $\eta_{\xi_{R,\pm}}$ vaguely converges to the measure $\eta_{\xi_{\pm}}$ as $R \rightarrow \infty$, respectively. Moreover, by (4.54),

$$\lim_{R \rightarrow \infty} \eta_{\xi_{R,\pm}}(\mathbb{R}) = \eta_{\xi_{\pm}}(\mathbb{R}). \quad (4.67)$$

Thus, by Theorem 4.5, one concludes weak convergence of the sequence of measures $\eta_{\xi_{R,\pm}}$ to the measure $\eta_{\xi_{\pm}}$ as $R \rightarrow \infty$. That (4.65) holds for every bounded Borel measurable function that is $\eta_{\xi_{\pm}}$ -almost everywhere continuous on \mathbb{R} now follows directly from Theorem 4.6. Finally, convergence in (4.66) is also a direct consequence of Theorem 4.6. \square

As immediate consequences of Theorem 4.12, we have the following two results:

Corollary 4.13. *Assume (2.8). Then convergence in (4.65) holds for any bounded Borel measurable function that is Lebesgue-almost everywhere continuous. In particular, (4.66) holds for any set S that is boundaryless with respect to Lebesgue measure (i.e., any set S for which the boundary of S has Lebesgue measure equal to zero).*

Proof. Noting that $\eta_{\xi_{\pm}}$ are absolutely continuous with respect to Lebesgue measure, the statements follow directly from Theorem 4.12. \square

Corollary 4.14. *Assume (2.8). If g is a bounded Borel measurable function that is compactly supported and Lebesgue almost everywhere continuous on \mathbb{R} , then*

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}} \xi(\lambda; H_{(0,R),\alpha,\beta}, H_{(0,R),\alpha,\beta}^{(0)}) d\lambda g(\lambda) = \int_{\mathbb{R}} \xi(\lambda; H_{(0,\infty),\alpha}, H_{(0,\infty),\alpha}^{(0)}) d\lambda g(\lambda). \quad (4.68)$$

Proof. If g satisfies the hypotheses of Corollary 4.14, then choosing $f(\lambda) := (\lambda^2 + 1)g(\lambda)$ in (4.65) yields the result, noting that f is a bounded (g has compact support) Borel measurable function and is continuous Lebesgue-almost everywhere (and thus $\eta_{\xi_{\pm}}$ -almost everywhere) on \mathbb{R} . \square

We should perhaps point out that the regularized trace formula discussed in [14], which also involves an infinite volume limit $R \rightarrow \infty$, differs from the result (4.65).

For completeness we now also show the weak convergence result in Theorem 4.12 can be obtained via Helly's Selection Theorem combined with Theorem 4.4 (without using Theorems 4.5, 4.6, and 4.12).

Theorem 4.15 (Convergence via Helly's Selection Theorem). *Assume (2.8), then*

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}} \frac{\xi_R(\lambda) d\lambda}{\lambda^2 + 1} f(\lambda) = \int_{\mathbb{R}} \frac{\xi(\lambda) d\lambda}{\lambda^2 + 1} f(\lambda), \quad f \in C_b(\mathbb{R}). \quad (4.69)$$

Proof. Once more we use the decompositions (4.46)–(4.51) and introduce the associated non-decreasing continuous functions $\sigma_{R,\pm}$, $R > 0$, and σ_{\pm} on $\mathbb{R} \cup \{\infty\}$ defined by

$$\begin{aligned}\sigma_{R,\pm}(\lambda) &= \int_{(-\infty,\lambda)} \frac{\xi_{R,\pm}(\lambda')d\lambda'}{(\lambda')^2 + 1}, \quad R > 0, \\ \sigma_{\pm}(\lambda) &= \int_{(-\infty,\lambda)} \frac{\xi_{\pm}(\lambda')d\lambda'}{(\lambda')^2 + 1}, \quad \lambda \in \mathbb{R} \cup \{\infty\}.\end{aligned}\tag{4.70}$$

By (4.58) one has

$$\lim_{R \rightarrow \infty} \sigma_{R,\pm}(\infty) = \sigma_{\pm}(\infty).\tag{4.71}$$

Next, we aim at proving

$$\lim_{R \rightarrow \infty} \sigma_{R,\pm}(\lambda) = \sigma_{\pm}(\lambda), \quad \lambda \in \mathbb{R}.\tag{4.72}$$

Having already shown vague convergence of $\{\eta_{R,\pm}\}$ to η_{\pm} (cf. (4.64) and Corollary 4.8), we now verify

$$\lim_{E \downarrow -\infty} \left(\limsup_{R \rightarrow \infty} (\sigma_{R,\pm}(E)) \right) = 0,\tag{4.73}$$

as (4.72) then follows from Theorem 4.4. One notes that (4.73) follows if one can show that

$$\begin{aligned}\text{for all } \varepsilon > 0, \text{ there exists } E(\varepsilon) \in \mathbb{R} \text{ such that } \sigma_{R,\pm}(E) &\leq \varepsilon \\ \text{for all } E \leq E(\varepsilon) \text{ and all } R > 0.\end{aligned}\tag{4.74}$$

Let $\varepsilon > 0$ be given. By (4.71) there exists an interval $K_{\varepsilon} = [a(\varepsilon), b(\varepsilon)]$ such that

$$\eta_{R,\pm}(\mathbb{R} \setminus K_{\varepsilon}) = \int_{\mathbb{R} \setminus K_{\varepsilon}} \frac{\xi_{R,\pm}(\lambda)d\lambda}{\lambda^2 + 1} \leq \varepsilon, \quad R > 0,\tag{4.75}$$

which is to say that

$$\int_{-\infty}^{a(\varepsilon)} \frac{\xi_{R,\pm}(\lambda)}{1 + \lambda^2} d\lambda + \int_{b(\varepsilon)}^{\infty} \frac{\xi_{R,\pm}(\lambda)}{1 + \lambda^2} d\lambda \leq \varepsilon, \quad R > 0.\tag{4.76}$$

Thus, one sees that (4.74) is valid once one takes $E(\varepsilon) = a(\varepsilon)$.

Now, fix $f \in C_b(\mathbb{R})$; we may assume without loss that $\|f\|_{\infty} = \sup_{\lambda \in \mathbb{R}} |f(\lambda)| \neq 0$ (otherwise, $f(\lambda) = 0$ for all $\lambda \in \mathbb{R}$, and the convergence (4.69) is trivial). By (4.71), for any $\varepsilon > 0$, there exists an $A(\varepsilon) > 0$ such that for all $a, b > A(\varepsilon)$,

$$\int_{-\infty}^a \frac{\xi_{R,\pm}(\lambda)d\lambda}{\lambda^2 + 1} + \int_b^{\infty} \frac{\xi_{R,\pm}(\lambda)d\lambda}{\lambda^2 + 1} \leq \frac{\varepsilon}{\|f\|_{\infty}}, \quad R > 0.\tag{4.77}$$

Thus, for all $a, b > A(\varepsilon)$ one has

$$\int_{-\infty}^a \frac{\xi_{R,\pm}(\lambda)d\lambda}{\lambda^2 + 1} |f(\lambda)| \leq \|f\|_{\infty} \int_{-\infty}^a \frac{\xi_{R,\pm}(\lambda)d\lambda}{\lambda^2 + 1} \leq \varepsilon, \quad R > 0,\tag{4.78}$$

$$\int_b^{\infty} \frac{\xi_{R,\pm}(\lambda)d\lambda}{\lambda^2 + 1} |f(\lambda)| \leq \|f\|_{\infty} \int_b^{\infty} \frac{\xi_{R,\pm}(\lambda)d\lambda}{\lambda^2 + 1} \leq \varepsilon, \quad R > 0.\tag{4.79}$$

At this point one has verified the assumptions (2.7) in the generalized version of Helly's second selection theorem in the form of [40, Sect. XIV.2], implying (4.69). \square

Remark 4.16. We conclude with a brief remark on an alternative approach to convergence of spectral shift functions. In cases where a fixed Hilbert space is naturally given, an abstract convergence result of spectral shift functions in the $L^1(\mathbb{R}; (\lambda^2 + 1)^{-1} d\lambda)$ -norm, based on trace norm convergence of resolvents, has been given in [53, Lemma 8.7.5] with a refinement in the case of semibounded self-adjoint operators in [38, Lemma 8.3]. This is then applied to derive relative oscillation theory results in essential spectral gaps between different Sturm–Liouville operators in [37] and [38]. This approach differs from the one developed in this paper as in our case one cannot expect subsequences of spectral shift functions to converge pointwise a.e. as discussed at the end of Remark 4.10.

APPENDIX A. BASIC FACTS ON SPECTRAL SHIFT FUNCTIONS

In this appendix we summarize basic properties of the spectral shift function as needed in the bulk of this paper (for details on this material we refer to [6], [53, Ch. 8], [54], [55, Sect. 0.9, Chs. 4, 5, 9]).

We start with the following basic assumptions:

Hypothesis A.1. *Suppose A and B are self-adjoint operators in \mathcal{H} with A bounded from below.*

(i) *Assume that B can be written as the form sum of A and a self-adjoint operator W in \mathcal{H}*

$$B = A +_q W, \quad (\text{A.1})$$

where W can be factorized into

$$W = W_1 W_2, \quad (\text{A.2})$$

such that

$$\text{dom}(W_j) \supseteq \text{dom}(|A|^{1/2}), \quad j = 1, 2, \quad (\text{A.3})$$

and

$$\overline{W_2(A - zI_{\mathcal{H}})^{-1}W_1} \in \mathcal{B}_1(\mathcal{H}), \quad z \in \rho(A). \quad (\text{A.4})$$

(ii) *In addition, we suppose that for some (and hence for all) $z \in \rho(B) \cap \rho(A)$,*

$$[(B - zI_{\mathcal{H}})^{-1} - (A - zI_{\mathcal{H}})^{-1}] \in \mathcal{B}_1(\mathcal{H}). \quad (\text{A.5})$$

Given Hypothesis A.1 (i), one observes that

$$\text{dom}(|B|^{1/2}) = \text{dom}(|A|^{1/2}), \quad (\text{A.6})$$

and that the resolvent of B can be written as (cf., e.g., the detailed discussion in [17] and the references therein)

$$\begin{aligned} (B - zI_{\mathcal{H}})^{-1} &= (A - zI_{\mathcal{H}})^{-1} \\ &\quad - \overline{(A - zI_{\mathcal{H}})^{-1}W_1} [I_{\mathcal{H}} + \overline{W_2(A - zI_{\mathcal{H}})^{-1}W_1}]^{-1} W_2 (A - zI_{\mathcal{H}})^{-1}, \end{aligned} \quad (\text{A.7})$$

$z \in \rho(B) \cap \rho(A).$

In particular, B is bounded from below in \mathcal{H} .

Moreover, assuming the full Hypothesis A.1 one infers that (cf. [20])

$$\begin{aligned} &\text{tr}_{\mathcal{H}}((B - zI_{\mathcal{H}})^{-1} - (A - zI_{\mathcal{H}})^{-1}) \\ &= -\frac{d}{dz} \ln \left(\det_{\mathcal{H}} \left(\overline{(B - zI_{\mathcal{H}})^{1/2} (A - zI_{\mathcal{H}})^{-1} (B - zI_{\mathcal{H}})^{1/2}} \right) \right) \\ &= -\frac{d}{dz} \ln \left(\det_{\mathcal{H}} \left(I_{\mathcal{H}} + \overline{W_2(A - zI_{\mathcal{H}})^{-1}W_1} \right) \right), \quad z \in \rho(B) \cap \rho(A). \end{aligned} \quad (\text{A.8})$$

Here $\det_{\mathcal{H}}(\cdot)$ denotes the Fredholm determinant (cf. [21, Ch. IV], [45], [47, Ch. 3]).

In addition, Hypothesis A.1 guarantees the existence of the real-valued spectral shift function $\xi(\cdot; B, A)$ satisfying

$$\begin{aligned} \operatorname{tr}_{\mathcal{H}}((B - zI_{\mathcal{H}})^{-1} - (A - zI_{\mathcal{H}})^{-1}) &= - \int_{\mathbb{R}} \frac{\xi(\lambda; B, A) d\lambda}{(\lambda - z)^2}, \\ z &\in \mathbb{C} \setminus [\inf(\sigma(B), \sigma(A)), \infty), \end{aligned} \quad (\text{A.9})$$

and

$$\xi(\lambda; B, A) = 0, \quad \lambda < \inf(\sigma(B), \sigma(A)), \quad (\text{A.10})$$

$$\xi(\cdot; B, A) \in L^1(\mathbb{R}; (\lambda^2 + 1)^{-1} d\lambda). \quad (\text{A.11})$$

Moreover, for a large class of functions f (e.g., any f s.t. $\widehat{f}(\cdot) \in L^1(\mathbb{R}; (|p| + 1) dp)$) one infers that $[f(B) - f(A)] \in \mathcal{B}_1(\mathcal{H})$ and

$$\operatorname{tr}_{\mathcal{H}}(f(B) - f(A)) = \int_{\mathbb{R}} f'(\lambda) \xi(\lambda; B, A) d\lambda. \quad (\text{A.12})$$

This applies, in particular, to powers of the resolvent, where $f(\cdot) = (\cdot - z)^{-n}$, $n \in \mathbb{N}$, and we refer to [53, Ch. 8] for more details.

Throughout this manuscript we assume that the normalization (A.10) is applied.

We also note the following monotonicity result: If

$$\begin{aligned} B \geq A \text{ (resp., } B \leq A) \text{ in the sense of quadratic forms, then} \\ \xi(\lambda; B, A) \geq 0 \text{ (resp., } \xi(\lambda; B, A) \leq 0). \end{aligned} \quad (\text{A.13})$$

Here, $B \geq A$ is meant in the sense of quadratic forms, that is,

$$\begin{aligned} \operatorname{dom}(|A|^{1/2}) \supseteq \operatorname{dom}(|B|^{1/2}) \text{ and} \\ (|B|^{1/2} f, \operatorname{sgn}(B)|B|^{1/2} f)_{\mathcal{H}} \geq (|A|^{1/2} f, \operatorname{sgn}(A)|A|^{1/2} f)_{\mathcal{H}}, \quad f \in \operatorname{dom}(|B|^{1/2}). \end{aligned} \quad (\text{A.14})$$

Next, suppose that the self-adjoint operator C in \mathcal{H} can be written as the form sum of B and a self-adjoint operator Q in \mathcal{H} , $C = B +_q Q$, where Q can be factored as $Q = Q_1 Q_2$, with Q, Q_1, Q_2 satisfying the assumptions of W, W_1, W_2 in Hypotheses A.1. Then the formula

$$\xi(\lambda; C, A) = \xi(\lambda; C, B) + \xi(\lambda; B, A) \text{ for a.e. } \lambda \in \mathbb{R}, \quad (\text{A.15})$$

holds.

Finally, we mention the connection between $\xi(\cdot; B, A)$ and the Fredholm determinant in (A.8),

$$\xi(\lambda; B, A) = \pi^{-1} \lim_{\varepsilon \downarrow 0} \operatorname{Im} \left(\ln \left(I_{\mathcal{H}} + \overline{W_2(A - (\lambda + i\varepsilon)I_{\mathcal{H}})^{-1} W_1} \right) \right) \text{ for a.e. } \lambda \in \mathbb{R}, \quad (\text{A.16})$$

choosing the branch of $\ln(\det_{\mathcal{H}}(\cdot))$ on \mathbb{C}_+ such that

$$\lim_{\operatorname{Im}(z) \uparrow +\infty} \ln \left(\det_{\mathcal{H}} \left(I_{\mathcal{H}} + \overline{W_2(A - zI_{\mathcal{H}})^{-1} W_1} \right) \right) = 0. \quad (\text{A.17})$$

APPENDIX B. A SPECTRAL SHIFT FUNCTION DECOMPOSITION

We conclude this paper with an interesting formula that provides a comparison of the spectral shift functions for the interval $[0, R_1]$ and $[0, R_2]$, where $0 < R_1 < R_2$, under the assumption of Dirichlet boundary conditions.

To simplify our notation a bit, the Dirichlet boundary conditions at $0, R_1$, and R_2 will be indicated by the subscript “D” and the symbol $H^{(0)}$ will be replaced by $-\Delta$.

We begin with the following result.

Lemma B.1. *Let $0 < R_1 < R_2$ and assume that $V \in L^1((0, R_2); dx)$ is real-valued. Then $\xi(\cdot; -\Delta_{(0, R_1) \cup (R_1, R_2), D} +_q V_{R_2}, -\Delta_{(0, R_1) \cup (R_1, R_2), D})$ can be decomposed into a sum of two spectral shift functions as follows,*

$$\begin{aligned} & \xi(\lambda; -\Delta_{(0, R_1) \cup (R_1, R_2), D} +_q V_{R_2}, -\Delta_{(0, R_1) \cup (R_1, R_2), D}) \\ &= \xi(\lambda; -\Delta_{(0, R_1), D} +_q V_{R_1}, -\Delta_{(0, R_1), D}) \\ & \quad + \xi(\lambda; -\Delta_{(R_1, R_2), D} +_q V|_{(R_1, R_2)}, -\Delta_{(R_1, R_2), D}) \text{ for a.e. } \lambda \in \mathbb{R}. \end{aligned} \quad (\text{B.1})$$

Proof. The proof relies essentially on the classical fact that

$$-\Delta_{(0, R_1) \cup (R_1, R_2), D} = -\Delta_{(0, R_1), D} \oplus -\Delta_{(R_1, R_2), D} \quad (\text{B.2})$$

(cf., e.g., [43, Proposition X.III.13.3]) and of course

$$V_{R_2} = V_{R_1} \oplus V|_{(R_1, R_2)}, \quad (\text{B.3})$$

both with respect to the decomposition

$$L^2((0, R_2); dx) = L^2((0, R_1); dx) \oplus L^2((R_1, R_2); dx). \quad (\text{B.4})$$

One computes

$$\det_{L^2((0, R_2); dx)} (I_{(0, R_2)} + u_{R_2} (-\Delta_{(0, R_1) \cup (R_1, R_2), D} - z)^{-1} v_{R_2}) \quad (\text{B.5})$$

$$\begin{aligned} &= \det_{L^2((0, R_2); dx)} (I_{(0, R_1)} \oplus I_{(R_1, R_2)} + u_{R_1} \oplus u|_{(R_1, R_2)}) \\ & \quad \times (-\Delta_{(0, R_1), D} \oplus -\Delta_{(R_1, R_2), D} - z)^{-1} (v_{R_1} \oplus v|_{(R_1, R_2)}) \\ &= \det_{L^2((0, R_1); dx)} (I_{(0, R_1)} + u_{R_1} (-\Delta_{(0, R_1), D} - z)^{-1} v_{R_1}) \\ & \quad \times \det_{L^2((R_1, R_2); dx)} (I + u|_{(R_1, R_2)} (-\Delta_{(R_1, R_2), D} - z)^{-1} v|_{(R_1, R_2)}). \end{aligned} \quad (\text{B.6})$$

Therefore, replacing z by $\lambda + i\varepsilon$, taking arguments, and finally the limit $\varepsilon \downarrow 0$ proves (B.1). \square

Krein’s resolvent formula then shows that the difference of the resolvents of $-\Delta_{(0, R_2), D}$ and $-\Delta_{(0, R_1) \cup (R_1, R_2), D}$ is a rank one operator. Explicitly, in terms of integral kernels, one has

$$\begin{aligned} G_{(0, R_1) \cup (R_1, R_2), D}^{(0)}(z, x, x') &= G_{(0, R_2), D}^{(0)}(z, x, x') \\ & \quad - \frac{G_{(0, R_2), D}^{(0)}(z, x, R) G_{(0, R_2), D}^{(0)}(z, R, x')}{G_{(0, R_2), D}^{(0)}(z, R, R)}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \end{aligned} \quad (\text{B.7})$$

where $G_{\Omega, D}^{(0)}(z, x, x')$ denotes the integral kernel of $(-\Delta_{\Omega, D} - zI_{\Omega})^{-1}$. Therefore, we write

$$(-\Delta_{(0, R_2), D} - z)^{-1} = (-\Delta_{(0, R_1) \cup (R_1, R_2), D} - z)^{-1} + T(z), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (\text{B.8})$$

where $T(z)$ is the rank-one operator generated by the integral kernel given by the second term on the right-hand side of (B.7). Thus, one computes

$$\begin{aligned}
& \det_{L^2((0,R_2);dx)} \left(I_{(0,R_2)} + u_{R_2} (-\Delta_{(0,R_2),D} - z)^{-1} v_{R_2} \right) \\
&= \det_{L^2((0,R_2);dx)} \left(I_{(0,R_2)} + u_{R_2} (-\Delta_{(0,R_1) \cup (R_1,R_2),D} - z)^{-1} v_{R_2} + u_{R_2} T(z) v_{R_2} \right) \\
&= \det_{L^2((0,R_2);dx)} \left([I_{(0,R_2)} + u_{R_2} (-\Delta_{(0,R_1) \cup (R_1,R_2),D} - z)^{-1} v_{R_2}] \right. \\
&\quad \times \left. \{ I_{(0,R_2)} + [I_{(0,R_2)} + u_{R_2} (-\Delta_{(0,R_1) \cup (R_1,R_2),D} - z)^{-1} v_{R_2}]^{-1} u_{R_2} T(z) v_{R_2} \} \right) \\
&= \det_{L^2((0,R_2);dx)} \left(I_{(0,R_2)} + u_{R_2} (-\Delta_{(0,R_1) \cup (R_1,R_2),D} - z)^{-1} v_{R_2} \right) \\
&\quad \times \det_{L^2((0,R_2);dx)} \left(I_{(0,R_2)} + [I_{(0,R_2)} + u_{R_2} (-\Delta_{(0,R_1) \cup (R_1,R_2),D} - z)^{-1} v_{R_2}]^{-1} \right. \\
&\quad \quad \left. \times u_{R_2} T(z) v_{R_2} \right), \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{B.9}
\end{aligned}$$

Since $\det_{\mathcal{H}}(I + A) = 1 + \operatorname{tr}_{\mathcal{H}}(A)$, whenever A is a rank-one operator in \mathcal{H} , the second determinant on the right-hand side of (B.9) can be computed as follows:

$$\begin{aligned}
& \det_{L^2((0,R_2);dx)} \left(I_{(0,R_2)} + [I_{(0,R_2)} + u_{R_2} (-\Delta_{(0,R_1) \cup (R_1,R_2),D} - z)^{-1} v_{R_2}]^{-1} \right. \\
&\quad \left. \times u_{R_2} T(z) v_{R_2} \right) \\
&= 1 + \frac{1}{G_{(0,R_2),D}^{(0)}(z, R, R)} \\
&\quad \times \left(v_{R_2} G_{(0,R_2),D}^{(0)}(z, R, \cdot), [I_{(0,R_2)} + u_{R_2} (-\Delta_{(0,R_1) \cup (R_1,R_2),D} - z)^{-1} v_{R_2}]^{-1} \right. \\
&\quad \quad \left. \times u_{R_2} G_{(0,R_2),D}^{(0)}(z, \cdot, R) \right)_{L^2((0,R_2);dx)} \\
&:= \mathcal{S}(z), \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{B.10}
\end{aligned}$$

Finally, replacing z by $\lambda + i\varepsilon$, taking arguments, and taking the limit $\varepsilon \downarrow 0$ throughout (B.9) yields

$$\begin{aligned}
& \xi(\lambda; -\Delta_{(0,R_2),D} +_q V_R, -\Delta_{(0,R_2),D}) \\
&= \xi(\lambda; -\Delta_{(0,R_1) \cup (R_1,R_2),D} +_q V|_{(0,R_2)}, -\Delta_{(0,R_1) \cup (R_1,R_2),D}) \\
&\quad + \lim_{\varepsilon \downarrow 0} \operatorname{Im}(\ln(\mathcal{S}(\lambda + i\varepsilon))) \text{ for a.e. } \lambda \in \mathbb{R}. \tag{B.11}
\end{aligned}$$

By Lemma B.1, the spectral shift function on the right-hand side of (B.11) is the sum of two spectral shift functions. Thus, in summary, we have the following decomposition formula:

Lemma B.2 (Spectral Shift Function Comparison Formula). *Let $0 < R_1 < R_2$ and assume that $V \in L^1((0, R_2); dx)$ is real-valued. Then*

$$\begin{aligned}
& \xi(\lambda; -\Delta_{(0,R_2),D} +_q V_R, -\Delta_{(0,R_2),D}) \\
&= \xi(\lambda; -\Delta_{(0,R_1),D} +_q V_R, -\Delta_{(0,R_1),D}) \\
&\quad + \xi(\lambda; -\Delta_{(R_1,R_2),D} +_q V|_{(R_1,R_2)}, -\Delta_{(R_1,R_2),D}) \\
&\quad + \lim_{\varepsilon \downarrow 0} \operatorname{Im}(\ln(\mathcal{S}(\lambda + i\varepsilon))) \text{ for a.e. } \lambda \in \mathbb{R}. \tag{B.12}
\end{aligned}$$

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