

ON A PROBLEM IN EIGENVALUE PERTURBATION THEORY

FRITZ GESZTESY, SERGEY N. NABOKO, AND ROGER NICHOLS

ABSTRACT. We consider additive perturbations of the type $H_t = H_0 + tV$, $t \in [0, 1]$, where H_0 and V are self-adjoint operators in a separable Hilbert space \mathcal{H} and V is bounded. In addition, we assume that the range of V is a generating (i.e., cyclic) subspace for H_0 . If λ_0 is an eigenvalue of H_0 , then under the additional assumption that V is nonnegative, the Lebesgue measure of the set of all $t \in [0, 1]$ for which λ_0 is an eigenvalue of H_t is known to be zero. We recall this result with its proof and show by explicit counterexample that the nonnegativity assumption $V \geq 0$ cannot be removed.

1. INTRODUCTION

The focus of this paper is a natural conjecture concerning the eigenvalues of a one-parameter family of self-adjoint perturbations H_t , $t \in [0, 1]$, of the form

$$H_t = H_0 + tV, \quad \text{dom}(H_t) = \text{dom}(H_0), \quad t \in [0, 1], \quad (1.1)$$

where H_0 is a (possibly unbounded) self-adjoint operator in a separable Hilbert space \mathcal{H} and V is a bounded self-adjoint operator in \mathcal{H} . If $\lambda_0 \in \sigma_p(H_0)$ (with $\sigma_p(T)$ denoting the point spectrum, that is, the set of eigenvalues, of a densely defined closed operator T in \mathcal{H}), a natural question is, “For which values of $t \in [0, 1]$ is λ_0 also an eigenvalue of H_t ?” In the general context, a direct answer to this question is likely out of reach. This is not a problem, however, as many of the applications where this question naturally arises (e.g., in studying the eigenvalues of Anderson-type models, see [5]) do not require one to explicitly determine the set of t . Instead, one only needs the set of such t to be “small” in a certain sense.

Without further assumptions on H_0 or V , it is easy to construct explicit examples for which H_t , $t \in [0, 1]$, share a common eigenvalue. For example, choose $\phi \in \mathcal{H} \setminus \{0\}$ and set $H_0 = V = (\phi, \cdot)_{\mathcal{H}} \phi$, with $(\cdot, \cdot)_{\mathcal{H}}$ denoting the inner product in \mathcal{H} . Obviously, $0 \in \sigma_p(H_0 + tV)$ for all $t \in [0, 1]$. The problem with this example is of course that the range of V is too small.

To exclude such examples, in the general setting we shall henceforth assume that the range of V is a generating subspace for H_0 , that is,

$$\mathcal{H} = \overline{\text{lin. span} \{(H_0 - zI_{\mathcal{H}})^{-1}V e_n \in \mathcal{H} \mid n \in \mathcal{I}, z \in \mathbb{C} \setminus \mathbb{R}\}}, \quad (1.2)$$

for a complete orthonormal system $\{e_n\}_{n \in \mathcal{I}}$, with $\mathcal{I} \subseteq \mathbb{N}$ an appropriate index set. Then, by appealing to a special form of spectral averaging [2, Corollary 4.2], it was

Date: June 11, 2020.

2010 Mathematics Subject Classification. Primary 15A22, 47A55, 47A75; Secondary 15A18.

Key words and phrases. Eigenvalues, linear pencils, perturbation theory.

S.N. is supported by grants NCN t 2013/09/BST1/04319, RFBR 12-01-00215-a, and Marie Curie grant PIFI-GA-2011-299919.

R.N. gratefully acknowledges support from an AMS–Simons Travel Grant.

Published in *J. Math. Anal. Appl.* **428**, No. 1, 295–305 (2015).

shown in [5, Lemma 4] that the Lebesgue measure of the set of $t \in [0, 1]$ for which λ_0 is an eigenvalue of H_t is equal to zero, that is,

$$|\{t \in [0, 1] \mid \lambda_0 \in \sigma_p(H_t)\}| = 0, \quad (1.3)$$

under the additional assumption that V is *nonnegative*, that is, $V \geq 0$. Here, by some abuse of notation, we abbreviate the Lebesgue measure on \mathbb{R} by $|\cdot|$.

Actually, (1.3) is only a special case of the result obtained in [5, Lemma 4] as, more generally, the authors of [5] show that for any measurable set M of Lebesgue measure zero, the set of all t in $[0, 1]$ for which H_t has an eigenvalue in M has Lebesgue measure equal to zero, which is to say

$$|\{t \in [0, 1] \mid \sigma_p(H_t) \cap M \neq \emptyset\}| = 0. \quad (1.4)$$

Obviously, (1.3) is obtained by choosing $M = \{\lambda_0\}$ in (1.4).

Upon seeing the measure zero result in (1.3), it is natural to inquire about the extent to which the assumption that $V \geq 0$ is necessary. This leads to the question: *Can one remove the assumption $V \geq 0$ and still retain the conclusion in (1.3)?*

A careful examination of the proofs to [5, Lemma 4] and its key ingredient [2, Corollary 4.2] reveals that nonnegativity of V is crucial for both, so dispensing with the assumption $V \geq 0$ (if possible) would require entirely new ideas. However, as it turns out, the nonnegativity assumption on V is absolutely crucial. We show that without $V \geq 0$, (1.3) breaks down in dramatic fashion for in this case one can actually construct a counterexample in the finite dimensional Hilbert space $\mathcal{H} = \mathbb{C}^3$ (cf. Example 2.9).

Next, we briefly summarize the contents of this paper. In Section 2 we present Conjecture 2.2, the natural conjecture that (1.3) continues to hold with the assumption $V \geq 0$ removed, and in Lemma 2.3, we consider perturbations of the form $H_t = H_0 + tV$, $t \in [0, 1]$, where H_0 and V are self-adjoint and V is compact. By applying the Birman–Schwinger principle, it is shown that there exist at most finitely many $t \in [0, 1]$ for which a given point $E_0 \in \mathbb{R} \cap \rho(H_0)$ belongs to $\sigma_p(H_t)$. (The assumption that $E_0 \in \rho(H_0)$, the resolvent set of H_0 , which is necessary in order to apply the Birman–Schwinger principle to $H_0 + tV$, means that this result is fundamentally different from results like (1.3), where $\lambda_0 \in \sigma_p(H_0)$ is assumed.) In Lemma 2.5, we recall the result of [5, Lemma 4] tailored to the present context, and we provide its proof for completeness. Following Lemma 2.5, we provide a general discussion which relates the eigenvalue problem

$$H_t \psi = \lambda \psi, \quad \psi \in \mathcal{H}, \lambda \in \mathbb{R}, t \in [0, 1], \quad (1.5)$$

to a linear pencil eigenvalue problem with respect to the t parameter. Example 2.7 provides an example where spectra are computed by looking at the corresponding linear pencil. In our main Example 2.9, we put Conjecture 2.2 to rest, showing by counterexample, that one cannot remove the assumption $V \geq 0$ from Lemma 2.5 and retain (1.3).

Finally, we briefly summarize some of the notation used in this paper: Let \mathcal{H} be a separable complex Hilbert space, $(\cdot, \cdot)_{\mathcal{H}}$ the scalar product in \mathcal{H} (linear in the second entry), and $I_{\mathcal{H}}$ the identity operator in \mathcal{H} . Next, let T be a linear operator mapping (a subspace of) a Banach space into another, with $\text{dom}(T)$, $\text{ran}(T)$, and $\text{ker}(T)$ denoting the domain, range, and kernel (i.e., null space) of T . If T is densely defined, then T^* denotes the Hilbert space adjoint of T . The closure of a closable

operator S is denoted by \overline{S} . If A is self-adjoint in \mathcal{H} , we denote $A_{\pm} = [|A| \pm A]/2$, employing the spectral theorem for A .

The spectrum, discrete spectrum, point spectrum, continuous spectrum, residual spectrum, and resolvent set of a closed linear operator in \mathcal{H} will be denoted by $\sigma(\cdot)$, $\sigma_d(\cdot)$, $\sigma_p(\cdot)$, $\sigma_c(\cdot)$, $\sigma_r(\cdot)$, and $\rho(\cdot)$, respectively (cf., e.g., [4, p. 451–452]).

The Banach spaces of bounded and compact linear operators in \mathcal{H} are denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_{\infty}(\mathcal{H})$, respectively.

We denote by $E_A(\cdot)$ the family of strongly right-continuous spectral projections of a self-adjoint operator A in \mathcal{H} (in particular, $E_A(\lambda) = E_A((-\infty, \lambda])$, $\lambda \in \mathbb{R}$).

2. ON AN EIGENVALUE PERTURBATION PROBLEM

Denoting by $|M|$ the Lebesgue measure of a measurable subset M of \mathbb{R} , and supposing that $\{e_n\}_{n \in \mathcal{I}}$ (with $\mathcal{I} \subseteq \mathbb{N}$ an appropriate index set) represents a complete orthonormal system in \mathcal{H} , we will always assume that we assume the following:

Hypothesis 2.1. *Suppose the range of V is a generating (i.e., cyclic) subspace for H_0 , that is,*

$$\mathcal{H} = \overline{\text{lin. span} \{(H_0 - zI_{\mathcal{H}})^{-1}Ve_n \in \mathcal{H} \mid n \in \mathcal{I}, z \in \mathbb{C} \setminus \mathbb{R}\}} \quad (2.1)$$

(equivalently, $\mathcal{H} = \overline{\text{lin. span} \{E_{H_0}(\lambda)Ve_n \in \mathcal{H} \mid n \in \mathcal{I}, \lambda \in \mathbb{R}\}}$).

The principal purpose of this paper is to **disprove** the following somewhat naturally sounding conjecture:

Conjecture 2.2. *Assume Hypothesis 2.1, let H_0 be self-adjoint in \mathcal{H} , and suppose $V = V^* \in \mathcal{B}(\mathcal{H})$. Consider*

$$H_t = H_0 + tV, \quad \text{dom}(H_t) = \text{dom}(H_0), \quad t \in [0, 1], \quad (2.2)$$

and suppose that $\lambda_0 \in \sigma_p(H_0)$. Then

$$|\{t \in [0, 1] \mid \lambda_0 \in \sigma_p(H_t)\}| = 0. \quad (2.3)$$

Here is an elementary result in this context:

Lemma 2.3. *Let H_0 and V be self-adjoint in \mathcal{H} such that $V(H_0 - z_0I_{\mathcal{H}})^{-1} \in \mathcal{B}_{\infty}(\mathcal{H})$ for some (and hence for all) $z_0 \in \rho(H_0)$. Consider a fixed $E_0 \in \mathbb{R} \cap \rho(H_0)$ and introduce $H_t = H_0 + tV$, $t \in [0, 1]$. Then there exist at most finitely many $t \in [0, 1]$ such that $E_0 \in \sigma_p(H_t)$.*

Proof. Applying the Birman–Schwinger principle (cf., e.g., [6, §III.2]),

$$E_0 \in \sigma_p(H_t) \text{ is equivalent to } (-1/t) \in \sigma_p(V(H_0 - E_0I_{\mathcal{H}})^{-1}). \quad (2.4)$$

Since by hypothesis, $V(H_0 - E_0I_{\mathcal{H}})^{-1} \in \mathcal{B}_{\infty}(\mathcal{H})$, the nonzero eigenvalues of $V(H_0 - E_0I_{\mathcal{H}})^{-1} \in \mathcal{B}_{\infty}(\mathcal{H})$ are either finite in number, or else, converge to zero (they need not be real). Either way, there can only be finitely many $t \in [0, 1]$ such that $(-1/t)$ is an eigenvalue of $V(H_0 - E_0I_{\mathcal{H}})^{-1}$. \square

However, since we had to assume $E_0 \in \rho(H_0) \cap \mathbb{R}$, this basically renders Lemma 2.3 irrelevant in connection with Conjecture 2.2.

The next lemma is inspired by a result due to Friedlander [3], who applied it in connection with his proof of purely absolutely continuous spectrum for a certain class of 2nd order elliptic differential operators with periodic coefficients satisfying a symmetry condition.

Lemma 2.4. ([3], *main theorem and its proof*) *Let H_0 be self-adjoint in \mathcal{H} , and suppose that $V^* = V \in \mathcal{B}(\mathcal{H})$ and $V(H_0 + z_0 I_{\mathcal{H}})^{-1} \in \mathcal{B}_{\infty}(\mathcal{H})$ for some (and hence for all) $z_0 \in \rho(H_0)$. In addition, assume that $0 \in \rho(H_0) \cup \sigma_a(H_0)$, and*

(i) $\text{rank}(V_{-}) < \infty$,

(ii) $\ker(V) = \{0\}$.

Then,

$$\{t \in \mathbb{C} \mid 0 \in \sigma_p(H_0 + tV)\} \text{ is discrete in } \mathbb{C} \quad (2.5)$$

(i.e., a subset of \mathbb{C} without finite accumulation point).

Proof. (The case $0 \in \rho(H_0)$ is just Lemma 2.3, but we will not use this in this proof.) We will prove the folklore-type result that for some discrete set $\mathcal{E} \subset \mathbb{C}$, and for some $N_0 \in \mathbb{N} \cup \{0\}$, $\dim(\ker(H_0 + tV)) = N_0$ for all $t \in \mathbb{C} \setminus \mathcal{E}$. If $N_0 = 0$, the proof is finished. In the case $N_0 \in \mathbb{N}$, we will seek to derive a contradiction.

Introducing $P_0 = E_{H_0}(\{0\})$ and $P_0^{\perp} = I_{\mathcal{H}} - P_0$, then $H_0 u(t) = -tV u(t)$ implies $P_0^{\perp} H_0 P_0^{\perp} u(t) = -tP_0^{\perp} V u(t)$, and hence, $P_0^{\perp} u(t) = -t(H_0^{\perp})^{-1} P_0^{\perp} V u(t)$, where $H_0^{\perp} = P_0^{\perp} H_0 P_0^{\perp}$. Thus, assuming $|t| \leq R$ for some $R > 0$, we decompose,

$$\begin{aligned} u(t) &= P_0 u(t) + P_0^{\perp} u(t) = P_0 u(t) - t(H_0^{\perp})^{-1} P_0^{\perp} V u(t) \\ &= P_0 u(t) - t[(H_0^{\perp})^{-1} P_0^{\perp} V P_0 + (H_0^{\perp})^{-1} P_0^{\perp} V P_0^{\perp}] u(t) \\ &= [P_0 - tF_0 - t\tilde{F}_n - tG_n] u(t) \\ &= [P_0 - tF_n - tG_n] u(t), \end{aligned} \quad (2.6)$$

where

$$F_0 = (H_0^{\perp})^{-1} P_0^{\perp} V P_0, \quad \text{rank}(F_0) < \infty, \quad (2.7)$$

$$(H_0^{\perp})^{-1} P_0^{\perp} V P_0^{\perp} = [\tilde{F}_n + G_n] \in \mathcal{B}_{\infty}(\mathcal{H}), \quad (2.8)$$

$$F_n = F_0 + \tilde{F}_n, \quad \text{rank}(\tilde{F}_n) < \infty, \quad \text{rank}(F_n) < \infty, \quad (2.9)$$

$$G_n = GP_n^{\perp} \text{ for } G := (H_0^{\perp})^{-1} P_0^{\perp} V P_0^{\perp} \in \mathcal{B}_{\infty}(\mathcal{H}), \quad \|G_n\|_{\mathcal{B}(\mathcal{H})} \leq (2R)^{-1}, \quad (2.10)$$

and P_n is a strictly monotone increasing sequence of orthogonal projections, $P_n \leq P_{n+1}$, $n \in \mathbb{N}$, $s\text{-}\lim_{n \rightarrow \infty} P_n = I_{\mathcal{H}}$, $P_n^{\perp} = I_{\mathcal{H}} - P_n$, $n \in \mathbb{N}$, such that for $n \in \mathbb{N}$ sufficiently large,

$$F_n P_n^{\perp} = 0, \quad F_0 P_n^{\perp} = 0. \quad (2.11)$$

Thus,

$$u(t) = [I_{\mathcal{H}} + tG_n]^{-1} [P_0 - tF_n] u(t), \quad |t| \leq R, \quad (2.12)$$

and finally, applying P_n to (2.12), employing (2.11), the equation $[H_0 + tV]u(t) = 0$, $|t| \leq R$, is reduced to the matrix equation,

$$[P_n - P_n(I_{\mathcal{H}} + tG_n)^{-1} [P_0 - tF_n]] P_n u(t) = 0 \quad |t| \leq R. \quad (2.13)$$

A solution $P_n u(t)$ to (2.13) yields an element $0 \neq u(t) \in \ker(H_0 + tV)$ upon taking the component of $u(t)$ orthogonal to $P_n u(t)$ to be $[I_{\mathcal{H}} - P_n][I_{\mathcal{H}} + tG_n]^{-1} [P_0 - tF_n] P_n u(t)$, assuming without loss that the projections P_n are chosen so that $P_0 \leq P_n$, $n \in \mathbb{N}$, and hence,

$$u(t) = P_n u(t) + [I_{\mathcal{H}} - P_n][I_{\mathcal{H}} + tG_n]^{-1} [P_0 - tF_n] P_n u(t). \quad (2.14)$$

Thus, (2.13) either has a positive dimensional subspace of solutions $P_n u(t_k)$, $1 \leq k \leq N_R$, in each disk $|t| \leq R$ and $\dim(\ker(H_0 + tV)) = 0$ outside the discrete set $\{t_k\}$ in the disc (and (2.5) follows in the disk), or else, $\det(P_n - P_n(I_{\mathcal{H}} + tG_n)^{-1} [P_0 - tF_n])$

vanishes identically in $|t| \leq R$. In the latter case, constancy of the kernel dimension outside a discrete set follows easily from the analyticity of the minors in t , including those of maximal size that are not identically zero. It is enough to notice that in the intersection of discs D_r (see below), the dimension of the space of solutions of (2.13) should be the same constant for both discs despite of choosing different projections P_n associated with them. This follows immediately from the coincidence of the dimension of the solution space for (2.13) with the kernel dimension for the operator $H_0 + tV$. We intend to assume that the constant kernel dimension, call it N_0 , is nonzero and then ultimately arrive at a contradiction.

Denoting $D_r := \{t \in \mathbb{C} \mid |t| \leq r\}$, $r > 0$, one may choose a sequence of radii $\{R_j\}_{j=1}^\infty$ diverging monotonically to $+\infty$ for which the corresponding matrix equation (2.13) has a nontrivial solution $u_j(t)$ for all $|t| < R_j$ apart from a discrete set $t \in \mathcal{E}_j \subset D_{R_j}$. By taking $u(t) = u_1(t)$ if $0 \leq |t| < R_1$ and $u(t) = u_j(t)$ if $R_j \leq |t| < R_{j+1}$ for some $j \in \mathbb{N}$, one concludes that there is a discrete set \mathcal{E} in \mathbb{C} such that for $t \in \mathbb{C} \setminus \mathcal{E}$, there exists an N_0 -dimensional subspace of solutions $u(t) \in \text{dom}(H_0)$, satisfying $H_0 u(t) = -tV u(t)$, $t \in \mathbb{C} \setminus \mathcal{E}$. If N_0 is positive one may choose a family of vectors $u(t) \in \ker(H_0 + tV)$ with $\|u(t)\|_{\mathcal{H}} = 1$, $t \in \mathbb{C} \setminus \mathcal{E}$. Note that this family need not to be continuous in t . (The exceptional points from \mathcal{E} in the disc D_{R_j} are simply those in the disc $D_{R_{j-1}}$ together with any additional ones in the annulus $D_{R_j} \setminus D_{R_{j-1}}$. This procedure allows to avoid the possibility of exceptional points accumulating locally as the radii diverge to infinity.)

Let $s, t \in \mathbb{C} \setminus \mathcal{E}$, then

$$0 = (u(s), (H_0 + tV)u(t))_{\mathcal{H}} = (t - \bar{s})(u(s), Vu(t))_{\mathcal{H}}, \quad (2.15)$$

that is,

$$(u(s), Vu(t))_{\mathcal{H}} = 0, \quad s, t \in \mathbb{C} \setminus \mathcal{E}, \quad \bar{s} \neq t. \quad (2.16)$$

The identity in (2.16) extends to

$$(u(s), Vu(s))_{\mathcal{H}} = 0, \quad s \in \mathbb{C} \setminus \mathcal{E}. \quad (2.17)$$

Indeed, (2.17) follows immediately from (2.15) if $s \in \mathbb{C} \setminus \mathcal{E}$ and $\text{Im}(s) \neq 0$. If $s \in \mathbb{R} \setminus \mathcal{E}$ is fixed, then in a sufficiently small neighborhood $|t - s| < \varepsilon$ in \mathbb{C} , it is possible to choose a norm-continuous vector-valued function $v(t) \in \ker(H_0 + tV)$ for which $v(s) = u(s)$. Then, in analogy to (2.15), one obtains

$$0 = (u(s), (H_0 + tV)v(t))_{\mathcal{H}} = (t - s)(u(s), Vv(t))_{\mathcal{H}}, \quad 0 < |t - s| < \varepsilon, \quad (2.18)$$

so that

$$(u(s), Vv(t))_{\mathcal{H}} = 0, \quad 0 < |t - s| < \varepsilon. \quad (2.19)$$

Now (2.17) follows immediately upon taking the limit $t \rightarrow s$ in (2.19). From this point on one can follow Friedlander's argument in [3, p. 54]. Setting

$$\mathcal{U} = \overline{\text{lin. span} \{u(t) \in \ker(H_0 + tV) \mid t \in \mathbb{C} \setminus \mathcal{E}\}}, \quad (2.20)$$

one concludes that

$$(u, Vv)_{\mathcal{H}} = 0, \quad u, v \in \mathcal{U}, \quad \text{and hence, } (u, Vu)_{\mathcal{H}} = 0, \quad u \in \mathcal{U}. \quad (2.21)$$

Thus,

$$(u, Vu)_{\mathcal{H}} \leq 0, \quad u \in \mathcal{U}, \quad \text{and } \text{rank}(V_-) < \infty \text{ imply } \dim(\mathcal{U}) < \infty. \quad (2.22)$$

Next, let $u(t_k) \in \mathcal{U}$, $t_k \in \mathbb{C} \setminus \mathcal{E}$, $k \in \mathbb{N}$, such that $|t_k| \rightarrow \infty$ as $k \rightarrow \infty$. Then,

$$-t_k^{-1} (H|_{\mathcal{U}})u(t_k) = Vu(t_k), \quad k \in \mathbb{N}, \quad (2.23)$$

together with $\|u(t_k)\|_{\mathcal{H}} = 1$ and $\dim(\mathcal{U}) < \infty$ yields the existence of a subsequence $\{u(t_{k_j})\}_{j \in \mathbb{N}}$ of $\{u(t_k)\}_{k \in \mathbb{N}}$ and a $u_\infty \in \mathcal{U}$, with $\|u_\infty\|_{\mathcal{H}} = 1$, such that

$$\lim_{j \rightarrow \infty} \|u(t_{k_j}) - u_\infty\|_{\mathcal{H}} = 0 \quad \text{and} \quad -t_{k_j}^{-1}(H_0|_{\mathcal{U}})u(t_{k_j}) = Vu(t_{k_j}), \quad j \in \mathbb{N}. \quad (2.24)$$

Since $H_0|_{\mathcal{U}}$ and V are bounded, taking the limit $j \rightarrow \infty$ in the second equality in (2.24) then yields

$$0 = Vu_\infty. \quad (2.25)$$

Since $\ker(V) = \{0\}$ by assumption, one finally concludes $u_\infty = 0$ and hence $\mathcal{U} = \{0\}$, contradicting $N_0 \in \mathbb{N}$. \square

We note that condition (i) in Lemma 2.4 is necessary because of Remark 2.10, whereas condition (ii) is necessary because of Example 2.9.

We continue with a relevant positive result in the special case where $0 \leq V \in \mathcal{B}(\mathcal{H})$, that is derived in the proof of [5, Lemma 4].

Lemma 2.5. ([2], [5, Lemma 4 and its proof]) *Assume Hypothesis 2.1, let H_0 be self-adjoint in \mathcal{H} , and $0 \leq V \in \mathcal{B}(\mathcal{H})$. Consider*

$$H_t = H_0 + tV, \quad \text{dom}(H_t) = \text{dom}(H_0), \quad t \in [0, 1], \quad (2.26)$$

and assume that $\text{ran}(V)$ is a generating (i.e., cyclic) subspace for H_0 , that is,

$$\mathcal{H} = \overline{\text{lin. span} \{(H_0 - zI_{\mathcal{H}})^{-1}Ve_n \in \mathcal{H} \mid n \in \mathcal{I}, z \in \mathbb{C} \setminus \mathbb{R}\}} \quad (2.27)$$

(equivalently, $\mathcal{H} = \overline{\text{lin. span} \{E_{H_0}(\lambda)Ve_n \in \mathcal{H} \mid n \in \mathcal{I}, \lambda \in \mathbb{R}\}}$). Suppose that $\lambda_0 \in \sigma_p(H_0)$. Then

$$|\{t \in [0, 1] \mid \lambda_0 \in \sigma_p(H_t)\}| = 0. \quad (2.28)$$

Proof. Fix $\lambda_0 \in \mathbb{R}$. Extend H_t from $t \in [0, 1]$ to $t \in \mathbb{R}$ by

$$H_t = H_0 + tV, \quad t \in \mathbb{R}. \quad (2.29)$$

If

$$E_{H_t}(\{\lambda_0\}) := E_{H_t}(\lambda_0) - E_{H_t}(\lambda_0 - 0), \quad t \in \mathbb{R}, \quad (2.30)$$

then spectral averaging (cf., e.g., [2, Corollary 4.2 and its proof]) immediately yields

$$\int_{\mathbb{R}} \frac{(v, V^{1/2}E_{H_t}(\{\lambda_0\})V^{1/2}v)_{\mathcal{H}}}{1+t^2} dt = 0, \quad v \in \mathcal{H}. \quad (2.31)$$

Since $V^{1/2}E_{H_t}(\{\lambda_0\})V^{1/2} \geq 0$, $t \in \mathbb{R}$, (2.31) implies

$$V^{1/2}E_{H_t}(\{\lambda_0\})V^{1/2}v = 0, \quad t \in \mathbb{R} \setminus N_v, \quad v \in \mathcal{H}, \quad (2.32)$$

where $N_v \subseteq \mathbb{R}$ is (Lebesgue) measurable with $|N_v| = 0$, $v \in \mathcal{H}$. Applying (2.32) to each element of a complete orthonormal system for \mathcal{H} , we obtain a (Lebesgue) measurable set $N \subseteq \mathbb{R}$ with $|N| = 0$ for which

$$V^{1/2}E_{H_t}(\{\lambda_0\})V^{1/2} = 0, \quad t \in \mathbb{R} \setminus N. \quad (2.33)$$

Now, (2.33) implies $E_{H_t}(\{\lambda_0\})|_{\text{ran}(V)} = 0$, $t \in \mathbb{R} \setminus N$. Indeed, by (2.33),

$$\begin{aligned} \|E_{H_t}(\{\lambda_0\})Vv\|_{\mathcal{H}}^2 &= (Vv, E_{H_t}(\{\lambda_0\})Vv)_{\mathcal{H}} \\ &= (V^{1/2}v, [V^{1/2}E_{H_t}(\{\lambda_0\})V^{1/2}]V^{1/2}v)_{\mathcal{H}} = 0, \\ & \quad t \in \mathbb{R} \setminus N, \quad v \in \mathcal{H}. \end{aligned} \quad (2.34)$$

Next one notes,

$$E_{H_t}(\{\lambda_0\})(H_t - zI_{\mathcal{H}})^{-1}Ve_n = (H_t - zI_{\mathcal{H}})^{-1}E_{H_t}(\{\lambda_0\})Ve_n = 0, \quad (2.35)$$

$$n \in \mathcal{I}, z \in \mathbb{C} \setminus \mathbb{R}, t \in \mathbb{R} \setminus N,$$

where $\{e_n\}_{n \in \mathcal{I}}$, $\mathcal{I} \subseteq \mathbb{N}$ an appropriate index set, is any complete orthonormal system in \mathcal{H} . By a standard resolvent identity, one obtains t -invariance of the cyclic subspace for H_t generated by $\text{ran}(V)$ in the form:

$$\begin{aligned} & \overline{\text{lin. span}\{(H_0 - zI_{\mathcal{H}})^{-1}Ve_n \mid n \in \mathcal{I}, z \in \mathbb{C} \setminus \mathbb{R}\}} \\ & = \overline{\text{lin. span}\{(H_t - zI_{\mathcal{H}})^{-1}Ve_n \mid n \in \mathcal{I}, z \in \mathbb{C} \setminus \mathbb{R}\}}, \quad t \in \mathbb{R}. \end{aligned} \quad (2.36)$$

Since the first subspace in (2.36) coincides with \mathcal{H} , (2.35) and boundedness of the spectral projections $E_{H_t}(\{\lambda_0\})$ imply

$$E_{H_t}(\{\lambda_0\})u = 0, \quad u \in \mathcal{H}, t \in \mathbb{R} \setminus N. \quad (2.37)$$

Since N is a set with zero Lebesgue measure, $\lambda_0 \notin \sigma_p(H_t)$ for a.e. $t \in \mathbb{R}$. The result in (2.28) follows immediately. \square

Next, we continue this line of thought and relate it to spectral theory for linear pencils of operators. Considering the standard decomposition of V

$$V = V_+ - V_-, \quad 0 \leq V_{\pm} = [|V| \pm V]/2 \in \mathcal{B}(\mathcal{H}), \quad (2.38)$$

provided by the spectral theorem for V , we now modify this to a more general (highly nonunique) decomposition

$$V = V_1 - V_2, \quad 0 \leq V_j \in \mathcal{B}(\mathcal{H}). \quad (2.39)$$

In particular, we may assume that either

$$V_1 \geq \varepsilon I_{\mathcal{H}}, \quad (2.40)$$

or, alternatively,

$$V_2 \geq \varepsilon I_{\mathcal{H}}, \quad (2.41)$$

(e.g., by adding an appropriate multiple of the identity to V_1 or V_2). In this case, the basic eigenvalue equation

$$H_t \psi = \lambda_0 \psi, \quad \psi \in \mathcal{H}, \lambda_0 \in \mathbb{R}, t \in [0, 1], \quad (2.42)$$

is equivalent to

$$\begin{aligned} [V_1^{-1/2} H_0 V_1^{-1/2} - \lambda_0 V_1^{-1}] (V_1^{1/2} \psi) &= -t [I_{\mathcal{H}} - V_1^{-1/2} V_2 V_1^{-1/2}] (V_1^{1/2} \psi) \\ &\text{if } V_1^{-1} \in \mathcal{B}(\mathcal{H}), \end{aligned} \quad (2.43)$$

or, alternatively, to

$$\begin{aligned} [V_2^{-1/2} H_0 V_2^{-1/2} - \lambda_0 V_2^{-1}] (V_2^{1/2} \psi) &= t [I_{\mathcal{H}} - V_2^{-1/2} V_1 V_2^{-1/2}] (V_2^{1/2} \psi) \\ &\text{if } V_2^{-1} \in \mathcal{B}(\mathcal{H}). \end{aligned} \quad (2.44)$$

Thus, the standard self-adjoint eigenvalue problem (2.42) is equivalent to a linear pencil eigenvalue problem (w.r.t. the parameter $t \in [0, 1]$) of the form

$$A(\lambda_0)f = tBf, \quad f \in \mathcal{H}, t \in [0, 1], \quad (2.45)$$

where

$$A(\lambda_0) = A(\lambda_0)^* \in \mathcal{B}(\mathcal{H}), \quad B = B^* \in \mathcal{B}(\mathcal{H}). \quad (2.46)$$

That is, the underlying linear self-adjoint pencil is of the form $A(\lambda_0) - tB$, $t \in [0, 1]$.

Generally speaking, the pencil $A(\lambda_0) - tB$, $t \in [0, 1]$, readily leads to non-real eigenvalues (i.e., becomes an inequivalent non-self-adjoint spectral problem) even if $A(\lambda_0)$ as well as B are self-adjoint, as the following elementary example illustrates.

Example 2.6. [7, §VI.1.1, Example 1.14] *In the Hilbert space \mathbb{C}^2 , the following linear pencil*

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} f = t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (2.47)$$

has eigenvalues $\pm i$ even though both matrix coefficients are self-adjoint.

While the eigenvalues of a standard self-adjoint eigenvalue problem in a separable Hilbert space are necessarily countable, no such result holds for pencil eigenvalue problems, as demonstrated by the following example kindly communicated to us by T. Azizov [1].

Example 2.7 ([1]). *In the Hilbert space \mathcal{H} consider the complete orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$ and introduce the shift operator S_+*

$$S_+ e_k = e_{k+1}, \quad k \in \mathbb{N}, \quad (2.48)$$

such that

$$S_+^* e_1 = 0, \quad S_+^* e_k = e_{k-1}, \quad k \in \mathbb{N}, \quad k \geq 2. \quad (2.49)$$

Then, with $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, the open unit disk in \mathbb{C} , one has the following facts (cf., e.g., [4, p. 468–469]),

$$\sigma_p(S_+) = \sigma_r(S_+^*) = \emptyset, \quad (2.50)$$

$$\sigma_r(S_+) = \sigma_p(S_+^*) = \mathbb{D}, \quad (2.51)$$

$$\sigma_c(S_+) = \sigma_c(S_+^*) = \partial\mathbb{D}, \quad (2.52)$$

$$\sigma(S_+) = \sigma(S_+^*) = \overline{\mathbb{D}}. \quad (2.53)$$

Next, introduce in $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$ the self-adjoint 2×2 block operator matrices

$$H_0 = \begin{pmatrix} 0 & S_+ \\ S_+^* & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & I_{\mathcal{H}} \\ I_{\mathcal{H}} & 0 \end{pmatrix}. \quad (2.54)$$

Then $V^2 = I_{\mathcal{K}} = \begin{pmatrix} I_{\mathcal{H}} & 0 \\ 0 & I_{\mathcal{H}} \end{pmatrix}$ shows that the linear pencil eigenvalue problem

$$H_0 f = tVf, \quad f \in \mathcal{K}, \quad t \in \mathbb{C}, \quad (2.55)$$

is equivalent to the standard (non-self-adjoint) eigenvalue problem

$$VH_0 f = tf, \quad f \in \mathcal{K}, \quad t \in \mathbb{C}, \quad VH_0 = \begin{pmatrix} S_+^* & 0 \\ 0 & S_+ \end{pmatrix}. \quad (2.56)$$

Together with (2.50)–(2.53) this yields

$$\sigma(VH_0) = \overline{\mathbb{D}}, \quad \sigma_p(VH_0) = \mathbb{D}, \quad (2.57)$$

in particular, each $t \in \mathbb{D}$ is an eigenvalue for (2.55).

Remark 2.8. Further generalizations of Example 2.7 are possible:

(i) In Example 2.7 the operator V is invertible and therefore not compact. This may unintentionally lead to the misunderstanding that the eigenvalue phenomenon is related to the noncompactness of V . However, a suitable reconstruction of Example 2.7, replacing the identity operators in V (see (2.54)) by a compact self-adjoint diagonal operator Λ_2 (in the basis $\{e_k\}_{k \in \mathbb{N}}$) and the operators S_+ (resp., S_+^*) in H_0

by $S_+\Lambda_1$ (resp., $\Lambda_1 S_+^*$), settles this issue. Here Λ_1 is again a self-adjoint diagonal operator in \mathcal{H} . A suitable choice of the diagonal operators leads to an example where the operator V is compact, but the set of eigenvalues t of the spectral problem (2.55) covers the whole complex plane. Surely, the critical condition that the range of V is a generating subspace for H_0 can be satisfied.

(ii) Another generalization is related to the case where the positive part of the self-adjoint operator V is invertible (on a suitable subspace of \mathcal{H}) and its negative part is a compact operator. In that case, as explicit examples show, the point spectrum of the spectral problem (2.55) may cover the whole disc \mathbb{D} again. We note that the vanishing of the negative part according to the Lemma 2.5 leads to the fact that the set of eigenvalues has Lebesgue measure 0.

Next, to put Conjecture 2.2 to rest once and for all, we offer the following elementary three-dimensional counterexample:

Example 2.9. Consider $\mathcal{H} = \mathbb{C}^3$,

$$H_0 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} = H_0^*, \quad V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = V^*. \quad (2.58)$$

Then,

$$\ker(H_0) = \text{lin.span} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \text{ran}(V) = \mathbb{C}^2 \oplus \{0\}, \quad (2.59)$$

and since

$$H_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad (2.60)$$

one infers that

$$\text{ran}(V) \dot{+} H_0 \text{ran}(V) = \mathbb{C}^3 \quad (2.61)$$

(with $\dot{+}$ denoting the direct, though, not necessarily orthogonal direct sum, which we denote by \oplus). Finally, introducing

$$f_t = \begin{pmatrix} 1 \\ -1 \\ -t \end{pmatrix}, \quad t \in \mathbb{C}, \quad (2.62)$$

one obtains

$$H_0 f_0 = 0, \quad (H_0 + tV)f_t = 0 \text{ for each } t \in \mathbb{C}, \quad (2.63)$$

illustrating in dramatic fashion that nonnegativity of V cannot be omitted in Lemma 2.5.

To cast (2.63) as an equivalent problem for a linear pencil of operators, decompose V according to $V = V_1 - V_2$ with

$$V_1 = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.64)$$

which yields a decomposition of V of the form (2.39). By (2.43), the eigenvalue problem (2.63) is equivalent to the linear pencil eigenvalue problem

$$H_0 f = -t(I_3 - V_2)f, \quad f \in \mathbb{C}^3, \quad t \in \mathbb{C}. \quad (2.65)$$

Remark 2.10. In Example 2.9, both matrices H_0 and V fail to be invertible. Surely in a finite dimensional space \mathcal{H} , the invertibility of at least one of H_0 or V immediately leads to the reduction of the spectral problem (2.55) to the standard eigenvalue problem in the spectral parameter t or t^{-1} and therefore to the finiteness of the set of t values.

Acknowledgments. We are indebted to Fedor Sukochev for raising a question that naturally led to what we called Conjecture 2.2. Helpful discussions with Tomas Azizov, Rostyslav Hryniv, Vladimir Derkach, Heinz Langer, Michael Levitin, Mark Malamud, Marco Marletta, Leonid Parnovski, Fritz Philipp, Andrei Shkalikov, Barry Simon, Günter Stolz, Vadim Tkachenko, and Christiane Tretter are gratefully acknowledged. Especially, we thank Tomas Azizov for sending us Example 2.7 and Leonid Parnovski for pointing out Friedlander’s paper [3] to us.

S. N. is grateful to the Department of Mathematics of the University of Missouri where part of this work was completed while on a Miller Scholar Fellowship in February–March of 2014.

REFERENCES

- [1] T. Ya. Azizov, private communication, May 20, 2013.
- [2] J.-M. Combes and P. D. Hislop, *Localization for some continuous, random Hamiltonians in d -dimensions*, *J. Funct. Anal.* **124**, 149–180 (1994).
- [3] L. Friedlander, *On the spectrum of a class of second order periodic elliptic differential operators*, *Commun. Math. Phys.* **229**, 49–55 (2002).
- [4] C. S. Kubrusly, *Elements of Operator Theory*, Birkhäuser, Boston, 2001.
- [5] S. Naboko, R. Nichols, and G. Stolz, *Simplicity of eigenvalues in Anderson-type models*, *Ark. Mat.* **51**, 157–183 (2013).
- [6] B. Simon, *Quantum Mechanics for Hamiltonians Defined as Quadratic Forms*, Princeton University Press, Princeton, NJ, 1971.
- [7] G. W. Stewart and J. G. Sun, *Matrix Perturbation Theory*, Academic Press, Boston, MA, 1990.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211, USA

E-mail address: `gesztesyf@missouri.edu`

URL: <http://www.math.missouri.edu/personnel/faculty/gesztesyf.html>

DEPARTMENT OF MATHEMATICAL PHYSICS, ST. PETERSBURG STATE UNIVERSITY, ULIANOVSKAIA 1, NIIF, ST. PETERHOF, ST. PETERSBURG, RUSSIAN FEDERATION, 198504

E-mail address: `sergey.naboko@gmail.com`

MATHEMATICS DEPARTMENT, THE UNIVERSITY OF TENNESSEE AT CHATTANOOGA, 415 EMCS BUILDING, DEPT. 6956, 615 MCCALLIE AVE, CHATTANOOGA, TN 37403, USA

E-mail address: `Roger-Nichols@utc.edu`

URL: <http://www.utc.edu/faculty/roger-nichols/index.php>