STABILITY OF SQUARE ROOT DOMAINS ASSOCIATED WITH ELLIPTIC SYSTEMS OF PDES ON NONSMOOTH DOMAINS

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Abstract. We discuss stability of square root domains for uniformly elliptic partial differential operators $L_{a,\Omega,\Gamma} = -\nabla \cdot a\nabla$ in $L^2(\Omega)$, with mixed boundary conditions on $\partial\Omega$, with respect to additive perturbations. We consider open, bounded, and connected sets $\Omega \in \mathbb{R}^n$, $n \in \mathbb{N}\backslash\{1\}$, that satisfy the interior corkscrew condition and prove stability of square root domains of the operator $L_{a,\Omega,\Gamma}$ with respect to additive potential perturbations $V \in L^p(\Omega) + L^\infty(\Omega)$, $p > n/2$.

Special emphasis is put on the case of uniformly elliptic systems with mixed boundary conditions.

1. Introduction

The aim of this note is to provide applications of a recently developed abstract approach to the stability of square root domains of non-self-adjoint operators with respect to additive perturbations to elliptic partial differential operators with mixed boundary conditions on a class of open, bounded, connected sets $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}\backslash\{1\}$, that satisfy the corkscrew condition (and hence go beyond bounded Lipschitz domains).

More precisely, if T_0 is an appropriate non-self-adjoint m-accretive operator in a separable, complex Hilbert space $\mathcal{H}$, we developed an abstract approach in [18] to determine conditions under which non-self-adjoint additive perturbations W of T_0 yield the stability of square root domains in the form

$$\operatorname{dom}\big((T_0 + W)^{1/2}\big) = \operatorname{dom}\big(T_0^{1/2}\big). \tag{1.1}$$

In fact, driven by applications to PDEs, we were particularly interested in the following variant of this stability problem for square root domains with respect to additive perturbations: if T_0 is an appropriate non-self-adjoint operator for which it is known that Kato's square root problem in the following abstract form, that is,

$$\operatorname{dom}\big(T_0^{1/2}\big) = \operatorname{dom}\big((T_0^*)^{1/2}\big) \tag{1.2}$$

is valid, for which non-self-adjoint additive perturbations W of T_0 can one conclude that also

$$\operatorname{dom}\big((T_0+W)^{1/2}\big) = \operatorname{dom}\big(T_0^{1/2}\big) = \operatorname{dom}\big((T_0^*)^{1/2}\big) = \operatorname{dom}\big(((T_0+W)^*)^{1/2}\big) \tag{1.3}$$

Date: June 11, 2020.

2010 *Mathematics Subject Classification.* Primary 35J10, 35J25, 47A07, 47A55; Secondary 47B44, 47D07, 47F05.

Key words and phrases. Square root domains, Kato problem, additive perturbations, systems, uniformly elliptic second-order differential operators.

S.H. was supported by NSF grant DMS-1101244.

R.N. gratefully acknowledges support from an AMS–Simons Travel Grant.

Published in *J. Differential Equations* **258**, 1749–1764 (2015).

holds?

Without going into details at this point we note that $T_0 + W$ will be viewed as a form sum of T_0 and W.

Formally speaking, the role of the operator T_0 in $\mathcal{H}$ in this note will be played by $L_{a,\Omega,\Gamma}$, an m-sectorial realization of the uniformly elliptic differential expression in divergence form, $-\nabla \cdot a\nabla$, in $L^2(\Omega)$, with $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}\backslash\{1\}$ an open, bounded, connected set that satisfies the corkscrew condition, and the coefficients $a_{j,k}$, $1 \leqslant j,k \leqslant n$, assumed to be essentially bounded. Moreover, $L_{a,\Omega,\Gamma}$ is constructed in such a manner via quadratic forms so that it satisfies a Dirichlet boundary condition along the closed (possibly empty) subset $\Gamma \subseteq \partial\Omega$ and a Neumann boundary condition on the remainder of the boundary, $\partial\Omega\backslash\Gamma$. The additive perturbation W of T_0 then is given by a potential term V, that is, by an operator of multiplication in $L^2(\Omega)$ by an element

$$V \in L^p(\Omega) + L^\infty(\Omega) \text{ for some } p > n/2. \tag{1.4}$$

(For simplicity we will not consider the one-dimensional case $n = 1$ in this note as that has been separately discussed in [19].)

In fact, we will go a step further and consider uniformly elliptic systems in $L^2(\Omega)^N$, $N \in \mathbb{N}$, where T_0 in $\mathcal{H}$ is represented by $\mathbf{L}_{a,\Omega,\mathbb{G}}$ in $L^2(\Omega)^N$, the m-sectorial realization of the $N \times N$ matrix-valued differential expression $\mathbf{L}$ which acts as

$$\mathbf{L}u = -\left(\sum_{j,k=1}^{n} \partial_j \left(\sum_{\beta=1}^{N} a_{j,k}^{\alpha,\beta} \partial_k u_\beta\right)\right)_{1\leqslant\alpha\leqslant N}, \quad u = (u_1, \dots, u_N), \tag{1.5}$$

with $a_{j,k}^{\alpha,\beta} \in L^\infty(\Omega)$, $1 \leqslant j,k \leqslant n$, $1 \leqslant \alpha,\beta \leqslant N$. Here $\mathbb{G}$ represents the collection $\mathbb{G} = (\Gamma_1, \Gamma_2, \dots, \Gamma_N)$, with $\Gamma_\alpha \subseteq \partial\Omega$ a closed (possibly empty) subset of $\partial\Omega$, and intuitively, $\mathbf{L}$ acts on vectors $u = (u_1, u_2, \dots, u_N)$, where each component u_α formally satisfies a Dirichlet boundary condition along Γ_α and a Neumann condition along the remainder of the boundary, $\partial\Omega\backslash\Gamma_\alpha$, $1 \leqslant \alpha \leqslant N$. The additive perturbation W of T_0 then corresponds to an $N \times N$ matrix-valued operator of multiplication in $L^2(\Omega)^N$ of the form

$$(\mathbf{V}f)_\alpha = \sum_{\beta=1}^{N} V_{\alpha,\beta} f_\beta, \quad 1 \leqslant \alpha \leqslant N, \ f \in \operatorname{dom}(\mathbf{V}) = \big\{f \in L^2(\Omega)^N \,\big|\, \mathbf{V}f \in L^2(\Omega)^N\big\}. \tag{1.6}$$

with

$$V_{\alpha,\beta} \in L^p(\Omega) + L^\infty(\Omega) \text{ for some } p > n/2,\ 1 \leqslant \alpha,\beta \leqslant N. \tag{1.7}$$

The considerable amount of literature on Kato's square root problem in the concrete case where T_0 represents a uniformly elliptic differential operator in divergence form $-\nabla \cdot a\nabla$ in $L^2(\Omega)$ with various boundary conditions on $\partial\Omega$, has been reviewed in great detail in [18]. Thus, in this note we now confine ourselves to refer, for instance, in addition to [2], [3], [4], [5], [6], [7], [8], [12], [9], [10], [11], [13], [14], [15], [21], [23], [24], [27], and the references cited in these sources.

The starting point for this note was a recent paper by Egert, Haller-Dintelmann, and Tolksdorf [14] (cf. Theorem 2.5), which permits us to go beyond the class of strongly Lipschitz domains considered in [18] and now consider open, bounded, and connected sets $\Omega \in \mathbb{R}^n$ that satisfy the interior corkscrew condition. In Section 2 we first consider uniformly elliptic partial differential operators with mixed boundary conditions on Ω, closely following [14], and subsequently study the quadratic forms associated with $L_{a,\Omega,\Gamma}$ and V. We then prove stability of square root domains of the operator $L_{a,\Omega,\Gamma}$ with respect to additive perturbations $V \in L^p(\Omega) + L^\infty(\Omega)$, $p > n/2$, for this more general class of domains Ω. The extension of these results to elliptic systems governed by (1.5) and perturbed by the matrix-valued potentials in (1.6) then is the content of Section 3.

Finally, we briefly summarize some of the notation used in this paper: Let $\mathcal{H}$ be a separable, complex Hilbert space with scalar product (linear in the second argument) and norm denoted by $(\cdot,\cdot)_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$, respectively. Next, if T is a linear operator mapping (a subspace of) a Hilbert space into another, then $\operatorname{dom}(T)$ denotes the domain of T. The closure of a closable operator S is denoted by $\overline{S}$. The form sum of two (appropriate) operators T_0 and W is abbreviated by $T_0 +_{\mathfrak{q}} W$.

The Banach space of bounded linear operators on a separable complex Hilbert space $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. The notation $\mathcal{X}_1 \hookrightarrow \mathcal{X}_2$ is used for the continuous embedding of the Banach space $\mathcal{X}_1$ into the Banach space $\mathcal{X}_2$.

If $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ is a bounded set, then $\operatorname{diam}(\Omega) = \sup_{x,y\in\Omega} |x-y|$ denotes the diameter of Ω. We use $m_{\ell,n}$ to denote the ℓ-dimensional Hausdorff measure on $\mathbb{R}^n$ (and hence $m_{n,n}$, also denoted by $|\cdot|$, represents the n-dimensional Lebesgue measure on $\mathbb{R}^n$). If $x \in \mathbb{R}^n$ and $r > 0$, then $B(x,r)$ denotes the open ball of radius r centered at x. In addition, I_n denotes the $n \times n$ identity matrix in $\mathbb{C}^n$, and the set of $k \times \ell$ matrices with complex-valued entries is denoted by $\mathbb{C}^{k\times\ell}$. Finally, we abbreviate $L^p(\Omega; d^n x) := L^p(\Omega)$ and $L^p(\Omega, \mathbb{C}^N; d^n x) := L^p(\Omega)^N$, $N \in \mathbb{N}$.

2. Elliptic Partial Differential Operators with Mixed Boundary Conditions

In this section we discuss stability of square root domains for uniformly elliptic partial differential operators $L_{a,\Omega,\Gamma} = -\nabla \cdot a \nabla$ in $L^2(\Omega)$, with mixed boundary conditions on $\partial\Omega$, with respect to additive perturbations in [18], by employing a recent result due to Egert, Haller-Dintelmann, and Tolksdorf [14] (recorded in Theorem 2.5 below). This permits us to go beyond the class of strongly Lipschitz domains considered in [18] and now consider open, bounded, and connected sets $\Omega \in \mathbb{R}^n$ that satisfy the interior corkscrew condition. We then prove stability of square root domains of the operator $L_{a,\Omega,\Gamma}$ with respect to additive potential perturbations $V \in L^p(\Omega) + L^\infty(\Omega)$, $p > n/2$, for this more general class of domains Ω.

We start with the following definitions:

Definition 2.1. *Let $n \in \mathbb{N}\backslash\{1\}$.*
(i) A nonempty, bounded, open, and connected set $\Omega \subset \mathbb{R}^n$ is said to satisfy the interior corkscrew condition if there exists a constant $\kappa \in (0,1)$ with the property that for each $x \in \overline{\Omega}$ and each $r \in (0, \operatorname{diam}(\Omega))$, there exists a point $y \in \overline{B(x,r)}$ such that $\overline{B(y,\kappa r)} \subseteq \Omega$.
(ii) Let Ω be a nonempty, proper, open subset of $\mathbb{R}^n$. One calls Ω a Lipschitz domain if for every $x_0 \in \partial\Omega$ there exist $r > 0$, a rigid transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, and a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ with the property that

$$T\big(\Omega \cap B(x_0,r)\big) = T\big(B(x_0,r)\big) \cap \big\{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,\big|\, x_n > \varphi(x')\big\}. \tag{2.1}$$

We recall our convention that $m_{\ell,n}$ denotes the ℓ-dimensional Hausdorff measure (for the basics on Hausdorff measure, see, e.g., [17, Ch. 2], [25, Ch. 2]) and hence $m_{n,n} = |\cdot|$ represents n-dimensional Lebesgue measure on $\mathbb{R}^n$.

The following proposition records some basic results in connection with the interior corkscrew condition.

Proposition 2.2. *Let $n \in \mathbb{N}\backslash\{1\}$ and suppose that $\Omega \subset \mathbb{R}^n$ is nonempty, bounded, open, and connected. Then the following items (i) and (ii) hold:*
(i) If Ω satisfies the interior corkscrew condition with constant $\kappa \in (0,1)$, then

$$\kappa^n r^n \leqslant |\Omega \cap B(x,r)| \leqslant r^n, \quad x \in \Omega,\ 0 < r < \operatorname{diam}(\Omega). \tag{2.2}$$

(ii) If Ω is a bounded Lipschitz domain, then Ω satisfies the interior corkscrew condition.

Definition 2.3. *Suppose $n \in \mathbb{N}$ is fixed and $0 < \ell \leqslant n$. A non-empty Borel set $M \subseteq \mathbb{R}^n$ is an ℓ-set if there exist constants $c_j = c_j(M) > 0$, $j = 1,2$, for which*

$$c_1 r^\ell \leqslant m_{\ell,n}(M \cap B(x,r)) \leqslant c_2 r^\ell, \quad x \in M,\ 0 < r \leqslant 1. \tag{2.3}$$

One notes that $\overline{M}$ is an ℓ-set if M is an ℓ-set and $m_{\ell,n}\big(\overline{M}\backslash M\big) = 0$ in this case.

Hypothesis 2.4. *Let $n \in \mathbb{N}\backslash\{1\}$.*
(i) Assume that $\Omega \subset \mathbb{R}^n$ is a nonempty, bounded, open, and connected set satisfying the interior corkscrew condition in Definition 2.1 (i).
(ii) Suppose $\Gamma \subset \partial\Omega$ is closed and for every $x \in \overline{\partial\Omega\backslash\Gamma}$, there exists an open neighborhood $U_x \subset \mathbb{R}^n$ and a bi-Lipschitz map $\Phi_x : U_x \to (-1,1)^n$ such that

$$\Phi_x(x) = 0, \tag{2.4}$$

$$\Phi_x(\Omega \cap U_x) = (-1,1)^{n-1} \times (-1,0), \tag{2.5}$$

$$\Phi_x(\partial\Omega \cap U_x) = (-1,1)^{n-1} \times \{0\}. \tag{2.6}$$

(iii) Suppose $\Gamma = \emptyset$ or Γ is an $(n-1)$-set.
(iv) Assume that $a : \Omega \to \mathbb{C}^{n\times n}$ is a Lebesgue measurable, matrix-valued function which is essentially bounded and uniformly elliptic, that is, there exist constants $0 < a_1 \leqslant a_2 < \infty$ such that for a.e. $x \in \Omega$,

$$a_1 \|\xi\|^2_{\mathbb{C}^n} \leqslant \operatorname{Re}\big[(\xi, a(x)\xi)_{\mathbb{C}^n}\big] \ \text{ and } \ |(\zeta, a(x)\xi)_{\mathbb{C}^n}| \leqslant a_2 \|\xi\|_{\mathbb{C}^n} \|\zeta\|_{\mathbb{C}^n}, \quad \xi, \zeta \in \mathbb{C}^n. \tag{2.7}$$

(v) With $C^\infty_\Gamma(\Omega)$ defined by

$$C^\infty_\Gamma(\Omega) := \{u|_\Omega \,|\, u \in C^\infty(\mathbb{R}^n),\ \operatorname{dist}(\operatorname{supp}(u), \Gamma) > 0\}, \tag{2.8}$$

denote by $W^{1,2}_\Gamma(\Omega)$ the closure of $C^\infty_\Gamma(\Omega)$ in $W^{1,2}(\Omega)$, that is,

$$W^{1,2}_\Gamma(\Omega) = \overline{C^\infty_\Gamma(\Omega)}^{W^{1,2}(\Omega)}, \tag{2.9}$$

and introduce the densely defined, accretive, and closed sesquilinear form in $L^2(\Omega)$,

$$\mathfrak{q}_{a,\Omega,\Gamma}(f,g) = \int_\Omega d^n x \, ((\nabla f)(x), a(x)(\nabla g)(x))_{\mathbb{C}^n}, \quad f, g \in \operatorname{dom}(\mathfrak{q}_{a,\Omega,\Gamma}) := W^{1,2}_\Gamma(\Omega). \tag{2.10}$$

We denote by $L_{a,\Omega,\Gamma}$ the m-sectorial operator in $L^2(\Omega)$ uniquely associated to $\mathfrak{q}_{a,\Omega,\Gamma}$.
(vi) Suppose that $V : \Omega \to \mathbb{C}$ is (Lebesgue) measurable and factored according to

$$V(x) = u(x)v(x), \quad v(x) = |V(x)|^{1/2}, \quad u(x) = e^{i\arg(V(x))}v(x) \ \text{ for a.e. } x \in \Omega, \tag{2.11}$$

such that

$$W^{1,2}_\Gamma(\Omega) \subseteq \operatorname{dom}(v). \tag{2.12}$$

In the special case where $a(x) = I_n$ for a.e. $x \in \Omega$, with I_n the $n \times n$ identity matrix in $\mathbb{C}^n$, we simplify notation and write

$$L_{I_n,\Omega,\Gamma} = -\Delta_{\Omega,\Gamma}. \tag{2.13}$$

Note that $-\Delta_{\Omega,\Gamma}$ is self-adjoint and non-negative.

For an example of a bounded, open, and connected set that satisfies the conditions (i)–(iii) of Hypothesis 2.4 and is not Lipschitz, see [14, Figure 1]. One notes that Hypothesis 2.4 (i) permits *inward-pointing* cusps.

Formally speaking, the operator $L_{a,\Omega,\Gamma}$ is of uniform elliptic divergence form $L_{a,\Omega,\Gamma} = -\nabla \cdot a \nabla$, satisfying a Dirichlet boundary condition along Γ and a Neumann (or, natural) boundary condition on the remainder of the boundary, $\partial\Omega \backslash \Gamma$.

The quadratic form $\mathfrak{q}_V$ in $L^2(\Omega)$, uniquely associated with V, is defined by

$$\mathfrak{q}_V(f,g) = \big(vf, e^{i\arg(V)}vg\big)_{L^2(\Omega)}, \quad f,g \in \operatorname{dom}(\mathfrak{q}_V) = \operatorname{dom}(v). \tag{2.14}$$

Under appropriate assumptions on V (see Hypotheses 2.6 and 2.7 below), the form sum of $\mathfrak{q}_{a,\Omega,\Gamma}$ and $\mathfrak{q}_V$ will define a sectorial form on $W^{1,2}_\Gamma(\Omega)$ and the operator uniquely associated to $\mathfrak{q}_{a,\Omega,\Gamma} + \mathfrak{q}_V$ will be denoted by $L_{a,\Omega,\Gamma} +_{\mathfrak{q}} V$ (see also the paragraph following [18, eq. (A.42)]).

The principal aim of this section is to prove stability of square root domains in the form

$$\operatorname{dom}\big((L_{a,\Omega,\Gamma} +_{\mathfrak{q}} V)^{1/2}\big) = \operatorname{dom}\big(L^{1/2}_{a,\Omega,\Gamma}\big) = W^{1,2}_\Gamma(\Omega), \tag{2.15}$$

under appropriate (integrability) assumptions on V, thereby extending the recent results on stability of square root domains obtained in [18] to the setting of certain classes of non-Lipschitz domains with mixed boundary conditions as discussed in [14]. As a basic input, we rely on the following result which is Theorem 4.1 in [14].

Theorem 2.5 (Egert–Haller-Dintelmann–Tolksdorf [14])**.** *Assume items* (i)–(v) *of Hypothesis* 2.4*. Then*

$$\operatorname{dom}\big(L^{1/2}_{a,\Omega,\Gamma}\big) = \operatorname{dom}\big((L^*_{a,\Omega,\Gamma})^{1/2}\big) = W^{1,2}_\Gamma(\Omega). \tag{2.16}$$

Next, we introduce various hypotheses corresponding to the potential coefficient V.

Hypothesis 2.6. *Let $n \in \mathbb{N}\backslash\{1\}$, assume that $\Omega \subseteq \mathbb{R}^n$ is nonempty and open, and let $V \in L^p(\Omega) + L^\infty(\Omega)$ for some $p > n/2$.*

In addition, we also discuss the critical L^p-index $p = n/2$ for V for $n \geqslant 3$:

Hypothesis 2.7. *Let $n \in \mathbb{N}\backslash\{1,2\}$, assume that $\Omega \subseteq \mathbb{R}^n$ is nonempty and open, and let $V \in L^{n/2}(\Omega) + L^\infty(\Omega)$.*

Here $V \in L^q(\Omega) + L^\infty(\Omega)$ means as usual that V permits a decomposition $V = V_q + V_\infty$ with $V_q \in L^q(\Omega)$ for some $q \geqslant 1$ and $V_\infty \in L^\infty(\Omega)$.

Theorem 2.8. *Assume Hypotheses* 2.4 *and* 2.6*. Then the following items* (i) *and* (ii) *hold:*

(i) *V is infinitesimally form bounded with respect to $-\Delta_{\Omega,\Gamma}$, and there exist constants $M > 0$ and $\varepsilon_0 > 0$ such that*

$$\begin{aligned} \big\||V|^{1/2}f\big\|^2_{L^2(\Omega)} \leqslant \varepsilon\big\|(-\Delta_{\Omega,\Gamma})^{1/2}f\big\|^2_{L^2(\Omega)} + M\varepsilon^{-n/(2p-n)}\|f\|^2_{L^2(\Omega)}, \\ f \in W^{1,2}_\Gamma(\Omega),\ 0 < \varepsilon < \varepsilon_0. \end{aligned} \tag{2.17}$$

(ii) V *is infinitesimally form bounded with respect to* $L_{a,\Omega,\Gamma}$ *and*

$$\big\||V|^{1/2}f\big\|_{L^2(\Omega)}^2 \leqslant \varepsilon \mathrm{Re}[\mathfrak{q}_{a,\Omega,\Gamma}(f,f)] + M a_1^{-n/(2p-n)}\varepsilon^{-n/(2p-n)}\|f\|_{L^2(\Omega)}^2, \quad f \in W_\Gamma^{1,2}(\Omega),\ 0<\varepsilon<a_1^{-1}\varepsilon_0. \tag{2.18}$$

The form sum $L_{a,\Omega,\Gamma} +_{\mathfrak{q}} V$ *is an m-sectorial operator which satisfies*

$$\begin{split} &\mathrm{dom}\,\big((L_{a,\Omega,\Gamma} +_{\mathfrak{q}} V)^{1/2}\big) = \mathrm{dom}\,\big(((L_{a,\Omega,\Gamma} +_{\mathfrak{q}} V)^*)^{1/2}\big) \\ &\quad = \mathrm{dom}\,\big(L_{a,\Omega,\Gamma}^{1/2}\big) = \mathrm{dom}\,\big((L_{a,\Omega,\Gamma}^*)^{1/2}\big) = W_\Gamma^{1,2}(\Omega). \end{split} \tag{2.19}$$

Proof. For notational simplicity, and without loss of generality, we put the L^∞-part V_∞ of V equal to zero for the remainder of this proof. Under Hypothesis 2.4, there exists an extension operator $\mathcal{E}$ satisfying

$$(\mathcal{E}f)(x) = f(x) \text{ for a.e. } x\in\Omega,\ f\in L^2(\Omega), \tag{2.20}$$

with

$$\begin{split} &\mathcal{E} : W_\Gamma^{1,2}(\Omega) \to W^{1,2}(\mathbb{R}^n), \\ &\|\mathcal{E}f\|_{W^{1,2}(\mathbb{R}^n)}^2 \leqslant C_1\|f\|_{W^{1,2}(\Omega)}^2, \quad f\in W^{1,2}(\Omega), \end{split} \tag{2.21}$$

and

$$\begin{split} &\mathcal{E} : L^2(\Omega) \to L^2(\mathbb{R}^n), \\ &\|\mathcal{E}f\|_{L^2(\mathbb{R}^n)}^2 \leqslant C_2\|f\|_{L^2(\Omega)}^2, \quad f\in L^2(\Omega), \end{split} \tag{2.22}$$

for some constants $C_j>0$, $j=1,2$ (cf., e.g., [4, Lemma 3.3], [16, Lemma 3.4]).

Let $V_{\rm ext}$ and $v_{\rm ext}$ denote the extensions of V and v, respectively, to all of $\mathbb{R}^n$ defined by setting $V_{\rm ext}$ and $v_{\rm ext}$ identical to zero on $\mathbb{R}^n\backslash\Omega$. Evidently, $V_{\rm ext}\in L^p(\mathbb{R}^n)$, so there exists a constant $M>0$ for which (cf., e.g., [18, Lemma 3.7])

$$\big\|v_{\rm ext}f\big\|_{L^2(\mathbb{R}^n)}^2 \leqslant \varepsilon\big\|(-\Delta)^{1/2}f\big\|_{L^2(\mathbb{R}^n)}^2 + M\varepsilon^{-n/(2p-n)}\|f\|_{L^2(\mathbb{R}^n)}^2, \quad f\in W^{1,2}(\mathbb{R}^n),\ \varepsilon>0. \tag{2.23}$$

Consequently, using (2.20)–(2.23), one estimates

$$\begin{split} \|vf\|_{L^2(\Omega)}^2 &= \|v_{\rm ext}\mathcal{E}f\|_{L^2(\mathbb{R}^n)}^2 \\ &\leqslant \varepsilon_1\big\|(-\Delta)^{1/2}\mathcal{E}f\big\|_{L^2(\mathbb{R}^n)}^2 + M\varepsilon_1^{-n/(2p-n)}\|\mathcal{E}f\|_{L^2(\mathbb{R}^n)}^2 \\ &= \varepsilon_1\|\nabla\mathcal{E}f\|_{L^2(\mathbb{R}^n)^n}^2 + M\varepsilon_1^{-n/(2p-n)}\|\mathcal{E}f\|_{L^2(\mathbb{R}^n)}^2 \\ &\leqslant \varepsilon_1 C_1\|\nabla f\|_{L^2(\Omega)^n}^2 + \big(\varepsilon_1 C_1 + C_0 M\epsilon^{-n/(2p-n)}\big)\|f\|_{L^2(\Omega)}^2 \\ &\leqslant \varepsilon_1 C_1\|\nabla f\|_{L^2(\Omega)^n}^2 + \big(C_1 + C_0 M\big)\varepsilon_1^{-n/(2p-n)}\|f\|_{L^2(\Omega)}^2, \\ &\hspace{4cm} f\in W_\Gamma^{1,2}(\Omega),\ 0<\varepsilon_1<1. \end{split} \tag{2.24}$$

To obtain the first term in the second equality above, we applied the 2nd representation theorem (cf., e.g., [22, VI.2.23]) to the non-negative, self-adjoint operator $-\Delta$ on $W^{2,2}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$. The form bound in (2.17) now follows by choosing $\varepsilon=\varepsilon_1 C_1$ throughout (2.24) and noting that

$$\|\nabla f\|_{L^2(\Omega)^n}^2 = \big\|(-\Delta_{\Omega,\Gamma})^{1/2}f\big\|_{L^2(\Omega)}^2, \quad f\in W_\Gamma^{1,2}(\Omega), \tag{2.25}$$

by another application of the 2nd representation theorem (see (2.10) with $a(\,\cdot\,) = I_n$), proving item *(i)*.

In view of (2.16) and (2.17), one notes that the hypotheses of [18, Theorem 3.6] are met. The statements in item (*ii*) thus follow from a direct application of [18, Theorem 3.6]. □

Remark 2.9. The proof of Theorem 2.8 follows the proof of [18, Theorem 3.12] essentially verbatim with only one notable exception: In [18, Theorem 3.12], Ω is assumed to be a *strongly Lipschitz domain* and hence the Stein extension theorem (cf., e.g., [1, Theorem 5.24] or [26, Theorem 5 in §VI.3.1]) is applied to obtain a total extension operator. In the present case, under the weaker assumptions on Ω, appealing to the Stein extension theorem is *not* permitted; so instead, we apply [4, Lemma 3.3], [16, Lemma 3.4] to obtain the extension operator in (2.20)–(2.22) (cf. [16, Remark 3.5]).

Next, we discuss infinitesimal form boundedness for potential coefficients in the critical exponent case in dimensions $n \geqslant 3$.

Theorem 2.10. *Assume Hypotheses* 2.4 *and* 2.7. *Then V is infinitesimally form bounded with respect to $-\Delta_{\Omega,\Gamma}$,*

$$\big\||V|^{1/2}f\big\|_{L^2(\Omega)}^2 \leqslant \varepsilon\big\|(-\Delta_{\Omega,\Gamma})^{1/2}f\big\|_{L^2(\Omega)}^2 + \eta(\varepsilon)\|f\|_{L^2(\Omega)}^2, \quad f \in W_\Gamma^{1,2}(\Omega),\ \varepsilon > 0. \tag{2.26}$$

As a result, V is infinitesimally form bounded with respect to $L_{a,\Omega,\Gamma}$,

$$\big\||V|^{1/2}f\big\|_{L^2(\Omega)}^2 \leqslant \varepsilon\mathrm{Re}[\mathfrak{q}_{a,\Omega,\Gamma}(f,f)] + \widetilde{\eta}(\varepsilon)\|f\|_{L^2(\Omega)}^2, \quad f \in W_\Gamma^{1,2}(\Omega),\ \varepsilon > 0. \tag{2.27}$$

Here η and $\widetilde{\eta}$ are non-negative functions defined on $(0,\infty)$, generally depending on Ω, n, and Γ.

Proof. Again, for simplicity, we put the L^∞-part V_∞ of V equal to zero. The proof is a straightforward modification of the proof of the corresponding result for Lipschitz domains given in [18, Theorem 3.14 (*iii*)], and we present the modified argument here for completeness. By Sobolev embedding (cf., e.g., [4, Remark 3.4 (*ii*)]),

$$W_\Gamma^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega), \quad 2^* = 2n/(n-2), \tag{2.28}$$

where "$\hookrightarrow$" abbreviates continuous (and dense) embedding, and hence there exists a constant $c > 0$ such that

$$\|f\|_{L^{2^*}(\Omega)}^2 \leqslant c\big(\|\nabla f\|_{L^2(\Omega)^n}^2 + \|f\|_{L^2(\Omega)}^2\big), \quad f \in W_\Gamma^{1,2}(\Omega). \tag{2.29}$$

Using Hölder's inequality, (2.29) implies

$$\begin{aligned}(f,|W|f)_{L^2(\Omega)} \leqslant c\|W\|_{L^{n/2}(\Omega)}\big(\|\nabla f\|_{L^2(\Omega)^n}^2 + \|f\|_{L^2(\Omega)}^2\big),&\\ f \in W_\Gamma^{1,2}(\Omega),\ W \in L^{n/2}(\Omega).&\end{aligned} \tag{2.30}$$

Next, let $\varepsilon > 0$ be given. Since $V \in L^{n/2}(\Omega)$, there exist functions $V_{n/2,\varepsilon} \in L^{n/2}(\Omega)$ and $V_{\infty,\varepsilon} \in L^\infty(\Omega)$ with

$$\|V_{n/2,\varepsilon}\|_{L^{n/2}(\Omega)} \leqslant \varepsilon/c, \quad V(x) = V_{n/2,\varepsilon}(x) + V_{\infty,\varepsilon}(x) \text{ for a.e. } x \in \Omega. \tag{2.31}$$

Applying (2.30) with $W = V_{n/2,\varepsilon}$, one estimates

$$\begin{aligned}\|vf\|_{L^2(\Omega)} &= (f,|V|f)_{L^2(\Omega)} \leqslant (f,[|V_{n/2,\varepsilon}| + \|V_{\infty,\varepsilon}\|_{L^\infty(\Omega)}]f)_{L^2(\Omega)}\\ &\leqslant \varepsilon\|\nabla f\|_{L^2(\Omega)^n}^2 + \eta(\varepsilon)\|f\|_{L^2(\Omega)}^2, \quad f \in W_\Gamma^{1,2}(\Omega),\end{aligned} \tag{2.32}$$

with

$$\eta(\varepsilon) := \varepsilon + \|V_{\infty,\varepsilon}\|_{L^\infty(\Omega)}. \tag{2.33}$$

Noting the fact that

$$\|\nabla f\|_{L^2(\Omega)^n}^2 = \big\|(-\Delta_{\Omega,\Gamma})^{1/2} f\big\|_{L^2(\Omega)}^2, \quad f \in W_\Gamma^{1,2}(\Omega), \tag{2.34}$$

by the 2nd representation theorem (cf., e.g., [22, Theorem VI.2.23]), (2.32) then also yields

$$\|vf\|_{L^2(\Omega)}^2 \leqslant \varepsilon\|(-\Delta_{\Omega,\Gamma})^{1/2} f\|_{L^2(\Omega)^n}^2 + \eta(\varepsilon)\|f\|_{L^2(\Omega)}^2, \quad f \in W_\Gamma^{1,2}(\Omega). \tag{2.35}$$

Since $\varepsilon > 0$ was arbitrary and $v = |V|^{1/2}$, (2.26) follows.

To prove (2.27), one notes that the uniform ellipticity condition on a implies

$$\|\nabla f\|_{L^2(\Omega)^n}^2 \leqslant a_1^{-1} \mathrm{Re}[\mathfrak{q}_{a,\Omega,\Gamma}(f,f)], \quad f \in W_\Gamma^{1,2}(\Omega). \tag{2.36}$$

Taking (2.36) together with (2.26) and (2.34), one infers that

$$\|vf\|_{L^2(\Omega)}^2 \leqslant a_1^{-1}\varepsilon_1 \mathrm{Re}[\mathfrak{q}_{a,\Omega,\Gamma}(f,f)] + \eta(\varepsilon_1)\|f\|_{L^2(\Omega)}^2, \quad f \in W_\Gamma^{1,2}(\Omega),\ \varepsilon_1 > 0. \tag{2.37}$$

The form bound in (2.27) follows by taking $\varepsilon_1 = a_1\varepsilon$, $\varepsilon > 0$, in (2.37). $\square$

3. The Case of Matrix-Valued Divergence Form Elliptic Partial Differential Operators

In this section we consider uniformly elliptic partial differential operators in divergence form in the vector-valued context, that is, we will focus on $N \times N$ matrix-valued differential expressions $\mathbf{L}$ which act as

$$\mathbf{L}u = -\left(\sum_{j,k=1}^{n} \partial_j \left(\sum_{\beta=1}^{N} a_{j,k}^{\alpha,\beta} \partial_k u_\beta\right)\right)_{1 \leqslant \alpha \leqslant N}, \quad u = (u_1, \dots, u_N), \tag{3.1}$$

and prove our principal result concerning stability of square root domains with respect to additive perturbations.

To set the stage, we introduce the following set of hypotheses.

Hypothesis 3.1. *Fix* $n \in \mathbb{N}\backslash\{1\}$, $N \in \mathbb{N}$.
(i) *Assume that* $\Omega \subset \mathbb{R}^n$ *is a non-empty, bounded, open, and connected set satisfying the interior corkscrew condition in Definition* 2.1 (i).
(ii) *For each* $1 \leqslant \alpha \leqslant N$, *suppose* $\Gamma_\alpha \subseteq \partial\Omega$ *is a closed subset of* $\partial\Omega$ *which is either empty or an* $(n-1)$*-set, and let* $\mathbb{G} = (\Gamma_1, \Gamma_2, \dots, \Gamma_N)$.
(iii) *Around every point* $x \in \overline{\partial\Omega\backslash \cap_{\alpha=1}^N \Gamma_\alpha}$, *suppose there exists an open neighborhood* $U_x \subset \mathbb{R}^n$ *and a bi-Lipschitz map* $\Phi_x : U_x \to (-1,1)^n$ *such that*

$$\Phi_x(x) = 0, \tag{3.2}$$

$$\Phi_x(\Omega \cap U_x) = (-1,1)^{n-1} \times (-1,0), \tag{3.3}$$

$$\Phi_x(\partial\Omega \cap U_x) = (-1,1)^{n-1} \times \{0\}. \tag{3.4}$$

(iv) *Define the set*

$$\mathcal{W}_{\mathbb{G}}(\Omega) = \prod_{\alpha=1}^{N} W_{\Gamma_\alpha}^{1,2}(\Omega), \tag{3.5}$$

where $W_{\Gamma_\alpha}^{1,2}(\Omega)$ *is defined as in* (2.9) *for each* $1 \leqslant \alpha \leqslant N$, *suppose that*

$$a_{j,k}^{\alpha,\beta} \in L^\infty(\Omega),\ 1 \leqslant j,k \leqslant n,\ 1 \leqslant \alpha,\beta \leqslant N, \tag{3.6}$$

and assume that the sesquilinear form in $L^2(\Omega)^N$,

$$\begin{aligned} \mathfrak{L}_{a,\Omega,\mathbb{G}}(f,g) = \sum_{j,k=1}^{n} \sum_{\alpha,\beta=1}^{N} \int_\Omega d^n x\, \overline{(\partial_j f_\alpha)(x)} a_{j,k}^{\alpha,\beta}(x)(\partial_k g_\beta)(x), \\ f,g \in \operatorname{dom}(\mathfrak{L}_{a,\Omega,\mathbb{G}}) := \mathcal{W}_{\mathbb{G}}(\Omega), \end{aligned} \tag{3.7}$$

satisfies a uniform ellipticity condition of the form, for some $\lambda > 0$,

$$\operatorname{Re}[\mathfrak{L}_{a,\Omega,\mathbb{G}}(f,f)] \geqslant \lambda \sum_{\alpha=1}^{N} \|\nabla f_\alpha\|^2_{L^2(\Omega)^n}, \quad f = (f_\alpha)_{\alpha=1}^N \in \mathcal{W}_{\mathbb{G}}(\Omega). \tag{3.8}$$

We denote by $\mathbf{L}_{a,\Omega,\mathbb{G}}$ *the m-sectorial operator in* $L^2(\Omega)^N$ *uniquely associated to the sesquilinear form* $\mathfrak{L}_{a,\Omega,\mathbb{G}}$.

Intuitively, $\mathfrak{L}_{a,\Omega,\mathbb{G}}$ acts on vectors $u = (u_1, u_2, \dots, u_N)$, where each component u_α formally satisfies a Dirichlet boundary condition along Γ_α and a Neumann condition along the remainder of the boundary, $\partial\Omega\backslash\Gamma_\alpha$, $1 \leqslant \alpha \leqslant N$ (cf., e.g., [14, Corollary 4.2]).

Hypothesis 3.2. *Let* $n \in \mathbb{N}\backslash\{1\}$, $N \in \mathbb{N}$, *and assume that* $\Omega \subset \mathbb{R}^n$ *is nonempty and open. Suppose, in addition, that* $p > n/2$ *and that* $V_{\alpha,\beta} \in L^p(\Omega) + L^\infty(\Omega)$ *for each* $1 \leqslant \alpha, \beta \leqslant N$.

Assuming Hypotheses 3.1 and 3.2, consider the operator of multiplication by the $N \times N$ matrix-valued function $\mathbf{V} = \{V_{\alpha,\beta}\}_{1\leqslant\alpha,\beta\leqslant N}$ in $L^2(\Omega)^N$ given by

$$(\mathbf{V}f)_\alpha = \sum_{\beta=1}^{N} V_{\alpha,\beta} f_\beta, \quad 1 \leqslant \alpha \leqslant N,\ f \in \operatorname{dom}(\mathbf{V}) = \big\{f \in L^2(\Omega)^N \,\big|\, \mathbf{V}f \in L^2(\Omega)^N\big\}. \tag{3.9}$$

Next, consider the generalized polar decomposition (cf. [20]) for $\mathbf{V}$:

$$\mathbf{V} = |\mathbf{V}^*|^{1/2} U |\mathbf{V}|^{1/2}, \tag{3.10}$$

where U is an appropriate partial isometry. The sesquilinear form corresponding to $\mathbf{V}$ is then given by

$$\begin{aligned} \mathfrak{V}(f,g) = \big(|\mathbf{V}^*|^{1/2} f, U|\mathbf{V}|^{1/2} g\big)_{L^2(\Omega)^N}, \\ f,g \in \operatorname{dom}(\mathfrak{V}) = \operatorname{dom}\big(|\mathbf{V}|^{1/2}\big) = \operatorname{dom}\big(|\mathbf{V}^*|^{1/2}\big). \end{aligned} \tag{3.11}$$

With the $L^p(\Omega)$ assumption on each entry $V_{\alpha,\beta}$, $\mathfrak{V}$ is infinitesimally form bounded with respect to $\mathbf{L}_{a,\Omega,\mathbb{G}}$. In order to prove this, it suffices to consider the case where the L^∞-part of each $V_{\alpha,\beta}$ is zero. In this case, one has the estimate

$$\begin{aligned} |\mathfrak{V}(f,f)|^2 &= \big|\big(|\mathbf{V}^*|^{1/2} f, U|\mathbf{V}|^{1/2} f\big)_{L^2(\Omega)^N}\big|^2 && (3.12) \\ &\leqslant \big|\big(|\mathbf{V}^*|^{1/2} f, |\mathbf{V}|^{1/2} f\big)_{L^2(\Omega)^N}\big|^2 && (3.13) \\ &\leqslant \big\||\mathbf{V}^*|^{1/2} f\big\|^2_{L^2(\Omega)^N} \big\||\mathbf{V}|^{1/2} f\big\|^2_{L^2(\Omega)^N} && (3.14) \\ &= \int_\Omega \big(|\mathbf{V}^*(x)|^{1/2} f(x), |\mathbf{V}^*(x)|^{1/2} f(x)\big)_{\mathbb{C}^N}\, d^n x \\ &\quad \times \int_\Omega \big(|\mathbf{V}(x)|^{1/2} f(x), |\mathbf{V}(x)|^{1/2} f(x)\big)_{\mathbb{C}^N}\, d^n x && (3.15) \end{aligned}$$

$$= \int_\Omega (f(x), |\mathbf{V}^*(x)| f(x))_{\mathbb{C}^N}\, d^n x \int_\Omega (f(x), |\mathbf{V}(x)| f(x))_{\mathbb{C}^N}\, d^n x \tag{3.16}$$

$$\leqslant \int_\Omega \|\mathbf{V}^*(x)\|_2 \|f(x)\|_{\mathbb{C}^N}^2\, d^n x \int_\Omega \|\mathbf{V}(x)\|_2 \|f(x)\|_{\mathbb{C}^N}^2\, d^n x \tag{3.17}$$

$$= \bigg[\int_\Omega \bigg(\sum_{\alpha=1}^N \sum_{\beta=1}^N |V_{\alpha,\beta}(x)|^2 \bigg)^{1/2} \|f(x)\|_{\mathbb{C}^N}^2\, d^n x \bigg]^2 \tag{3.18}$$

$$\leqslant \bigg[\int_\Omega W(x) \|f(x)\|_{\mathbb{C}^N}^2\, d^n x \bigg]^2, \quad f \in \operatorname{dom}\big(|\mathbf{V}|^{1/2}\big), \tag{3.19}$$

where we have set

$$W(x) = \sum_{\alpha=1}^N \sum_{\beta=1}^N |V_{\alpha,\beta}(x)| \text{ for a.e. } x \in \Omega, \tag{3.20}$$

and used $\|\cdot\|_2$ to denote the Hilbert–Schmidt norm of a matrix in $\mathbb{C}^{N\times N}$. The estimate in (3.19) subsequently implies

$$\begin{aligned} |\mathfrak{V}(f,f)| &\leqslant \int_\Omega \big\|W(x)^{1/2} f(x)\|_{\mathbb{C}^N}^2\, d^n x \\ &= \sum_{\alpha=1}^N \int_\Omega \big|W(x)^{1/2} f_\alpha(x)\big|^2\, d^n x \\ &= \sum_{\alpha=1}^N \big\|W^{1/2} f_\alpha\big\|_{L^2(\Omega)}^2, \quad f \in \operatorname{dom}\big(|\mathbf{V}|^{1/2}\big). \end{aligned} \tag{3.21}$$

By hypothesis, one infers that $W \in L^p(\Omega)$. Since $p > n/2$, W is infinitesimally form bounded with respect to $-\Delta_{\Omega,\Gamma_\alpha}$ for each $1 \leqslant \alpha \leqslant N$ (recalling the notational convention $-\Delta_{\Omega,\Gamma} = L_{I_n,\Omega,\Gamma}$ set forth in (2.13)), with a form bound of the following type:

$$\begin{aligned} \big\|W^{1/2} f\big\|_{L^2(\Omega)}^2 \leqslant \varepsilon \big\|(-\Delta_{\Omega,\Gamma_\alpha})^{1/2} f\big\|_{L^2(\Omega)} + M_\alpha \varepsilon^{-n/(2p-n)} \|f\|_{L^2(\Omega)}^2, \\ f \in W^{1,2}_{\Gamma_\alpha}(\Omega),\ 0 < \varepsilon < 1,\ 1 \leqslant \alpha \leqslant N, \end{aligned} \tag{3.22}$$

for appropriate constants $M_\alpha > 0$, $1 \leqslant \alpha \leqslant N$. Setting $M = \max_{1\leqslant\alpha\leqslant N} M_\alpha$ and applying (3.22) to each term of the summation in (3.21), one obtains

$$\begin{aligned} |\mathfrak{V}(f,f)| &\leqslant \varepsilon \sum_{\alpha=1}^N \big\|(-\Delta_{\Omega,\Gamma_\alpha})^{1/2} f_\alpha\big\|_{L^2(\Omega)}^2 + M \sum_{\alpha=1}^N \varepsilon^{-n/(2p-n)} \|f_\alpha\|_{L^2(\Omega)}^2 \\ &= \varepsilon \sum_{\alpha=1}^N \|\nabla f_\alpha\|_{L^2(\Omega)}^2 + M\varepsilon^{-n/(2p-n)} \|f\|_{L^2(\Omega)^N}^2, \\ &\qquad f \in \mathcal{W}_{\mathbb{G}}(\Omega),\ 0 < \varepsilon < 1. \end{aligned} \tag{3.23}$$

Finally, applying the uniform ellipticity condition (3.8) to (3.23), one obtains the form bound,

$$\begin{aligned} |\mathfrak{V}(f,f)| \leqslant \lambda^{-1} \varepsilon \operatorname{Re}[\mathfrak{L}_{a,\Omega,\mathbb{G}}(f,f)] + M\varepsilon^{-n/(2p-n)} \|f\|_{L^2(\Omega)^N}^2, \\ f \in \mathcal{W}_{\mathbb{G}}(\Omega),\ 0 < \varepsilon < 1. \end{aligned} \tag{3.24}$$

By suitably rescaling ε throughout (3.24), one infers that $\mathbf{V}$ is infinitesimally form bounded with respect to $\mathbf{L}_{a,\Omega,\mathbb{G}}$. Infinitesimal form boundedness of $\mathbf{V}$ with respect to $\mathbf{L}_{a,\Omega,\mathbb{G}}$ is summarized in the following result.

Theorem 3.3. *Assume Hypotheses* 3.1 *and* 3.2. *Then* $\mathbf{V}$ *is infinitesimally form bounded with respect to* $\mathbf{L}_{a,\Omega,\mathbb{G}}$ *and there exist constants* $M > 0$ *and* $\varepsilon_0 > 0$ *such that*

$$|\mathfrak{V}(f,f)| \leqslant \varepsilon \operatorname{Re}[\mathfrak{L}_{a,\Omega,\mathbb{G}}(f,f)] + M\varepsilon^{-n/(2p-n)}\|f\|^2_{L^2(\Omega)^N}, \quad f \in \mathcal{W}_{\mathbb{G}}(\Omega),\ 0<\varepsilon<\varepsilon_0. \tag{3.25}$$

In view of Theorem 3.3, the form sum $\mathbf{L}_{a,\Omega,\mathbb{G}} +_{\mathfrak{q}} \mathbf{V}$ is well-defined and represents an m-sectorial operator in $L^2(\Omega)^N$. Our next result extends stability of square root domains to $\mathbf{L}_{a,\Omega,\mathbb{G}}$ and $\mathbf{L}_{a,\Omega,\mathbb{G}} +_{\mathfrak{q}} \mathbf{V}$.

Theorem 3.4. *Assume Hypotheses* 3.1 *and* 3.2 *and let* $\mathbf{V}$ *denote the operator of component-wise multiplication in* $L^2(\Omega)^N$ *defined in* (3.9). *Then*

$$\begin{split}\operatorname{dom}\big((\mathbf{L}_{a,\Omega,\mathbb{G}} +_{\mathfrak{q}} \mathbf{V})^{1/2}\big) &= \operatorname{dom}\big(((\mathbf{L}_{a,\Omega,\mathbb{G}} +_{\mathfrak{q}} \mathbf{V})^*)^{1/2}\big)\\ &= \operatorname{dom}\big(\mathbf{L}^{1/2}_{a,\Omega,\mathbb{G}}\big) = \operatorname{dom}\big((\mathbf{L}^*_{a,\Omega,\mathbb{G}})^{1/2}\big) = \mathcal{W}_{\mathbb{G}}(\Omega).\end{split} \tag{3.26}$$

Proof. Let $\mathbf{V}$ be factored into the form $\mathbf{V} = \mathbf{B}^*\mathbf{A}$

$$\mathbf{A} = U|\mathbf{V}|^{1/2}, \quad \mathbf{B} = |\mathbf{V}^*|^{1/2}, \quad \operatorname{dom}(\mathbf{A}) = \operatorname{dom}(\mathbf{B}) = \operatorname{dom}(|\mathbf{V}|^{1/2}), \tag{3.27}$$

according to the generalized polar decomposition in (3.10). One observes that $\mathcal{W}_{\mathbb{G}}(\Omega) \subset \operatorname{dom}(\mathbf{A}) = \operatorname{dom}(\mathbf{B})$. Therefore, [14, Theorem 9.2] implies

$$\operatorname{dom}(\mathbf{A}) \supseteq \operatorname{dom}\big(\mathbf{L}^{1/2}_{a,\Omega,\mathbb{G}}\big), \quad \operatorname{dom}(\mathbf{B}) \supseteq \operatorname{dom}\big((\mathbf{L}^*_{a,\Omega,\mathbb{G}})^{1/2}\big). \tag{3.28}$$

Next, let $\mathbf{D}_{\Omega,\mathbb{G}}$ denote the non-negative self-adjoint operator uniquely associated to the sesquilinear form

$$\begin{split}\mathfrak{D}_{\Omega,\mathbb{G}}(f,g) = \sum_{j,k=1}^{n}\sum_{\alpha,\beta=1}^{N}\int_{\Omega}\overline{(\partial_j f_\alpha)(x)}\delta_{j,k}\delta_{\alpha,\beta}(\partial_k g_\beta)(x)\,d^n x,\\ f,g \in \operatorname{dom}(\mathfrak{D}_{\Omega,\mathbb{G}}) := \mathcal{W}_{\mathbb{G}}(\Omega),\end{split} \tag{3.29}$$

where $\delta_{j,k}$ denotes the Kronecker delta symbol. (One notes that (3.29) is simply (3.7) with tensor coefficients $a^{\alpha,\beta}_{j,k} = \delta_{j,k}\delta_{\alpha,\beta}$, $1 \leqslant j,k \leqslant n$, $1 \leqslant \alpha,\beta \leqslant N$.) Then (cf., e.g., the discussion preceding [14, Theorem 9.2])

$$(\mathbf{D}_{\Omega,\mathbb{G}}f)_\alpha = -\Delta_{\Omega,\Gamma_\alpha}f_\alpha, \quad 1 \leqslant \alpha \leqslant N,\ f \in \operatorname{dom}(\mathbf{D}_{\Omega,\mathbb{G}}) = \prod_{\beta=1}^{N}\operatorname{dom}(-\Delta_{\Omega,\Gamma_\beta}), \tag{3.30}$$

where we have used the notational convention $-\Delta_{\Omega,\Gamma_\alpha} = L_{I_n,\Omega,\Gamma_\alpha}$ set forth in (2.13). In addition (cf. the discussion in the proof to [14, Theorem 9.2]),

$$\big(\mathbf{D}^{1/2}_{\Omega,\mathbb{G}}f\big)_\alpha = (-\Delta_{\Omega,\Gamma_\alpha})^{1/2}f_\alpha, \quad 1 \leqslant \alpha \leqslant N,\ f \in \operatorname{dom}\big(\mathbf{D}^{1/2}_{\Omega,\mathbb{G}}\big) = \mathcal{W}_{\mathbb{G}}(\Omega). \tag{3.31}$$

As a result of (3.31), the bound in (3.23) actually implies

$$\begin{split}\|\mathbf{A}f\|^2_{L^2(\Omega)^N} \leqslant \varepsilon\big\|\mathbf{D}^{1/2}_{\Omega,\mathbb{G}}f\big\|^2_{L^2(\Omega)^N} + M\varepsilon^{-n/(2p-n)}\|f\|^2_{L^2(\Omega)^N},\\ f \in \mathcal{W}_{\mathbb{G}}(\Omega),\ 0<\varepsilon<1.\end{split} \tag{3.32}$$

Hence, [18, Lemma 2.12] guarantees the existence of constants $M_1 > 0$, $q > 0$, and $E_0 \geqslant 1$ such that

$$\big\|\mathbf{A}(\mathbf{D}_{\Omega,\mathbb{G}} + EI_{L^2(\Omega)^N})^{-1/2}\big\|_{\mathcal{B}(L^2(\Omega)^N)} \leqslant M_1 E^{-q}, \quad E > E_0. \tag{3.33}$$

Since $\operatorname{dom}\big(\mathbf{L}_{a,\Omega,\mathbb{G}}^{1/2}\big) = \operatorname{dom}\big(\mathbf{D}_{\Omega,\mathbb{G}}^{1/2}\big)$, [18, Lemma 2.11] yields the existence of a constant $C > 0$ such that

$$\sup_{E\geqslant 1} \big\|(\mathbf{L}_{a,\Omega,\mathbb{G}} + EI_{L^2(\Omega)^N})^{1/2}(\mathbf{D}_{\Omega,\mathbb{G}} + EI_{L^2(\Omega)^N})^{-1/2}\big\|_{\mathcal{B}(L^2(\Omega)^N)} \leqslant C. \tag{3.34}$$

The estimates in (3.33) and (3.34) imply

$$\big\|\mathbf{A}(\mathbf{L}_{a,\Omega,\mathbb{G}} + EI_{L^2(\Omega)^N})^{-1/2}\big\|_{\mathcal{B}(L^2(\Omega)^N)} \leqslant \widehat{M}_1 E^{-q}, \quad E > E_0, \tag{3.35}$$

for an appropriate constant $\widehat{M}_1 > 0$. A similar argument involving adjoints can be used to show

$$\big\|\overline{(\mathbf{L}_{a,\Omega,\mathbb{G}} + EI_{L^2(\Omega)^N})^{-1/2}\mathbf{B}^*}\big\|_{\mathcal{B}(L^2(\Omega)^N)} \leqslant \widehat{M}_2 E^{-q}, \quad E > E_0, \tag{3.36}$$

for an appropriate constant $\widehat{M}_2 > 0$.

Finally, in light of (3.28), (3.35), (3.36), and the fact that (cf. [14, Theorem 9.2])

$$\operatorname{dom}\big(\mathbf{L}_{a,\Omega,\mathbb{G}}^{1/2}\big) = \operatorname{dom}\big((\mathbf{L}_{a,\Omega,\mathbb{G}}^*)^{1/2}\big), \tag{3.37}$$

the string of equalities in (3.26) follows from an application of [18, Corollary 2.7]. We note that [18, Hypothesis 2.1 (*iii*)] holds in the present setting since

$$\begin{aligned} &\Big\|\overline{\mathbf{A}(\mathbf{L}_{a,\Omega,\mathbb{G}} + EI_{L^2(\Omega)^N})^{-1}\mathbf{B}^*}\Big\|_{\mathcal{B}(L^2(\Omega)^N)} \\ &\quad \leqslant \big\|\mathbf{A}(\mathbf{L}_{a,\Omega,\mathbb{G}} + EI_{L^2(\Omega)^N})^{-1/2}\big\|_{\mathcal{B}(L^2(\Omega)^N)} \\ &\qquad \times \big\|\overline{(\mathbf{L}_{a,\Omega,\mathbb{G}} + EI_{L^2(\Omega)^N})^{-1/2}\mathbf{B}^*}\big\|_{\mathcal{B}(L^2(\Omega)^N)}, \quad E > 0, \end{aligned} \tag{3.38}$$

and the estimates in (3.35), (3.36) yield decay to zero as $E \uparrow \infty$ of the factors on the right-hand side of (3.38). □

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