On controllability of a rotating string

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Abstract

We consider the problem of controllability of a rotating string. We say that a set of initial data of the string is controllable if, for any initial data of this set by suitable manipulation of the exterior force, the string goes to rest. The main result of the paper is a description of controllable sets of initial data. The equation is not strongly hyperbolic. To get exact controllability of such equation in the sharp time interval, we use control functions from Sobolev spaces with noninteger indices. To prove our results, we apply the method of moments that has been widely used in control theory of distributed parameter systems since the classical papers of H.O. Fattorini and D.L. Russell. We use recent results about exponential bases in Sobolev spaces with noninteger indices.

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1. Introduction

A disk in a computer is rotating at a very high speed. It also performs small transverse oscillations. As a consequence of those oscillations, disk might hit surrounding surfaces and lose...
the information that it carries. Therefore it would be helpful to minimize transverse oscillations. Another important mechanical model of the similar type is a rotating propeller of a helicopter; its transversal oscillations may cause crash of helicopter. In this paper, we consider the simplest distributed rotating system, namely a rotating string. We say that a set of initial data of the string is controllable if, for any initial data of this set by suitable manipulation of the exterior force, the string goes to rest. The main result of the paper is a description of controllable sets of initial data.

The equation describing a rotating string is not strongly hyperbolic: one of its leading coefficients tends to zero at the end of the string. If we restrict ourselves by controls from $L^2(0,T)$, then, typically, equations with singular coefficients lack exact controllability in the sharp time interval, however, can preserve approximate controllability [4]. To get exact controllability of such equations in the sharp time interval, we use control functions from Sobolev spaces with noninteger indices $H^s(0,T)$ (see, e.g., [19] for the definition and properties of these spaces). Controls from $H^s(0,T)$ with noninteger $s$ were used for the first time in [6]. In the present paper we extend this approach for a new class of systems.

To prove our results, we apply the method of moments that has been widely used in control theory of distributed parameter systems since the classical papers of H.O. Fattorini and D.L. Russell in the late 60s to early 70s (see the survey [21] and the book [3] for the history of the subject and complete references). The method is based on properties of exponential families (usually in the space $L^2(0,T)$), the most important of which for control theory are minimality, the Riesz basis property, and also, as in the current paper, the $L^2$-basis property. The latter is defined as a Riesz basis property in the closure of the linear span of the family. Recent investigations into new classes of distributed systems such as hybrid systems, systems with parameters that vary in time, and damped systems as well as problems of simultaneous control have raised a number of new difficult problems in the theory of exponential families (see, e.g., [1–10,14]).

In the present paper the family of nonharmonic exponentials $e^{\pm i\alpha n}$ arises with $a_n \sim n + 3/4, n = 0, 1, \ldots$ (up to a scaling). Such a family does not form a Riesz basis in $L^2(0,2\pi)$; it is not also a Riesz basis in $L^2(0,T)$ for any other $T$. However, using recent results about exponential bases in Sobolev spaces with noninteger indices [5], we can show that this family forms (after normalization) a Riesz basis in some Sobolev space and that, together with the duality arguments [12,18], enables studying controllability of the system in the time interval $[0,2\pi]$.

Note, the asymptotic formula $a_n \sim n + 3/4$ is the consequence of the absence of the strong hyperbolicity. For the string with fixed end points we would get $a_n \sim n + 1$ and for the string with one free and one fixed end $a_n \sim n + 1/2$. The asymptotic formula $a_n \sim n + 3/4$ implies the Riesz basis property of the family of nonharmonic exponentials in the Sobolev space with noninteger index, which, in turn, makes natural involving such a control space on the “critical” time interval.

We now briefly derive our physical model. A homogeneous string with no resistance to bending is attached to the vertical axis at one end and free at the other end. In the equilibrium position, it is located vertically. If the axis is rotating with the constant angular velocity $\omega$, the string does the same. If the weight of the string is neglected, then it is located horizontally in the equilibrium position. If we hit it, it starts oscillating. The oscillations are described by the wave equation. To derive it, we first need to find the tension of the string. Let the tension at the point $x$ be $H(x)$. Then $dH(x)$ is the centripetal force acting on the element $(x, x + dx)$ of the string. We find from the elementary mechanics

$$dH = -\rho \omega^2 x \, dx.$$ (1.1)
Here $\rho(x)$ is the density of the string (which is supposed to be constant in this paper). The end $x = l$ of the string is free and hence

$$H(l) = 0.$$  

Integrating (1.1) and using (1.2) yields finally

$$H(x) = \omega^2 \int_x^l \rho x \, dx.$$  

Hence the equation of the small transverse oscillations is

$$\rho(x)y_{tt} = (Hy_x)_x + F_e(x, t).$$  

Here $F_e(x, t)$ is an exterior force. We suppose that

$$F_e(x, t) = g(x)f(t),$$  

where $f(t)$ is considered as a control and the distribution of the force along the string $g(x)$ is supposed to be given. It is assumed that $g \in L^2(0, l)$.

For simplicity of the exposition we suppose that $\rho = \text{const} > 0$. The controllability results that we prove are valid also for $x$-dependent $\rho$ since they are based on the $L$-basis property of (normalized) exponentials $e^{\pm i\alpha_n t}$. This property, in turn, depends on asymptotic representation of eigenfrequencies $\alpha_n$, and that representation can be easily determined for $x$-dependent $\rho$.

For a constant $\rho$, (1.3) implies

$$H(x) = \rho \omega^2 \frac{l^2 - x^2}{2}.$$  

\section{Statement of the initial boundary value problem and of the main results}

For any $T > 0$, we consider the following initial boundary value problem for the hyperbolic type PDE:

$$\rho y_{tt} = \frac{\rho \omega^2}{2} (l^2 - x^2) y_x + g(x)f(t), \quad (x, t) \in Q_T = (0, l) \times (0, T),$$  

$$y(0, t) = 0; \quad y(x, t), y_x(x, t) < \infty \quad \text{as} \quad x \to l^-,$$

$$y(0, x) = y_0(x), \quad y_t(x, 0) = y_1(x).$$  

Here $y_0$ and $y_1$ are the initial displacement and velocity. This problem describes the transverse oscillations of the rotating string. It is well known [16] that the initial boundary value problem (2.1)–(2.3) has a unique solution if the functions $y_0$, $y_1$, $g$, and $f$ are smooth enough. Our goal is to find the conditions on the system under consideration that guarantee the existence of the controlling force $f(t)$ such that the string goes to rest in a given time $T$ and construct this force.
The differential operator (with respect to \(x\)) that appears in (2.1) is the Legendre operator (see, e.g., [15])

\[
L_z := -((l^2 - x^2)z')', \quad 0 < x < l, \quad z(0) = 0; \quad z(x), z'(x) < \infty \quad \text{as} \quad x \to l^{-}
\]

so that its eigenfunctions (that are the solutions of \(L_z = \lambda z\)) and eigenvalues have the form

\[
z_n(x) = P_{2n+1}\left(\frac{x}{l}\right), \quad \lambda_n = (2n + 1)(2n + 2), \quad n \geq 0. \tag{2.4}
\]

The Legendre polynomials are orthogonal

\[
\int_0^l P_{2n+1}\left(\frac{x}{l}\right) P_{2m+1}\left(\frac{x}{l}\right) \, dx = \frac{l}{4n + 3} \delta_{nm} \tag{2.5}
\]

and the functions \(\psi_n(x) = \sqrt{(4n + 3)/l} P_{2n+1}(x/l)\) form an orthonormal basis in \(L^2(0, l)\).

For any real \(s\) we introduce the space \(W_s\):

\[
W_s = \left\{ a(x) = \sum_{n=0}^{\infty} a_n \psi_n(x) : \|a\|_s^2 := \sum_{n=0}^{\infty} |a_n|^2 \lambda_n^s < \infty \right\}.
\]

Evidently, \(W_0 = L^2(0, l)\). For \(s > 0\) the space \(W_s\) is the domain of the corresponding (namely, \(s/2\)) power of the Legendre operator.

We need also the weight spaces depending on the function \(g(x)\) that appears in (1.5). To define these spaces, let us represent the function \(g(x)\) as a series

\[
g(x) = \rho \sum_{n \geq 0} g_n \psi_n(x), \tag{2.6}
\]

where the factor \(\rho\) is introduced for convenience.

We introduce the space \(W_{s-g}^{-s} \subset \mathbb{R}\),

\[
W_{s-g}^{-s} = \left\{ a(x) = \sum_{n=0}^{\infty} a_n \psi_n(x) : a_n = 0 \quad \text{for} \quad g_n = 0, \quad \|a\|_{s-g}^2 := \sum_{n: g_n \neq 0} |a_n|^2 |g_n|^{-2} \lambda_n^{-s} < \infty \right\}.
\]

The space \(W_{s-g}^{g} \subset \mathbb{R}\), dual to \(W_{s-g}^{-g}\), is defined as follows:

\[
W_{s-g}^{g} = \left\{ a(x) = \sum_{n=0}^{\infty} a_n \psi_n(x) : \|a\|_s^2 := \sum_{n=0}^{\infty} |a_n|^2 |g_n|^{-2} \lambda_n^{-s} < \infty \right\}. \tag{2.7}
\]

A solution of the initial boundary value problem (2.1)–(2.3) is a sum of the solutions of two problems. The first one is Eq. (2.1) with boundary conditions (2.2) and zero initial conditions

\[
y(0, x) = 0, \quad y_t(x, 0) = 0. \tag{2.8}
\]

The second initial boundary value problem consists of the equation

\[
\rho y_{tt} = \frac{\rho \omega^2}{2} ((l^2 - x^2)y_x)_x, \quad (x, t) \in Q_T = (0, l) \times (0, T), \tag{2.9}
\]

with the boundary conditions (2.2) and initial conditions (2.3).
Theorem 1. The solution of the initial boundary value problem (2.9), (2.2), (2.3) exists, is unique and has the same regularity as the initial data. In particular, if \( y_0 \in W_{s+1}^{-}, y_1 \in W_s^{-} \), then \( y \in C([0, T]; W_{s+1}^{-}) \) and \( y_t \in C([0, T]; W_{s+1}^{-}) \) and \( y_t \in C([0, T]; W_{s+1}^{-}) \).

For the initial boundary value problem (2.1)–(2.3) with nonsmooth \( f \) and \( g \) we prove the following result.

Theorem 1. Let \( T > 0, g \in L^2(0, T) \).

(a) Let \( f \in L^2(0, T) \) and \( y_0 \in W_1^{-}, y_1 \in W_0^{-} \). Then there is a unique solution to the initial boundary value problem (2.1)–(2.3) such that \( y \in C([0, T]; W_1^{-}) \) and \( y_t \in C([0, T]; W_0^{-}) \).

(b) Let \( f \in H_{00}^{1/2}(0, T) \) and \( y_0 \in W_{3/2}^{-}, y_1 \in W_1^{-} \). Then there is a unique solution to the initial boundary value problem (2.1)–(2.3) such that \( y \in C([0, T]; W_{3/2}^{-}) \) and \( y_t \in C([0, T]; W_1^{-}) \).

Here the Banach space \( H_{00}^{1/2}(0, T) \) is defined as follows:

\[
H_{00}^{1/2}(0, T) = \{ u \in H^{1/2}(0, T) : \left[ t(T - t) \right]^{1/2} u(t) \in L^2(0, T) \}.
\]

The discussion of the properties of the space \( H_{00}^{1/2}(0, T) \) can be found in [19, Section I.11]. It is shown that this space is the subspace of \( H^{1/2}(\mathbb{R}) \) with the elements having a compact support in \([0, T] \). The space \( (H_{00}^{1/2}(0, T))' \), dual to \( H_{00}^{1/2}(0, T) \), plays an important role in our construction. The following result from [19, Section I.11] allows to better understand the structure of the dual space. Any element of it admits the representation

\[
f \in (H_{00}^{1/2}(0, T))' \iff f = f_0 + f_1,
\]

\[
f_0 \in H^{-1/2}(0, T), \quad \left[ t(T - t) \right]^{1/2} f_1 \in L^2(0, T).
\]

Represent the solution \( y(x, t) \) as a Fourier–Legendre series

\[
y(x, t) = \sum_{n \geq 0} c_n(t) \psi_n(x) \quad (2.10)
\]

with some unknown coefficients \( c_n(t) \). Also represent the initial data (2.3) as the similar series

\[
y_j(x) = \sum_{n \geq 0} c_n^j \psi_n(x), \quad j = 0, 1, \quad (2.11)
\]

with some (known) coefficients \( c_n^0, c_n^1 \). Substituting the representations (2.10) and (2.11) into Eqs. (2.1) and (2.3) and using the orthogonality condition (2.5) yields the Cauchy problem for the system of (independent) ODEs

\[
\dot{c}_n + 2\omega^2 N_n^2 c_n = g_n f(t), \quad (2.12)
\]

\[
c_n(0) = c_n^0, \quad \dot{c}_n(0) = c_n^1, \quad n \geq 0, \quad (2.13)
\]

where

\[
N_n = \sqrt{(n + 1/2)(n + 1)}. \quad (2.14)
\]
It is easy to check that \( N_{n+1} - N_n > 1 \) and, therefore, the set \( \{N_n\} \) is uniformly discrete or separated:
\[
\inf_{m \neq n} |N_m - N_n| > 0.
\]  
(2.15)

Introduce the (positive) number \( T^* \):
\[
T^* = \frac{\pi \sqrt{2}}{\omega}.
\]  
(2.16)

To consider controllability problem for the initial boundary value problem (2.1)–(2.3), we need the following

**Definition.** Let a moment \( T > 0 \) be given. The initial data \((y_0, y_1) \in W_{1/2}^{-g} \times W_0^{-g}\) of the string are said to be controllable in the time interval \([0, T]\) by \( L^2 \) controls if there exist a control function \( f \in L^2(0, T) \) such that the solution of the problem (2.1)–(2.3) satisfies the additional conditions at the moment \( t = T \):
\[
y(x, T) = y_t(x, T) = 0, \quad x \in [0, l].
\]  
(2.17)

The set of all such states is called the controllable set (in the time interval \([0, T]\)) and is denoted by \( G(T) \).

(The similar notion will be applied for controls \( f \in H_{00}^{1/2}(0, T) \) and initial data from \( W_{3/2}^{-g/2} \times W_{1/2}^{-g/2} \).)

The system is called spectrally controllable in the time interval \([0, T]\) if all initial data of the form \((y_0, y_1) = (\psi_m, \psi_n), m, n \in \mathbb{N}, \) belong to \( G(T) \).

The system (2.1)–(2.2) is called approximately controllable in the time interval \([0, T]\) if \( G(T) \) is dense in \( W_1^{-g} \times W_0^{-g} \).

We now formulate our main results on controllability of the rotating string.

**Theorem 2.** Consider the system (2.1)–(2.3). Let \( T > 0, g \in L^2(0, l), f \in L^2(0, T) \).

(a) If there exists \( n \) such that \( g_n = 0 \), then the system (2.1)–(2.3) is not approximately controllable for any \( T \).

(b) If \( T < T^* \), the system is not spectrally controllable for any \( g \in L^2(0, l) \).

(c) If \( T \geq T^* \) and \( g_n \neq 0 \) for all \( n \), then the system is spectrally controllable.

(d) If \( T > T^* \), then \( G(T) = W_{1/2}^{-g} \times W_0^{-g} \). The space of control functions \( f(t) \) that bring the system from rest to rest on the time interval \([0, T]\), \( T > T^* \), is infinite-dimensional.

(e) If \( T \leq T^* \), \( G(T) \) belongs to \( W_1^{-g} \times W_0^{-g} \) but not equal to this space.

Similar investigations can be done for control functions from \( H_{00}^{1/2}(0, T) \). The main advantage of such controls is the fact of exact controllability with respect to the “natural” state space \( W_{3/2}^{-g} \times W_{1/2}^{-g} \) in the “critical” time interval \([0, T^*]\).

**Theorem 3.** Consider the system (2.1)–(2.3). Let \( f \in H_{00}^{1/2}(0, T) \), \( g \in L^2(0, l) \). Then
\[
G(T) = W_{3/2}^{-g} \times W_{1/2}^{-g} \quad \text{for } T \geq T^*.
\]
3. The problem of moments

The (unique) solution of the Cauchy problem (2.12)–(2.13) may be found elementary

\[ c_n(t) = c_0^n \cos(\omega \sqrt{2} N_n t) + c_1^n \sin(\omega \sqrt{2} N_n t) + g_n \int_0^t \frac{\sin(\omega \sqrt{2} N_n (t - \tau))}{\omega \sqrt{2} N_n} f(\tau) d\tau. \]

(3.1)

The exact controllability conditions (2.17) along with orthogonality condition (2.5) lead to the conditions

\[ c_n(T) = c_n'(T) = 0 \quad \text{for all } n \geq 0. \]

(3.2)

We find

\[ g_n \int_0^T \frac{\sin(\omega \sqrt{2} N_n (T - \tau))}{\omega \sqrt{2} N_n} f(\tau) d\tau = -c_0^n \cos(\omega \sqrt{2} N_n T) - c_1^n \frac{\sin(\omega \sqrt{2} N_n T)}{\omega \sqrt{2} N_n}, \]

(3.3)

\[ g_n \int_0^T \cos(\omega \sqrt{2} N_n (T - \tau)) f(\tau) d\tau \]

\[ = \omega \sqrt{2} N_n c_0^n \sin(\omega \sqrt{2} N_n T) - c_1^n \cos(\omega \sqrt{2} N_n T), \quad n \geq 0. \]

(3.4)

Equations (3.3)–(3.4) form the moment problem for the function \( f(\tau) \). Multiplying (3.3) by \( \pm i \omega \sqrt{2} N_n \) and adding to (3.4) yields after some simplifications

\[ g_n \int_0^T e^{\pm i \omega \sqrt{2} N_n \tau} f(\tau) d\tau = -(\pm i \omega \sqrt{2} N_n c_0^n + c_1^n), \quad n \geq 0. \]

(3.5)

Introducing the system of functions \( G = \{ \Gamma_n^\pm(t) \} \),

\[ \Gamma_n^\pm(t) = e^{\pm i \omega_n t}, \quad \omega_n := \omega \sqrt{2} N_n, \quad n \geq 0, \]

(3.6)

the sequence

\[ \xi_n^\pm = -(\pm i \omega_n c_0^n + c_1^n), \quad n \geq 0, \]

(3.7)

and denoting the inner product in \( L^2(0, T) \) as \( (, ) \) yields instead of (3.5)

\[ g_n \left( f, \Gamma_n^\pm \right) = \xi_n^\pm \quad \text{for all } \Gamma_n^\pm \in G. \]

(3.8)

In Section 5, we will need an expression for \( \pm i \omega_n c_n(\tau) + \dot{c}_n(\tau) \). Using representation (3.1) and notations (3.6) yields after some simplifications

\[ \pm i \omega_n c_n(\tau) + \dot{c}_n(\tau) = (\pm i \omega_n c_0^n + c_1^n) e^{\pm i \omega_n \tau} + g_n \left( f, \Gamma_n^\pm \right)_{L^2(0, \tau)} e^{\pm i \omega_n \tau} \]

(3.9)

for any \( \tau \geq 0. \)
4. Basis properties of the family $\mathcal{G}$

Let us remind some notions from the Hilbert spaces theory. Let $\mathcal{E} = \{e_j\}_{j \in \mathbb{N}}$ be a family of elements (vectors) $e_j$ of a Hilbert space $\mathcal{H}$.

Family $\mathcal{E}$ is called minimal in $\mathcal{H}$ if for any $j$ element $e_j$ does not belong to the closure of the linear span of all the remaining elements.

If family $\mathcal{E}$ is minimal in $\mathcal{H}$, there exists a family $\mathcal{E}' = \{e'_i\} \subset \mathcal{H}$, which is said to be biorthogonal to $\mathcal{E}$, such that $(e_j, e'_i) = \delta_{ji}$.

Family $\mathcal{E}$ is said to be a Riesz basis in $\mathcal{H}$, if $\mathcal{E}$ is an image of an isomorphic mapping $V$ of some orthonormal basis $\mathcal{F} = \{f_j\}_{j \in \mathbb{N}} \subset \mathcal{H}$, i.e., $e_j = V f_j$, $j \in \mathbb{N}$, where $V$ is a bounded linear and boundedly invertible operator. Definition and properties of Riesz bases can be found in [3,13]. In particular, the following inequalities hold for any $f \in \mathcal{H}$:

\[ \sum_j |(f, e_j)|^2 \asymp \|f\|^2 \]  
(4.1)

and also

\[ \sum_j |\alpha_j|^2 \asymp \|f\|^2 \text{ if } f = \sum_j \alpha_j e_j, \]  
(4.2)

where $\asymp$ stands for the two sides inequality with constants independent of $f$. The inequalities (4.1) and (4.2) are used below in Section 5.

Note that a family biorthogonal to a Riesz basis also forms a Riesz basis in $\mathcal{H}$.

Family $\mathcal{E}$ is said to be an $L$-basis in $\mathcal{H}$ if it forms a Riesz basis in the closure of its linear span.

In this section we study minimality and the basis properties of the set of the functions $\mathcal{G}$ given by (3.6).

The following asymptotic relation is easy to be verified:

\[ N_n = n + \frac{3}{4} + \mathcal{O}\left(\frac{1}{n}\right) \text{ as } n \to \infty. \]  
(4.3)

Lemma 1. The family $\mathcal{G} = \{\Gamma_n^\pm(t)\}$ has the following properties:

(a) The family $\mathcal{G}$ is complete and minimal in $L^2(0, T^*)$, however, it is not a Riesz basis in this space.

(b) The family $\mathcal{G}$ forms an $L$-basis in $L^2(0, T)$ for $T > T^*$. Moreover, there exits an infinite family of exponentials $\mathcal{G}_1$ such that the family $\mathcal{G} \cup \mathcal{G}_1$ forms a Riesz basis in $L^2(0, T)$.

(c) For $T < T^*$, the family $\mathcal{G}$ is not minimal in $L^2(0, T)$. Moreover, it can be split into two subfamilies, $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1$, where $\mathcal{G}_0$ forms a Riesz basis in $L^2(0, T)$ and $\mathcal{G}_1$ is an infinite family.

(d) The family $\{a_n^{1/2} \Gamma_n^\pm(t)\}$ is a Riesz basis in $[H^{1/2}_{00}(0, T^*)]'$.

Here $[H^{1/2}_{00}(0, T^*)]'$ is the space dual to $H^{1/2}_{00}(0, T^*)$ provided $L^2(0, T^*)$ is identified with its dual (see [19, Section I.12]).

Proof. We begin with the proof of the assertions (b) and (c). To prove that the family $\mathcal{G}$ forms an $L$-basis we show that it can be extended to a Riesz basis. Then, as a subset of a Riesz basis it is, by definition, an $L$-basis.
We denote \( \Omega := \{ \pm \omega_n \}_{n \geq 0} \), and
\[
n^+(r) := \sup_{x \in \mathbb{R}} \# \{ \Omega \cap [x, x + r) \},
\]
where \( \# \mathcal{A} \) is the number of elements in the set \( \mathcal{A} \). Function \( n^+(r) \) is clearly sub-additive:
\[
n^+(r_1 + r_2) \leq n^+(r_1) + n^+(r_2).
\]
Let us define in a standard way (see, e.g., [11, p. 346]) the upper uniform density of \( \Omega \) to be
\[
D^+(\Omega) := \lim_{r \to \infty} \frac{n^+(r)}{r}.
\]
The limit exists due to the sub-additivity of \( n^+(r) \). Similarly, one can introduce the function
\[
n^-(r) := \inf_{x \in \mathbb{R}} \# \{ \Omega \cap [x, x + r) \}
\]
and the lower uniform density of \( \Omega \):
\[
D^-(\Omega) := \lim_{r \to \infty} \frac{n^-(r)}{r}
\]
(function \( n^-(r) \) is super-additive, and so, the limit exists).

It is easy to prove, using (4.3), that
\[
D^+(\Omega) = D^-(\Omega) = \frac{1}{\omega \sqrt{2}}.
\]

Statements (b) and (c) follow now from the results of Seip [22]:

For any \( T > 2\pi D^+(\Omega) \) (\( = \frac{\pi \sqrt{2}}{\omega} = T^* \)) the family \( \mathcal{G} \) can be extended to a family of exponentials that forms a Riesz basis in \( L^2(0, T) \) by adding an infinite family of exponentials.

For any \( T < 2\pi D^-(\Omega) \) (\( = \frac{\pi \sqrt{2}}{\omega} = T^* \)) the family \( \mathcal{G} \) can be turned into a Riesz basis in \( L^2(0, T) \) by extracting an infinite family of exponentials.

(a) Let us denote by \( F(z) \) the generating function of the family \( \mathcal{G} \):
\[
F(z) = \prod_{n=0}^{\infty} \left( 1 - \frac{z^2}{\omega_n^2} \right)
\]
and let
\[
\Phi(z) = \prod_{n=0}^{\infty} \left( 1 - \frac{z^2}{\mu_n^2} \right),
\]
where \( \mu_n := \omega \sqrt{2} (n + 1/2) \). Let \( \delta_n := \omega_n - \mu_n \). We put also \( \mu_{-n} = -\mu_n, \delta_{-n} = -\delta_n \).

A particular case of [1, Lemma 4] can be formulated as follows:

For any real \( h \neq 0 \),
\[
\left| \frac{F(x + ih)}{\Phi(x + ih)} \right| \asymp \exp \Re \left\{ - \sum_{|\mu_n| \leq |x|} \left( \frac{\delta_n}{\mu_n} + \frac{\delta_n}{x + ih - \mu_n} \right) \right\},
\]
where \( \asymp \) stands for the two sides inequality with some positive constants independent of \( x \in \mathbb{R} \).
From (4.3) it follows that
\[ \delta_n = \frac{\omega \sqrt{2}}{4} + O \left( \frac{1}{n} \right). \] (4.9)
Taking into account that \( \Phi(z) = \cos(\pi z / \omega \sqrt{2}) = \cos(z T^*/2) \) and, so, \( |\Phi(x + ih)| \asymp 1, \ x \in \mathbb{R} \), we can derive from (4.8), (4.9) (the more general result proved in [1, Theorem 3]) that for any real \( h \)
\[ |F(x + ih)| \asymp (1 + |x|)^{-1/2}, \ x \in \mathbb{R}. \] (4.10)
Hence
\[ \int_{-\infty}^{+\infty} \frac{|F(x)|^2}{1 + x^2} \, dx < \infty, \]
that proves minimality of \( \mathcal{G} \) in \( L^2(0, T^*) \) (see [20], [3, Section II.4]).
Since the generating function \( F(x) \) does not belong to \( L^2(\mathbb{R}) \), the family \( \mathcal{G} \) is complete in \( L^2(0, T^*) \) [17].
In the theory of exponential Riesz bases, an important role is played by the Muckenhoupt \( (A_2) \) condition
\[ \sup_{I} \left( \frac{1}{|I|} \int_{I} w(x) \, dx \cdot \frac{1}{|I|} \int_{I} \frac{1}{w(x)} \, dx \right) < \infty, \]
where the supremum is taken over all intervals \( I \) of the real axis.
The relation (4.10) implies that \( |F(x + ih)|^2 \) does not satisfy the Muckenhoupt condition on straight lines parallel to the real axis (it is the well-know fact, complete proof can be found in [3, Section II.3.4]). Therefore the family \( \mathcal{G} \) does not form a Riesz basis in \( L^2(0, T^*) \) (see, e.g., [3, Sections II.3.4, II.4.2]).

**Remark 1.** Family \( \mathcal{G} \) not only is minimal but also is uniformly minimal in \( L^2(0, T^*) \). It means that the norms of the family biorthogonal to \( \mathcal{G} \) are uniformly bounded. This fact can be proved similarly to the proof of [3, Theorem 3(v)] where a family with the same (as \( F \)) behavior of a generating function was studied.

To prove statement (d), we apply now the following theorem [5, Theorem 2].

*Let \( F \) be an entire function of exponential type, with indicator diagram of width \( a \), whose zero set \( \{\xi_k\} \) lies in a strip parallel to the real axis and satisfies the separation condition \( \inf_{k \neq j} |\xi_k - \xi_j| > 0 \). If for some real \( h \) and \( s \)
\[ c \left( 1 + |x| \right)^s \leq |F(x + ih)| \leq C \left( 1 + |x| \right)^s, \quad x \in \mathbb{R}, \]
with positive constants \( c \) and \( C \) independent of \( x \in \mathbb{R} \), then the family \( \{ e^{i \xi_k t} / (1 + |\xi_k|)^s \} \) forms a Riesz basis in \( H^s(0, a) \) for \( s \notin \{ -N + \frac{1}{2} \} \) and in \( [H_{00}^{n-1/2}(0, a)]' \) for \( s = -n + 1/2, n \in \mathbb{N} \).*

Here \( [H_{00}^{n-1/2}(0, a)]' \) is the dual space to \( H_{00}^{n-1/2}(0, a) \) relative to \( L^2(0, a) \) (see [19, Sections I.11, I.12] for detailed description of these spaces; we do not discuss this description because we actually do not use that space below).
Statement (d) follows now from this theorem (with \( a = T^*, s = -1/2 \) and (4.10). \( \square \)
5. Controllability: Proof of the main results

Proof of Proposition 1 and Theorem 1(a). From the definition of spaces $W_s$ and asymptotic representation of $\omega_n$ (see (3.6), (4.3)) it is clear that

$$\| \{ y(\cdot, \tau), y_\tau(\cdot, \tau) \} \|_{W_1 \times W_0} \simeq \sum_{n \geq 0} (\omega_n^2 |c_n(\tau)|^2 + |\dot{c}_n(\tau)|^2)$$

(5.1)

and

$$\| \{ y(\cdot, \tau), y_\tau(\cdot, \tau) \} \|_{W_1^{-s} \times W_0^{-s}} \simeq \sum_{n; g_n \neq 0} (\omega_n^2 |c_n(\tau)|^2 + |\dot{c}_n(\tau)|^2)|g_n|^2$$

(5.2)

(5.1) and (5.2) mean the equivalence of norms of functions and their Fourier representations in the corresponding spaces.

In Section 3 we have shown that

$$\pm i \omega_n c_n(\tau) + \dot{c}_n(\tau) = e^{\pm i \omega_n \tau} (\pm i \omega_n c_n^0 + c_n^1 + g_n(f, \Gamma_n^\pm)_{L^2(0, \tau)}).$$

(5.3)

Therefore, for any $\tau > 0$, the state $\{ y(\cdot, \tau), y_\tau(\cdot, \tau) \}$ belongs to the same space as the initial data $\{ y_0, y_1 \}$.

Let now $y_0 = y_1 = 0$. Then (5.3) implies

$$\sum_{n; g_n \neq 0} (\omega_n^2 |c_n(\tau)|^2 + |\dot{c}_n(\tau)|^2)|g_n|^2 \leq \sum_{n, \pm} |(f, \Gamma_n^\pm)_{L^2(0, \tau)}|^2.$$  

(5.5)

On the other hand, the inequality

$$\sum_{n, \pm} |(f, \Gamma_n^\pm)_{L^2(0, \tau)}|^2 \leq K \int_0^\tau |f(t)|^2 \, dt \quad (\leq K \| f \|_{L^2(0, T)}^2)$$

(5.6)

is valid for any positive $\tau$, any $f \in L^2(0, \tau)$, and some constant $K$ independent of $f$. For $\tau > T^*$ inequality (5.6) follows from Lemma 1(b) and (4.1). Therefore, it is obviously true for $\tau \leq T^*$.

Indeed, it is sufficient to put $f(t) = 0$ for $t > \tau$.

From (5.2), (5.5), and (5.6) it follows that $\{ y(\cdot, \tau), y_\tau(\cdot, \tau) \} \in W_1^{-g} \times W_0^{-g}$.

Using the relations (5.1), (5.2), (5.4)–(5.6), one can easily check that the sequence $Y_N(t) := \left\{ \sum_{n=0}^N c_n(t) \psi_n + \sum_{n=0}^N \dot{c}_n(t) \psi_n \right\}$ converges to $\{ y(\cdot, t), y_\tau(\cdot, t) \}$ in $W_1^{-g} \times W_0^{-g}$ norm uniformly on any finite interval $t \in [0, T]$. Therefore $\{ y(\cdot, t), y_\tau(\cdot, t) \}$ depends continuously on $t$ in $W_1^{-g} \times W_0^{-g}$ norm, which proves Proposition 1 and Theorem 1(a). \qed

The proof of the statement (b) of Theorem 1 will be provided below together with the proof of Theorem 3.
Proof of Theorem 2. (a) The first assertion of the theorem is a well-know result in control theory that we include for the completeness purposes only. Let for some integer \( m \geq 0 \),

\[ g_m = 0, \quad |c_m^0| + |c_{m}^1| \neq 0. \]

It follows directly from (3.7), (3.8) that, for any \( T > 0 \), the controllable set \( G(\tau) \) is orthogonal to the two-dimensional subspace of \( W_1 \times W_0 \) spanned by \((\psi_m, 0)\) and \((0, \psi_m)\).

(b) According to [3, Theorem III.3.10], spectral controllability of the system (2.1), (2.2) is equivalent to minimality of \( \{g_n \Gamma_{n}^\pm\} \) in \( L^2(0, T) \). In view of Lemma 1(c), for \( T < T^* \) the family \( \{\Gamma_{n}^\pm\} \) is not minimal. Therefore, the family \( \{g_n \Gamma_{n}^\pm\} \) is also not minimal even in the case when \( g_n \neq 0 \) for all \( n \).

(c) Due to Lemma 1((a) or (b)) the family \( \{\Gamma_{n}^\pm\} \) is minimal in \( L^2(0, T^*) \). If \( g_n \neq 0 \) for all \( n \), the same is true for the family \( \{g_n \Gamma_{n}^\pm\} \). Then the system is spectrally controllable in the time interval \([0, T^*]\) and hence, in any time interval \([0, T]\) with \( T > T^* \).

(d) Suppose \( T > T^* \). Then, according to Lemma 1(b) the family \( G \) forms an \( \mathcal{L}\)-basis in \( L^2(0, T) \). Introduce the basis \( \{\gamma_n^\pm\} \) that is biorthogonal to the basis \( \{\Gamma_{n}^\pm\} \), i.e.,

\[ (\Gamma_{n}^+, \gamma_{m}^-) = (\Gamma_{n}^-, \gamma_{m}^+) = 0, \quad (\Gamma_{n}^+, \gamma_{m}^+) = (\Gamma_{n}^-, \gamma_{m}^-) = \delta_{m} \] for any \( n, m \geq 0 \).

Therefore, a formal solution of the moment equalities (3.8) can be written as

\[ f(t) = \sum_{n, \pm} \frac{\xi_n^\pm}{g_n} \gamma_n^\pm(s). \] (5.7)

From the definitions of the coefficients \( \xi_n^\pm \) and spaces \( W_s^{\pm} \) it follows that this formula actually gives a function \( f \) from \( L^2(0, T) \) if and only if \((\gamma_0, \gamma_1) \in W_1^{-\pm} \times W_0^{-\pm} \).

Note, control is not unique in this case. Indeed, because the exponential family is not complete for \( T > T^* \), we can always add a function that is orthogonal to all exponentials; such a control drives the system from rest to rest. The dimension of the space of control sequences such that

\[ (f, \Gamma_n^\pm) = 0 \quad \text{for all } n \geq 0 \]

is equal to the codimension of the linear span of \( G \) in \( L^2(0, T) \). For \( T > T^* \) it is infinite (Lemma 1(b)).

(e) Inclusion \( G(\tau) \subset W_1^{-\pm} \times W_0^{-\pm} \) for \( T \leq T^* \), was really obtained in the proof of Theorem 1(a). If the equality \( G(T^*) = W_1^{-\pm} \times W_0^{-\pm} \) was true, then due to Theorem III.3.10(a) of [3] the family \( \{\Gamma_n^\pm\} \) would form Riesz basis or an \( \mathcal{L}\)-basis in \( L^2(0, T^*) \). This is impossible because of the statement (a) of Lemma 1. □

Remark 2. It is interesting to note that if \( g_n \neq 0 \) for all \( n \) and \( |g_n| < c_1 \exp(-c_2 n) \) (where \( c_1 \) and \( c_2 \) are positive constants independent of \( n \)), then the system (2.1), (2.2) is approximately controllable in any interval \([0, T]\). The proof of this fact is similar to the proof of [3, Theorem V.1.3(e)].

Proof of Theorems 3 and 1(b). Let us introduce the problem which is dual to problem (2.1)–(2.3):

\[ \rho u_{tt} = \frac{\rho \omega^2}{2}(\bar{l}^2 - x^2)u_x, \quad (x, t) \in Q_T = (0, l) \times (0, T^*), \] (5.8)

\[ u(0, t) = 0; \quad u(x, t), u_x(x, t) < \infty \quad \text{as } x \to l^-, \] (5.9)

\[ u(x, T^*) = u_0(x), \quad u_t(x, T^*) = u_1(x) \] (5.10)
with the observation
\[ \mathcal{O}\{u_0, u_1\} = z(t) := (u(\cdot, t), g)_{L^2(0, l)}. \]  

(5.11)

**Proposition 2.** Operator \( \mathcal{O} \) is an isomorphism between \( W_{-1/2}^g \times W_{-3/2}^g \) and \( [H_{00}^{1/2}(0, T*)]' \).

**Proof.** The (unique) solution of (5.8)–(5.10) may be constructed as in Section 2. Let
\[ u_j(x) = \sum_{n \geq 0} u_n^j \psi_n(x), \quad j = 0, 1, \quad \text{and} \quad g(x) = \sum_{n \geq 0} g_n \psi_n(x). \]

It can be easily checked that
\[ u(x, t) = \sum_{n \geq 0} \left( u_0^0 \cos \omega_n(t - T) + u_0^1 \sin \omega_n(t - T) \right) \psi_n(x). \]  

(5.12)

Orthogonality condition (2.5) yields the representation for the observation (5.11),
\[ z(t) = \sum_{n \geq 0} g_n \left( u_0^0 \cos \omega_n(t - T) + u_0^1 \sin \omega_n(t - T) \right) \]
\[ = \sum_{n, \pm} g_n b_n^\pm e^{\pm i\omega_n(t - T)} = \sum_{n, \pm} g_n b_n^\pm \omega_n^{-1/2} e^{\pm i\omega_n(t - T)}. \]

(5.13)

where \( b_n^\pm := (1/2)(u_0^0 \pm u_0^1/(i\omega_n)) \). Since the family \( \{\omega_n^{1/2} e^{\pm i\omega_n t}\} \) forms a Riesz basis in \( [H_{00}^{1/2}(0, T*)]' \) (see Lemma 1(d)), so does the family \( \{\omega_n^{1/2} e^{\pm i\omega_n(t - T)}\} \). Hence (5.13) implies (see (4.2))
\[ \|z\|_{[H_{00}^{1/2}(0, T*)]'}^2 \geq \sum_{n, \pm} |g_n b_n^\pm \omega_n^{-1/2}|^2 \times \sum_n |g_n|^2 \left( |u_0^0 \omega_n^{-1/2}|^2 + |u_0^1 \omega_n^{-3/2}|^2 \right). \]  

(5.14)

The proposition is proved.

\( \square \)

Going back to the proof of Theorems 3 and 1(b), we suppose that in (2.3) \( y_0 = y_1 = 0 \) and functions \( f, g, u_0, \) and \( u_1 \) are smooth enough, so the initial boundary value problems (2.1)–(2.3) and (5.8)–(5.10) have classical solutions.

Multiplying (2.1) by \( u(x, t) \), integrating over \( Q_T = (0, l) \times (0, T) \), integrating by parts, and using (5.8)–(5.11) and (2.2) yields
\[ \rho \int_0^l \left[ y_1(x, T^*) u_0(x) - y(x, T^*) u_1(x) \right] dx = \int_0^T f(t) z(t) dt. \]  

(5.15)

For nonsmooth data this equality is understood as an equality between linear functionals acting on elements from dual spaces and can be written as
\[ \rho[C f, \{u_0, u_1\}] = [f, \mathcal{O}\{u_0, u_1\}], \]  

(5.16)

where \( C \) is the (control) operator: \( C f = [y_1(\cdot, T^*), -y(\cdot, T^*)] \). From (5.16), \( \rho C = \mathcal{O}^* \) and Proposition 2 implies that operator \( C \) is an isomorphism of \( H_{00}^{1/2}(0, T*) \) and \( W_{1/2}^g \times W_{3/2}^g \).

Basing on this result we are able now to complete the proofs of Theorems 1(b) and 3. First turn to Theorem 1(b). Due to Proposition 1 it suffice to prove its statement for zero initial conditions.
Suppose that \( t \leq T^* \). From (3.1), (3.3), (3.4), and (3.6) and using the definition of spaces \( W^{−g}_3 \times W^{−g}_1 \) it follows that

\[
\| \{ y(\cdot, t), y_t(\cdot, t) \} \|_{W^{−g}_3 \times W^{−g}_1}^2 \lesssim \sum_{n: g_n \neq 0} |(f, \Gamma_n^\pm)_{L^2(0,t)}|^2 \lambda_n^{1/2}
\]

\[
\lesssim \sum_{n: g_n \neq 0} |(\tilde{f}, \Gamma_n^\pm)_{L^2(0,T^*)}|^2 \lambda_n^{1/2}
\]

\[
\lesssim \| \{ y(\cdot, T^*), y_t(\cdot, T^*) \} \|_{W^{−g}_3 \times W^{−g}_1}^2.
\]

Here

\[
\tilde{f}(t) = \begin{cases} f(s) & \text{if } 0 \leq s \leq t, \\ 0 & \text{if } t < s \leq T^* < 0. \end{cases}
\]

Since we have just proved that operator \( C \) is an isomorphism, and so

\[
\{ y(\cdot, T^*), y_l(\cdot, T^*) \} \in W^{−g}_3 \times W^{−g}_1,
\]

it follows that

\[
\{ y(\cdot, t), y_l(\cdot, t) \} \in W^{−g}_3 \times W^{−g}_1.
\]

The same arguments as in proving Theorem 1(a) lead to the continuity of the state in this space. Taking into account Proposition 1, we conclude that the statement of Theorem 1(b) is valid for all \( t \).

We have proved that, for all \( T \), the reachability set \( \mathcal{R}(T) \)—set of states reachable from zero initial state with the help of controls from \( H^{1/2}_{00}(0, T) \)—belongs to \( W^{−g}_3 \times W^{−g}_1 \). From Proposition 2 we know that \( \mathcal{R}(T^*) = W^{−g}_3 \times W^{−g}_1 \). Therefore \( \mathcal{R}(T) = W^{−g}_3 \times W^{−g}_1 \) for \( T \geq T^* \).

Because of invertibility of time in Eq. (2.1), \( \mathcal{G}(T) = \mathcal{R}(T) \) for \( T \geq T^* \), that proves Theorem 3. \( \square \)

References