THE UNIVERSITY OF TENNESSEE CHATTANOOGA

Benchmarking with DEA

Introduction to Data Envelopment Analysis

September 12

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Benchmarking

Evaluate by comparison with a standard = Benchmarking

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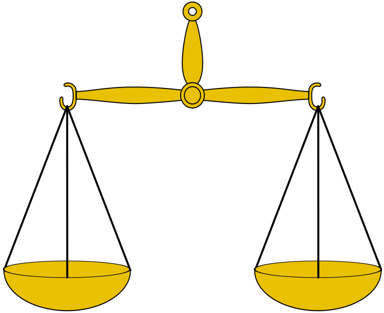
- Objective evaluation



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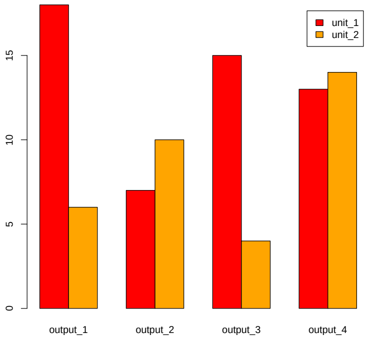
Benchmarking

Evaluate by comparison with a standard = Benchmarking

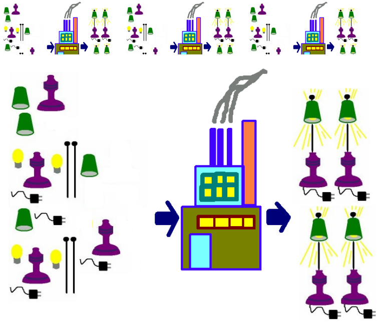
- Objective evaluation
- Relative or in comparasion to



Benchmarking

Evaluate by comparison with a standard = Benchmarking

- Objective evaluation
- Relative or in comparasion to
- Homogeneous results



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Motivation

- Improvement of the units

Motivation

- Improvement of the units
- Budget distribution

Motivation

- Improvement of the units
- Budget distribution
- Rewards establishment

Motivation

- Improvement of the units
- Budget distribution
- Rewards establishment
- Evaluation of the evolution

Results

- Knowledge

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- Knowledge
- Coordination

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- Knowledge
- Coordination
- Attribution

Results

- Knowledge
- Coordination
- Attribution
- Measures

Ratios

- Examples: $\dfrac{\text{profit}}{\text{investment}}, \dfrac{\text{sale}}{\text{agent}}$

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- Advantages: Easy calculation and interpretation

Ratios

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- Advantages: Easy calculation and interpretation
- Disadvantages: One-dimensionality, disparity of results, supposes there is no economy of scale

Frontier models

- One prefers: Higher outputs with lower inputs

Frontier models

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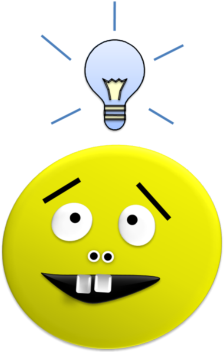
- One does not know: The best way to do that

Frontier models

- One prefers: Higher outputs with lower inputs
- One does not know: The best way to do that
- The way: Estimating the frontier



Frontier Models

Clasification

	Deterministic	Stochastic

Frontier Models
Clasification

	Deterministic	Stochastic
Parametric		
Non parametric		

Frontier Models

Clasification

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Parametric	COLS	
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- COLS = Corrected Ordinary Least Squares

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- DEA = Data Envelopment Analysis

Frontier Models
Clasification

	Deterministic	Stochastic
Parametric	COLS	SFA
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- COLS = Corrected Ordinary Least Squares
- SFA = Stochastic Frontier Analysis
- DEA = Data Envelopment Analysis
- SDEA = Stochastic Data Envelopment Analysis

Frontier models

Adventages and disadventages

- Non parametric: They are more flexible and do not need parameter estimation

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¿$\Longrightarrow$ SDEA?

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- Requires large data sets

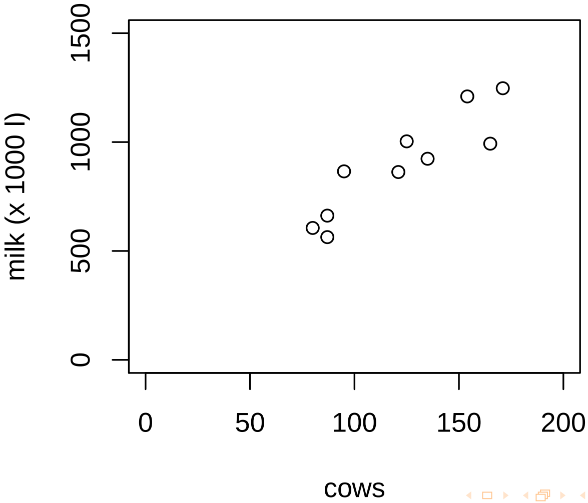
DEA and SFA are useful models and have many adventages over classical models

Technology: Given by data

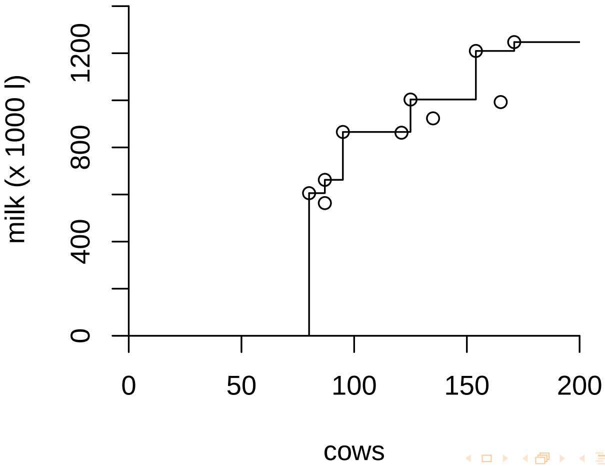
Data on milk production on livestock farms

	cows	milk
1	121	862.53
2	80	605.76
3	95	865.66
4	87	662.33
5	125	1003.44
6	135	923.51
7	87	563.68
8	171	1247.31
9	165	992.73
10	154	1209.69

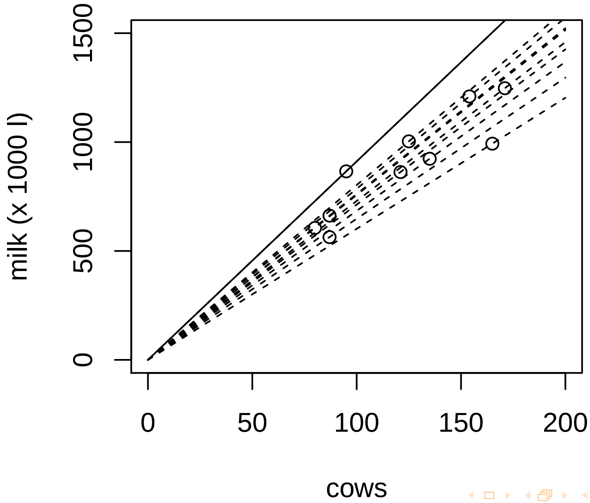
Technologic frontier



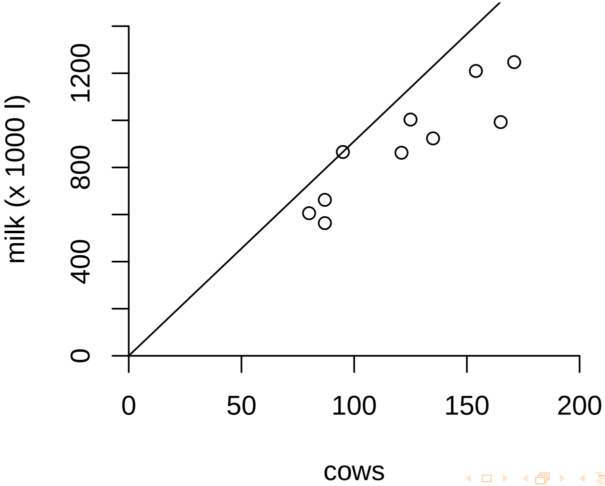
Technology: free disposal hull (fdh)



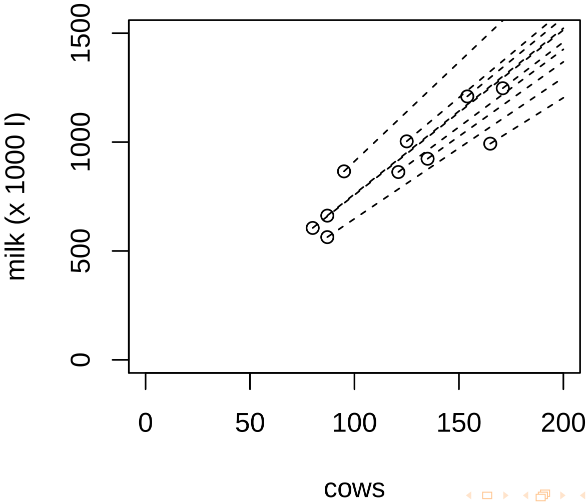
Technology: constant return to scale (crs)



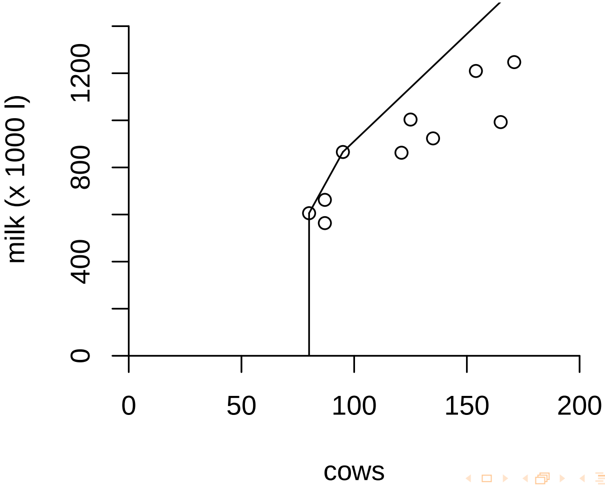
Frontier: crs technology



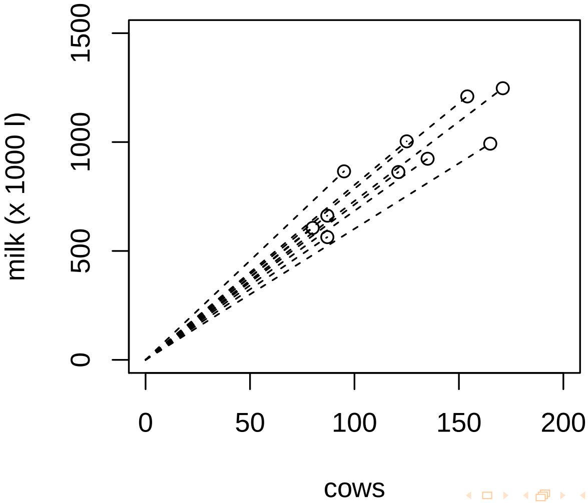
Tecnology: increasing return to scale (irs)



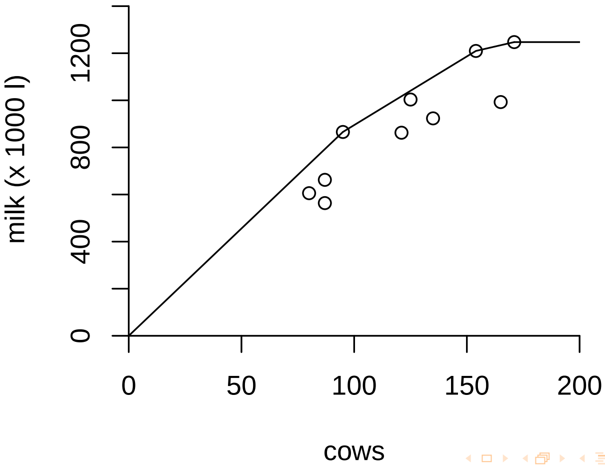
Frontier: irs technology



Tecnology: decreasing return to scale (drs)



Frontier: drs tecnology



Eficiency

	cows	milk
1	121	862.53
9	165	992.73
10	154	1209.69

- Farm 10 dominates to Farm 9
- Farms 1 and 9 do not dominate each other
- Farms 1 and 10 do not dominate each other

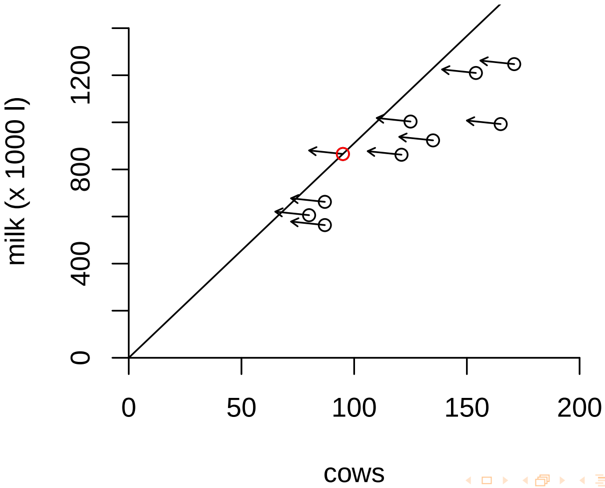
Definition

The non dominated units are said to be efficient in the sense of Pareto or in the sense of Koopmans.

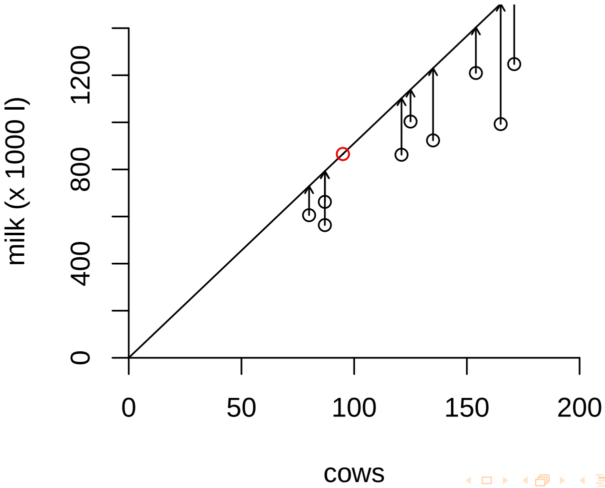
Remark

Efficient units are solutions to weighted problems and "reciprocally".

Technology dominance



Efficiency score: crs model



Efficiency score: crs model

	cows	milk	potencial	rate
1	121	862.53	1102.57	1.28
2	80	605.76	728.98	1.20
3	95	865.66	865.66	1.00
4	87	662.33	792.76	1.20
5	125	1003.44	1139.02	1.14
6	135	923.51	1230.15	1.33
7	87	563.68	792.76	1.41
8	171	1247.31	1558.18	1.25
9	165	992.73	1503.51	1.51
10	154	1209.69	1403.28	1.16

Definitions and DEA elements

DEA = Data Envelopment Analysis

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Elements:

- inputs

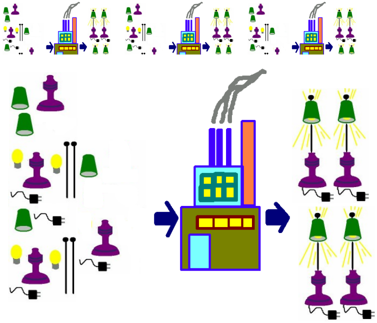

*Image from wikimedia

Definitions and DEA elements

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Elements:

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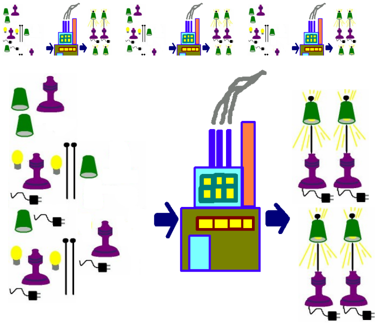

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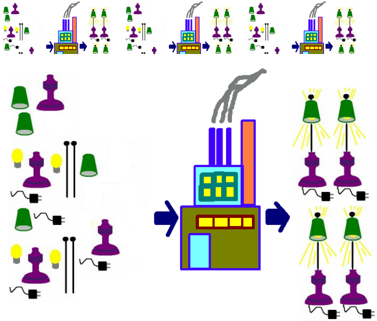
Definitions and DEA elements

DEA = Data Envelopment Analysis

Elements:

- inputs
- outputs
- DMU = Decision making units



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Definitions and DEA elements

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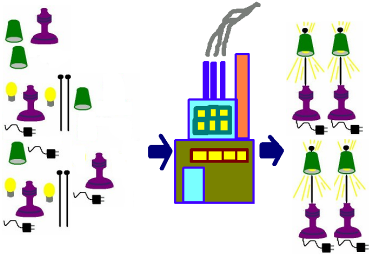

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The goal: Get the maximum amount of outputs using the minimum amount of the inputs.

Objetives of DEA

1. Identify the efficient DMUs
2. Get a rank of DMUs according to their efficiencies
3. Obtain the way that each DMU can be improve

DEA Methodology

Two convergent approaches:

1. Efficiency as a ration betwen outpus and inputs

DEA Methodology

Two convergent approaches:

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2. Efficiency is equivalent to scalar problems

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Score: Provides a score for each DMU.

Notation

- Inputs: x_{id} is the amount of input i used by DMU d.
- Outputs: y_{od} is the amount of output o produced by DMU d.
- Score: $s(d) = \frac{\sum_{o=1}^{n_O} u_{od} y_{od}}{\sum_{i=1}^{n_I} v_{id} x_{id}}$

Where u_{od} is the weight of output o in DMU d and v_{id} is the weight of input i in DMU d.

Remark

The numerator of the previous ratio is called virtual output and virtual input denominator.

CRS model formulation

The *CRS DEA model oriented to input* considers for each DMU, the following problem:

$$
\begin{aligned}
\max \quad & \sum_{o=1}^{n_O} u_o y_{o0} \\
\text{s.a} \quad & \\
& \sum_{i=1}^{n_I} v_i x_{i0} = 1 \\
& \sum_{o=1}^{n_O} u_o y_{od} \leq \sum_{i=1}^{n_I} v_i x_{id}, \quad \forall d = 1, 2, \ldots, n_D \\
& u_o, v_i \geq 0, \quad \forall o, \forall i
\end{aligned}
\qquad (P_0)
$$

where 0 is the evaluated unit.

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Fix the amount of input to 1.

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& u_o, v_i \geq 0, \quad \forall o, \forall i
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where 0 is the evaluated unit.

The score of each DMU must be under 1.

Example

We consider a set of libraries in Tokyo ("Data Envelopment Analysis", Cooper, Seiford and Tone), in which 23 DMUs with 4 inputs and 2 outputs are considered.

- These data appertain to public libraries located in 23 districts of the metropolitan area of Tokyo.
- As inputs we have: area, number of books, staff and population
- As outputs: number of people registered and books borrowed

`http://knuth.uca.es/shiny/DEA`

The selection problem

What happens if one drops population variable?

library	score4	rank4	score3	rank3
17	1.00	20.50	1.00	22.00
19	1.00	20.50	1.00	22.00
23	1.00	20.50	1.00	22.00
5	1.00	20.50	0.91	20.00
9	1.00	20.50	0.91	19.00
20	0.85	17.00	0.85	18.00
15	0.84	16.00	0.84	17.00
21	0.79	14.00	0.79	16.00

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library	score4	rank4	score3	rank3
17	1.00	20.50	1.00	22.00
19	1.00	20.50	1.00	22.00
23	1.00	20.50	1.00	22.00
5	1.00	20.50	0.91	20.00
9	1.00	20.50	0.91	19.00
20	0.85	17.00	0.85	18.00
15	0.84	16.00	0.84	17.00
21	0.79	14.00	0.79	16.00

So, one can move units up and down including or removing variables into the model

Significance measures

A significance measure has been defined in "Stepwise selection of variables in DEA using contribution loads". *Fernando Fernandez-Palacin, Maria Auxiliadora Lopez-Sanchez, M. Munoz-Marquez*. Pesquisa Operacional, 38:1, pg. 31-52. 2018, DOI: 10.1590/0101-7438.2018.038.01.0031.

Adventages

- The significance measure is objective
- It allows an automatic algorithm for variable selection
- It allows to compare diferent models

Global Model

$$\max \quad \sum_{d=1}^{n_D} \sum_{o=1}^{n_O} u_{od} y_{od}$$

s.a

$$\sum_{i=1}^{n_I} v_{id} x_{id} = 1, \quad \forall d = 1, 2, \ldots, n_D$$

$$\sum_{o=1}^{n_O} u_{oe} y_{od} \leq \sum_{i=1}^{n_I} v_{ie} x_{id}, \quad \forall e = 1, \ldots, n_D, \forall d = 1, \ldots, n_D$$

$$u_{od}, v_{id} \geq 0, \quad \forall o, \forall i, \forall d$$

$$(P)$$

$\bar{\alpha}$-ratios definition

For a set of inputs x and outputs y, u and v feasible weights for the global model, let:

$$\bar{\alpha}_i^I = \bar{\alpha}_i^I(u,v) = \frac{\sum_{d=1}^{n_D} v_{id} x_{id}}{\sum_{i=1}^{n_I} \sum_{d=1}^{n_D} v_{id} x_{id}} \qquad \text{para} \qquad i = 1, 2, \ldots, n_I$$

$$\bar{\alpha}_o^O = \bar{\alpha}_o^O(u,v) = \frac{\sum_{d=1}^{n_D} u_{od} y_{od}}{\sum_{o=1}^{n_O} \sum_{d=1}^{n_D} u_{od} y_{od}} \qquad \text{para} \qquad o = 1, 2, \ldots, n_O \tag{1}$$

Properties

$$\sum_{i=1}^{n_I} \bar{\alpha}_i^I = 1 \text{ and } 0 \leq \bar{\alpha}_i^I \leq 1, \qquad \forall i = 1, 2, \ldots, n_I$$
$$\sum_{o=1}^{n_O} \bar{\alpha}_o^O = 1 \text{ and } 0 \leq \bar{\alpha}_o^O \leq 1, \qquad \forall o = 1, 2, \ldots, n_O$$

Standarized definition:

$$\hat{\alpha}_i^I = \hat{\alpha}_i^I(u, v) = n_I \bar{\alpha}_i^I, \qquad \forall i = 1, 2, \ldots, n_I$$
$$\hat{\alpha}_o^O = \hat{\alpha}_o^O(u, v) = n_O \bar{\alpha}_o^O, \qquad \forall o = 1, 2, \ldots, n_O$$

$$
\begin{aligned}
\max \quad & \sum_{d=1}^{n_D} \sum_{o=1}^{n_O} u_{od} y_{od} + \epsilon(\hat{\alpha}_m^I + \hat{\alpha}_m^O) \\
\text{s.a} \quad & \\
& \sum_{i=1}^{n_I} v_{id} x_{id} = 1, \quad \forall d = 1, 2, \ldots, n_D \\
& \sum_{o=1}^{n_O} u_{oe} y_{od} \leq \sum_{i=1}^{n_I} v_{ie} x_{id}, \quad \forall e = 1, \ldots, n_D, \forall d = 1, \ldots, n_D \\
& \hat{\alpha}_i^I = \frac{n_I \sum_{d=1}^{n_D} v_{id} x_{id}}{\sum_{i=1}^{n_I} \sum_{d=1}^{n_D} v_{id} x_{id}}, \quad \forall i = 1, 2, \ldots, n_I \\
& \hat{\alpha}_o^O = \frac{n_O \sum_{d=1}^{n_D} u_{od} y_{od}}{\sum_{o=1}^{n_O} \sum_{d=1}^{n_D} u_{od} y_{od}}, \quad \forall i = 1, 2, \ldots, n_O \\
& 0 \leq \hat{\alpha}_m^I \leq \hat{\alpha}_i^I, \quad \forall i = 1, 2, \ldots, n_I \\
& 0 \leq \hat{\alpha}_m^O \leq \hat{\alpha}_o^O, \quad \forall o = 1, 2, \ldots, n_O \\
& u_{od}, v_{id} \geq 0, \quad \forall o, \forall i, \forall d
\end{aligned}
\qquad (P_{\hat{\alpha}})
$$

How to solve?

The problem can be solved in two steps:

- In the first step, P is solved and we get the scores.
- In the second step, the maximum value of $\hat{\alpha}$-ratios are computed taking the scores from the first step as constraints.

$$
\begin{aligned}
\max \quad & \alpha^I + \alpha^O \\
\text{s.a} \quad & \\
& \sum_{i=1}^{n_I} v_{id} x_{id} = 1, \quad \forall d = 1, 2, \ldots, n_D \\
& \sum_{o=1}^{n_O} u_{od} y_{od} \leq \sum_{i=1}^{n_I} v_i x_{id}, \quad \forall d = 1, 2, \ldots, n_D \\
& \sum_{o=1}^{n_O} u_{od} y_{od} = s(d), \quad \forall d = 1, 2, \ldots, n_D \\
& 0 \leq \alpha^I \leq \alpha_i^I = \frac{n_I}{n_D} \sum_{d=1}^{n_D} v_{id} x_{id}, \quad \forall i = 1, 2, \ldots, n_I \\
& 0 \leq \alpha^O \leq \alpha_o^O = \frac{n_O \sum_{d=1}^{n_D} u_{od} y_{od}}{\sum_{d=1}^{n_D} s(d)}, \quad \forall i = 1, 2, \ldots, n_O
\end{aligned}
\qquad (P_\alpha)
$$

Example of loads computation

The computed loads are:

Inputs

	Area	Books	Staff	Populations
First step	0.0553	1.4541	1.2858	1.2048
Second step	0.4011	1.3372	0.9922	1.2695

Outputs

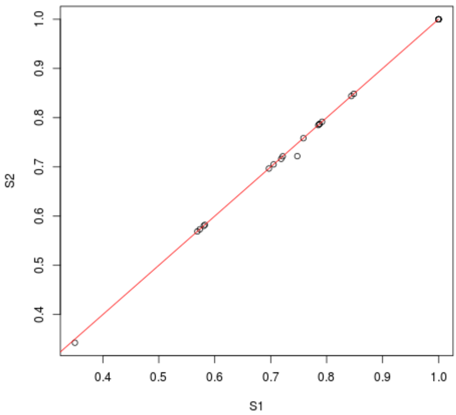
	Regist	Borrow
First step	0.7924	1.2076
Second step	1.0000	1.0000

Selecting variables in Tokyo data

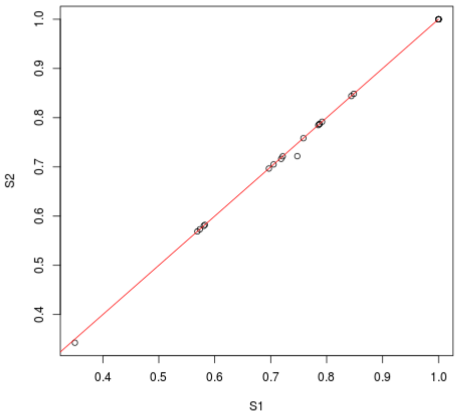
We consider three models:

- M1= Model with 4 inputs and 2 outputs
- M2= Model with 3 inputs and 2 outputs. The input "Area" has been dropped out
- M3= Model with 2 inputs and 2 outputs. The input "Books" has also dropped out

M1 vs M2 scores

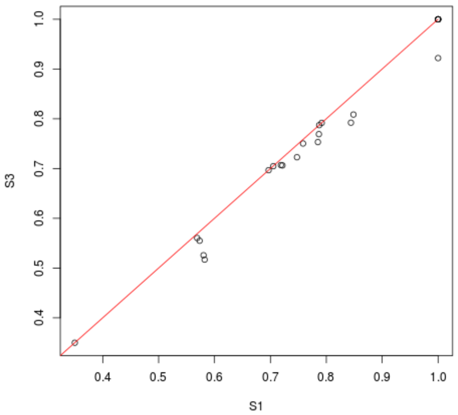


M1 vs M2 scores

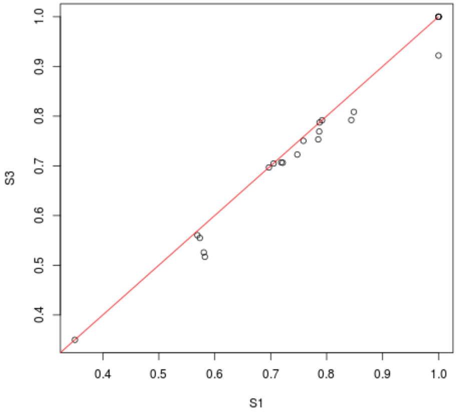


The load "Area" are low, 0.4011, and one can see little changes in scores

M1 vs M3 scores



M1 vs M3 scores



The load of "Books" are high, 1.3372, and one can see higher changes in scores

Changes in scores

	Maximum	Average
$\frac{\lvert S1 - S2 \rvert}{S1}$	0.034	0.002
$\frac{\lvert S1 - S3 \rvert}{S1}$	0.112	0.025

Conclusions

1. The significance measures introduced consistently measure the contribution of each input and each output to the total measure of efficiency.
2. These measures verify all the desirable properties for them.
3. An automatic procedure of selection of inputs and outputs variables has been established.

Future work

1. Continue the development of the software
2. Make a full computational study (in progress)
3. Extend the results to other DEA models

References

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Thank you for your attention

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