TRACE IDEAL PROPERTIES OF A CLASS OF INTEGRAL OPERATORS

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Dedicated with admiration to Emma Previato – Geometer Extraordinaire

Abstract. We consider a particular class of integral operators $T_{\gamma,\delta}$ in $L^2(\mathbb{R}^n)$, $n \in \mathbb{N}$, $n \geqslant 2$, with integral kernels $T_{\gamma,\delta}(\,\cdot\,,\,\cdot\,)$ bounded (Lebesgue) a.e. by

$$|T_{\gamma,\delta}(x,y)| \leqslant C\langle x\rangle^{-\delta}|x-y|^{2\gamma-n}\langle y\rangle^{-\delta}, \quad x,y \in \mathbb{R}^n,\ x \neq y,$$

for fixed $C \in (0,\infty)$, $0 < 2\gamma < n$, $\delta > \gamma$, and prove that

$$T_{\gamma,\delta} \in \mathcal{B}_p\big(L^2(\mathbb{R}^n)\big) \text{ for } p > n/(2\gamma),\ p \geqslant 2.$$

(Here $\langle x\rangle := (1+|x|^2)^{1/2}$, $x \in \mathbb{R}^n$, and $\mathcal{B}_p$ abbreviates the ℓ^p-based trace ideal.) These integral operators (and their matrix-valued analogs) naturally arise in the study of multi-dimensional Schrödinger and Dirac-type operators and we describe an application to the case of massless Dirac-type operators.

1. Introduction

The principal aim of this paper is to derive trace ideal properties of a class of (matrix-valued) integral operators that naturally arise in the context of multi-dimensional Schrödinger and Dirac-type operators. More precisely, we will focus on integral operators $T_{\gamma,\delta}$ in $L^2(\mathbb{R}^n)$, $n \in \mathbb{N}$, $n \geqslant 2$, which in the scalar context are associated with integral kernels $T_{\gamma,\delta}(\,\cdot\,,\,\cdot\,)$ that are bounded (Lebesgue) a.e. by

$$|T_{\gamma,\delta}(x,y)| \leqslant C\langle x\rangle^{-\delta}|x-y|^{2\gamma-n}\langle y\rangle^{-\delta}, \quad x,y \in \mathbb{R}^n,\ x \neq y, \tag{1.1}$$

for fixed $C \in (0,\infty)$, $0 < 2\gamma < n$, $\delta > \gamma$. We then prove in Theorem 3.1 that

$$T_{\gamma,\delta} \in \mathcal{B}_p\big(L^2(\mathbb{R}^n)\big) \text{ for } p > n/(2\gamma),\ p \geqslant 2. \tag{1.2}$$

This result is then applied to prove Theorem 3.4 which derives a uniform trace ideal norm bound with respect to the spectral parameter on the resolvent of massless Dirac-type operators if the Dirac-type resolvent is viewed as a map between suitable weighted $L^2(\mathbb{R}^n)$ spaces. Such estimates are well-known to imply limiting absorption principles for the Dirac-type operator in question which, in turn, have strong spectral implications (such as the absence of any singular continuous spectrum, etc.). Moreover, the fact that trace ideal bounds are involved can now be used to infer continuity properties of underlying spectral shift functions which yields

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further applications to the Witten index for a particular class of non-Fredholm operators as discussed, for instance, in [3]–[9], [12], [24] (see also Remark 3.5). We note that two-dimensional massless Dirac-type operators are also known to be relevant in the context of graphene, one more reason to study the massless case.

In Section 2 we collect a fair amount of background material, some of which is crucial for our main Section 3. In particular, we focus on integral operators with integral kernels closely related to the right-hand side of (1.1) and survey some of the pertinent literature in this context, including a fundamental criterion by Nirenberg and Walker [22] (we include a detailed proof of the latter), a result on absolute kernels, some well-known Schur tests, and a trace norm estimate due to Demuth, Stollmann, Stolz, and van Casteren [10] (again we supply the proof of the latter). Section 2 also recalls a version of the Sobolev inequality and a fundamental trace ideal interpolation result. Our principal Section 3 then proves the inclusion (1.2) in Theorem 3.1 and demonstrates its applicability to the case of massless Dirac-type operators in Theorem 3.4. Finally, Appendix A collects some useful results on pointwise domination of linear operators and its consequences in connection with boundedness, compactness, and Hilbert–Schmidt properties. We include a discussion of block matrix operator situations necessitated by the study of Dirac-type operators.

We conclude this introduction with some comments on the notation employed in this paper: Let $\mathcal{H}$ be a separable complex Hilbert space, $(\cdot,\cdot)_{\mathcal{H}}$ the scalar product in $\mathcal{H}$ (linear in the second argument), and $I_{\mathcal{H}}$ the identity operator in $\mathcal{H}$.

Next, if T is a linear operator mapping (a subspace of) a Hilbert space into another, then $\operatorname{dom}(T)$ and $\ker(T)$ denote the domain and kernel (i.e., null space) of T. The spectrum, point spectrum (the set of eigenvalues), the essential spectrum of a closed linear operator in $\mathcal{H}$ will be denoted by $\sigma(\,\cdot\,)$, $\sigma_p(\,\cdot\,)$, $\sigma_{ess}(\,\cdot\,)$, respectively. Similarly, the absolutely continuous and singularly continuous spectrum of a self-adjoint operator in $\mathcal{H}$ are denoted by $\sigma_{ac}(\,\cdot\,)$ and $\sigma_{sc}(\,\cdot\,)$.

The Banach spaces of bounded and compact linear operators on a separable complex Hilbert space $\mathcal{H}$ are denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_\infty(\mathcal{H})$, respectively; the corresponding ℓ^p-based trace ideals will be denoted by $\mathcal{B}_p(\mathcal{H})$, their norms are abbreviated by $\|\cdot\|_{\mathcal{B}_p(\mathcal{H})}$, $p\in[1,\infty)$. Moreover, $\operatorname{tr}_{\mathcal{H}}(A)$ denotes the corresponding trace of a trace class operator $A\in\mathcal{B}_1(\mathcal{H})$.

If $p\in[1,\infty)\cup\{\infty\}$, then $p'\in[1,\infty)\cup\{\infty\}$ denotes its conjugate index, that is, $p':=(1-1/p)^{-1}$. If Lebesgue measure is understood, we simply write $L^p(M)$, $M\subseteq\mathbb{R}^n$ measurable, $n\in\mathbb{N}$, instead of the more elaborate notation $L^p(M;d^nx)$. For $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, $n\in\mathbb{N}$, we abbreviate $\langle x\rangle:=(1+|x|^2)^{1/2}$.

Finally, $\lfloor\cdot\rfloor$ denotes the floor function on $\mathbb{R}$, that is, $\lfloor x\rfloor$ characterizes the largest integer less than or equal to $x\in\mathbb{R}$.

2. Some Background Material

This preparatory section is primarily devoted to various results of integral operators, but we also briefly recall Sobolev's inequality and some interpolation results for trace ideal operators.

We start by recalling the following version of Sobolev's inequality (see, e.g., [26, Corollary I.14]), to be employed in the proof of Theorem 3.1.

Theorem 2.1. *Let $n \in \mathbb{N}$, $r,s \in (1,\infty)$, $0 < \lambda < n$, $r^{-1} + s^{-1} + \lambda n^{-1} = 2$, $f \in L^r(\mathbb{R}^n)$, $h \in L^s(\mathbb{R}^n)$. Then, there exists $C_{r,s,\lambda,n} \in (0,\infty)$ such that*

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} d^n x\, d^n y\, \frac{|f(x)||h(y)|}{|x-y|^\lambda} \leqslant C_{r,s,\lambda,n} \|f\|_{L^r(\mathbb{R}^n)} \|h\|_{L^s(\mathbb{R}^n)}. \tag{2.1}$$

We continue this section with a special case of a very interesting result of Nirenberg and Walker [22] also to be employed in the proof of Theorem 3.1. For convenience of the reader we offer a detailed proof.

Theorem 2.2. *Let $n \in \mathbb{N}$, $c,d \in \mathbb{R}$, $c+d > 0$, and consider*

$$K_{c,d}(x,y) = |x|^{-c}|x-y|^{(c+d)-n}|y|^{-d}, \quad x,y \in \mathbb{R}^n,\ x \neq x'. \tag{2.2}$$

Then the integral operator $K_{c,d}$ in $L^2(\mathbb{R}^n)$ with integral kernel $K_{c,d}(\,\cdot\,,\,\cdot\,)$ in (2.2) is bounded if and only if

$$c < n/2 \quad \text{and} \quad d < n/2. \tag{2.3}$$

Proof. To prove the necessity of the conditions (2.3), assume $K \in \mathcal{B}(L^2(\mathbb{R}^n))$. Then

$$\int_{\mathbb{R}^n} d^n y\, K_{c,d}(\,\cdot\,,y) f(y) \in L^2(\mathbb{R}^n), \quad f \in L^2(\mathbb{R}^n). \tag{2.4}$$

In particular, choosing $f = \chi_{\overline{B_n(0;1)}}$, the characteristic function of the closed unit ball in $\mathbb{R}^n$,

$$\overline{B_n(0;1)} = \{x \in \mathbb{R}^n \,|\, |x| \leqslant 1\}, \tag{2.5}$$

which has finite Lebesgue measure, $\big|\overline{B_n(0;1)}\big| = \pi^{n/2}/\Gamma((n/2)+1)$, one obtains

$$\int_{\overline{B_n(0;1)}} d^n y\, K_{c,d}(\,\cdot\,,y) \in L^2(\mathbb{R}^n). \tag{2.6}$$

Since $K_{c,d}(\,\cdot\,,\,\cdot\,)$ is symmetric in x and y, one also infers

$$\int_{\overline{B_n(0;1)}} d^n x\, K_{c,d}(x,\,\cdot\,) \in L^2(\mathbb{R}^n). \tag{2.7}$$

In summary, if $K_{c,d} \in \mathcal{B}(L^2(\mathbb{R}^n))$, then

$$\int_{\overline{B_n(0;1)}} d^n y\, K_{c,d}(\,\cdot\,,y) \in L^2(\mathbb{R}^n) \quad \text{and} \quad \int_{\overline{B_n(0;1)}} d^n x\, K_{c,d}(x,\,\cdot\,) \in L^2(\mathbb{R}^n). \tag{2.8}$$

To investigate the behavior of

$$\int_{\overline{B_n(0;1)}} \frac{d^n y}{|x|^c |x-y|^{n-c-d} |y|^d}, \quad x \in \mathbb{R}^n \backslash \{0\}, \tag{2.9}$$

as $|x| \to \infty$, one writes

$$\int_{\overline{B_n(0;1)}} \frac{d^n y}{|x|^c |x-y|^{n-c-d} |y|^d} = \frac{1}{|x|^{n-d}} \int_{\overline{B_n(0;1)}} \frac{d^n y}{\left|\dfrac{x}{|x|} - \dfrac{y}{|x|}\right|^{n-c-d} |y|^d}, \quad x \in \mathbb{R}^n \backslash \{0\}. \tag{2.10}$$

If $x,y \in \mathbb{R}^n$ with $|x| \geqslant 2$ and $|y| \leqslant 1$, then the elementary estimates

$$\frac{1}{2} \leqslant 1 - \frac{1}{|x|} \leqslant \left|\frac{x}{|x|} - \frac{y}{|x|}\right| \leqslant 1 + \frac{1}{|x|} \leqslant \frac{3}{2} \tag{2.11}$$

imply

$$C_1 \leqslant \left|\frac{x}{|x|} - \frac{y}{|x|}\right|^{c+d-n} \leqslant C_2, \quad x, y \in \mathbb{R}^n,\, |x| \geqslant 2,\, |y| \leqslant 1, \tag{2.12}$$

for some constants $C_1, C_2 \in (0,\infty)$. In particular, the finiteness of the integral in (2.9) implies the finiteness of the integral

$$\int_{B_n(0;1)} \frac{d^n y}{|y|^d}. \tag{2.13}$$

By Lebesgue's dominated convergence theorem,

$$\begin{aligned} \lim_{|x|\to\infty} \int_{B_n(0;1)} \frac{d^n y}{\left|\frac{x}{|x|} - \frac{y}{|x|}\right|^{n-c-d} |y|^d} &= \int_{B_n(0;1)} d^n y \lim_{|x|\to\infty} \frac{1}{\left|\frac{x}{|x|} - \frac{y}{|x|}\right|^{n-c-d} |y|^d} \\ &= \int_{B_n(0;1)} \frac{d^n y}{|y|^d} =: I_{1,n,d}, \end{aligned} \tag{2.14}$$

since (2.11) implies

$$\lim_{|x|\to\infty} \left|\frac{x}{|x|} - \frac{y}{|x|}\right| = 1, \quad y \in \mathbb{R}^n. \tag{2.15}$$

Therefore, by (2.10) and (2.14),

$$\int_{B_n(0;1)} \frac{d^n y}{|x|^c |x-y|^{n-c-d} |y|^d} \sim I_{1,n,d} \cdot \frac{1}{|x|^{n-d}} \quad \text{as } |x| \to \infty, \tag{2.16}$$

and, similarly,

$$\int_{B_n(0;1)} \frac{d^n x}{|x|^c |x-y|^{n-c-d} |y|^d} \sim I_{1,n,c} \cdot \frac{1}{|y|^{n-c}} \quad \text{as } |y| \to \infty. \tag{2.17}$$

In light of (2.16) and (2.17), the containments in (2.8) hold only if $(n-d)2 > n$ and $(n-c)2 > n$, that is, only if $c < n/2$ and $d < n/2$.

To prove sufficiency of the conditions in (2.3), assume $c < n/2$ and $d < n/2$. It suffices to prove the claim $K_{c,d} \in \mathcal{B}(L^2(\mathbb{R}^n))$ in the special case where $c, d \in [0,\infty)$. The claim for general c and d then follows from this special case. Indeed, if c were negative, for example, then d would be positive, and the elementary inequality

$$\frac{|x|}{|x-y|} \leqslant 1 + \frac{|y|}{|x-y|}, \quad x, y \in \mathbb{R}^n,\, x \neq y, \tag{2.18}$$

implies

$$\begin{aligned} K_{c,d}(x,y) &= \left(\frac{|x|}{|x-y|}\right)^{-c} \frac{1}{|x-y|^{n-d}|y|^d} \leqslant M_{-c}\left(1 + \frac{|y|^{-c}}{|x-y|^{-c}}\right) \frac{1}{|x-y|^{n-d}|y|^d} \\ &= \frac{M_{-c}}{|x-y|^{n-d}|y|^d} + \frac{M_{-c}}{|x-y|^{n-(c+d)}|y|^{c+d}}, \quad x, y \in \mathbb{R}^n\backslash\{0\},\, x \neq y, \end{aligned} \tag{2.19}$$

where for each $\alpha \in [0,\infty)$, $M_\alpha \in (0,\infty)$ is a constant such that

$$(1+t)^\alpha \leqslant M_\alpha (1 + t^\alpha), \quad t \in [0,\infty). \tag{2.20}$$

Note that the existence of M_α is guaranteed by the fact that, for each fixed $\alpha \in [0,\infty)$, the function

$$\phi_\alpha(t) = \frac{(1+t)^\alpha}{1+t^\alpha}, \quad t \in [0,\infty), \tag{2.21}$$

is continuous on $[0,\infty)$ and has a finite limit as $t\to\infty$. The special case under consideration (viz., $c,d\in[0,\infty)$) then implies that the right-hand side of (2.19) is the sum of the kernels of two integral operators in $\mathcal{B}(L^2(\mathbb{R}^n))$, so that $K_{c,d}(\,\cdot\,,\,\cdot\,)$ generates a bounded operator on $L^2(\mathbb{R}^n)$. Therefore, for the remainder of this proof, we assume $0\leqslant c<n/2$ and $0\leqslant d<n/2$.

By the arithmetic-geometric mean inequality,

$$|x|\geqslant\prod_{j=1}^n|x_j|^{1/n},\quad x=(x_1,\dots,x_n)\in\mathbb{R}^n, \tag{2.22}$$

implying

$$K_{c,d}(x,y)\leqslant\prod_{j=1}^n\frac{1}{|x_j|^{c/n}|x_j-y_j|^{1-(c+d)/n}|y_j|^{d/n}}, \tag{2.23}$$
$$x=(x_j)_{j=1}^n, y=(y_j)_{j=1}^n\in\mathbb{R}^n,\ x_j\neq y_j,\ x_j\neq 0,\ y_j\neq 0,\ 1\leqslant j\leqslant n.$$

Therefore, by Lemma A.4, it suffices to show that the integral operator $J_{c,d}$ with integral kernel

$$J_{c,d}(s,t)=\frac{1}{|s|^{c/n}|s-t|^{1-(c+d)/n}|t|^{d/n}},\quad s,t\in\mathbb{R}\backslash\{0\},\ s\neq t, \tag{2.24}$$

belongs to $\mathcal{B}(L^2(\mathbb{R}))$.

The function $J_{c,d}(\,\cdot\,,\,\cdot\,)$ defined by (2.24) is homogeneous of degree (-1). In addition,

$$\int_0^\infty ds\,J(s,1)\,s^{-1/2}=\int_0^\infty\frac{ds}{s^{(1/2)+(c/n)}|s-1|^{1-(c+d)/n}}<\infty, \tag{2.25}$$

as the integrand on the right-hand side in (2.25) behaves like a constant times $s^{-[(1/2)+(c/n)]}$ $\big($resp., $|s-1|^{(c+d)/n-1}\big)$ as $s\to 0+$ (resp., $s\to 1$) and decays like a constant times $s^{(d/n)-(3/2)}$ as $s\to\infty$. Therefore, by Lemma A.5, the restriction of $J_{c,d}(\,\cdot\,,\,\cdot\,)$ to $(0,\infty)\times(0,\infty)$ generates an integral operator $J_{c,d,+}\in\mathcal{B}(L^2((0,\infty)))$. In particular,

$$\begin{aligned}
&\int_{\mathbb{R}}dx\left|\int_{\mathbb{R}}dy\,J_{c,d}(x,y)f(y)\right|^2\\
&\quad\leqslant\int_0^\infty dx\left(\int_0^\infty dy\,J_{c,d}(x,y)|f(-y)|\right)^2+\int_0^\infty dx\left(\int_0^\infty dy\,J_{c,d}(x,y)|f(-y)|\right)^2\\
&\qquad+\int_0^\infty dx\left(\int_0^\infty dy\,J_{c,d}(x,y)|f(y)|\right)^2+\int_0^\infty dx\left(\int_0^\infty dy\,J_{c,d}(x,y)|f(y)|\right)^2\\
&\quad\leqslant 4\|J_{c,d,+}\|_{\mathcal{B}(L^2((0,\infty)))}\|f\|^2_{L^2(\mathbb{R})},\quad f\in L^2(\mathbb{R}),
\end{aligned} \tag{2.26}$$

which proves $J_{c,d}\in\mathcal{B}(L^2(\mathbb{R}))$. To obtain (2.26), we have employed the elementary fact

$$\max\Big\{\big\|f|_{(0,\infty)}\big\|_{L^2((0,\infty))},\big\|f|_{(-\infty,0)}\big\|_{L^2((-\infty,0))}\Big\}\leqslant\|f\|_{L^2(\mathbb{R})},\quad f\in L^2(\mathbb{R}). \tag{2.27}$$

□

The following result is discussed in connection with the notion of absolute kernels in [18, p. 271].

Theorem 2.3. *Let $n \in \mathbb{N}$ and $\beta \in (0,n)$. Suppose $p_0, q_0, p, q \in (1,\infty)$ with*

$$\frac{1}{p'} + \frac{1}{p_0} < 1, \quad \frac{1}{q} + \frac{1}{q_0} < 1, \quad \frac{1}{p} = \frac{1}{q} + \frac{1}{p_0} + \frac{1}{q_0} - \frac{\beta}{n}. \tag{2.28}$$

If

$$a \in L^{p_0}(\mathbb{R}^n) \quad \text{and} \quad b \in L^{q_0}(\mathbb{R}^n), \tag{2.29}$$

then the kernel

$$k(x,y) = a(x)|x-y|^{\beta-n} b(y) \text{ for a.e. } x,y \in \mathbb{R}^n \tag{2.30}$$

generates a bounded integral operator $K \in \mathcal{B}(L^q(\mathbb{R}^n), L^p(\mathbb{R}^n))$.

While Theorem 2.3 permits a variety of functions a and b, it does not apply to the kernel $K_{c,d}$ in (2.2) due to the integrability requirements in (2.29).

Theorem 2.2 gives necessary and sufficient conditions for boundedness of the integral operator $K_{c,d}$. In general, there are no known practical necessary and sufficient conditions for the boundedness of an integral operator. However, there are various sufficient conditions which allow one to infer boundedness of an integral operator from appropriate bounds on the integral kernel itself. Illustrative examples are the well-known *Schur criteria* or *Schur tests* to which we briefly turn next for the sake of completeness.

The following well-known version of the Schur test (cf., e.g., [16, Theorem 5.2]) provides a sufficient condition for boundedness between L^2-spaces in terms of pointwise bounds on the integral kernel when integrated against a pair of measurable trial functions.

Theorem 2.4 (Schur test–first version)**.** *Let $(X, \mathcal{M}, d\mu)$ and $(Y, \mathcal{N}, d\nu)$ be σ-finite measure spaces and $k : X \times Y \to [0,\infty)$ a measurable function. If $\phi : X \to (0,\infty)$ and $\psi : Y \to (0,\infty)$ are measurable and if $\alpha, \beta \in (0,\infty)$ are such that*

$$\int_Y d\nu(y)\, k(x,y)\psi(y) \leqslant \alpha\phi(x) \text{ for a.e. } x \in X \tag{2.31}$$

and

$$\int_X d\mu(x)\, k(x,y)\phi(x) \leqslant \alpha\psi(y) \text{ for a.e. } y \in Y, \tag{2.32}$$

then k is the integral kernel of a bounded integral operator

$$K \in \mathcal{B}\big(L^2(Y; d\nu), L^2(X; d\mu)\big) \tag{2.33}$$

and

$$\|K\|_{\mathcal{B}(L^2(Y;d\nu),L^2(X;d\mu))} \leqslant (\alpha\beta)^{1/2}. \tag{2.34}$$

Proof. If $f \in L^2(Y; d\nu)$, then using the Cauchy–Schwarz inequality one obtains

$$\begin{aligned}
&\int_X d\mu(x) \left(\int_Y d\nu(y)\, k(x,y)|f(y)| \right) \\
&\quad = \int_X d\mu(x) \left(\int_Y d\nu(y)\, k(x,y)^{1/2}\psi(y)^{1/2} \left[\frac{k(x,y)}{\psi(y)}\right]^{1/2} |f(y)| \right)^2 \\
&\quad \leqslant \int_X d\mu(x) \left(\int_Y d\nu(y\, k(x,y)\psi(y)) \right) \left(\int_Y d\nu(y')\, \frac{k(x,y')}{\psi(y')} |f(y')|^2 \right) \\
&\quad \leqslant \int_X d\mu(x)\, \alpha\phi(x) \left(\int_Y d\nu(y)\, \frac{k(x,y)}{\psi(y)} |f(y)|^2 \right)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha \int_Y d\nu(y)\, \frac{|f(y)|^2}{\psi(y)} \bigg(\int_X d\mu(x)\, k(x,y)\phi(x) \bigg) \\
&\leqslant \alpha \int_Y d\nu(y)\, \frac{|f(y)|^2}{\psi(y)} \beta\psi(y) \\
&= \alpha\beta \int_Y d\nu(y)\, |f(y)|^2. \qquad (2.35)
\end{aligned}
$$

□

Example 2.5 (Abel kernel)**.** *The integral operator K in $L^2((0,1))$ generated by the kernel*

$$
k(x,y) = \begin{cases} 0, & x \leqslant y, \\ (x-y)^{-1/2}, & y < x, \end{cases} \qquad (2.36)
$$

belongs to $\mathcal{B}(L^2((0,1)))$. In fact, the Schur test applies with $\psi(x) = \phi(x) = 1$ for a.e. $x \in (0,1)$ and $\alpha = \beta = 2$.

The proof of the following L^p-based version of the Schur test relies on Hölder's inequality (cf., e.g., [29, Satz 6.9]).

Theorem 2.6 (Schur test–second version)**.** *Let $p, p' \in (1,\infty)$ with $p^{-1} + (p')^{-1} = 1$ and let $(X, \mathcal{M}, d\mu)$ and $(Y, \mathcal{N}, d\nu)$ be σ-finite measure spaces. Suppose $k : X \times Y \to \mathbb{C}$ is a measurable function and that there exist measurable functions $k_1, k_2 : X \times Y \to [0,\infty)$ such that*

$$
|k(x,y)| \leqslant k_1(x,y) k_2(x,y) \ \text{ for a.e. } (x,y) \in X \times Y, \qquad (2.37)
$$

and

$$
\|k_1(x,\,\cdot\,)\|_{L^{p'}(Y;d\nu)} \leqslant C_1, \quad \|k_2(\,\cdot\,,y)\|_{L^p(X;d\mu)} \leqslant C_2, \qquad (2.38)
$$

for μ-a.e. $x \in X$ and ν-a.e. $y \in Y$ for some constants $C_1, C_2 \in (0,\infty)$. Then k is the integral kernel of a bounded integral operator

$$
K \in \mathcal{B}\big(L^p(Y;d\nu), L^p(X;d\mu)\big) \qquad (2.39)
$$

and

$$
\|K\|_{\mathcal{B}(L^p(Y;d\nu), L^p(X;d\mu))} \leqslant C_1 C_2. \qquad (2.40)
$$

While Theorems 2.4 and 2.6 provide useful sufficient conditions for an integral operator to be bounded over an L^p-space, they do not yield information about possible compactness or trace ideal properties of the integral operator. (For compactness properties, see, e.g., [16, § 13, 14], [18, § 11], [20, Ch. 2], [28, Sect. 6.3], [32, Ch. V].) In particular, neither Theorem 2.4 nor Theorem 2.6 implies the trace ideal property 1.2. As an example of a result which provides sufficient conditions for an integral operator to belong to the trace class, we mention the following result on general integral operators due to [10] and provide its short proof.

Theorem 2.7. *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and suppose that $A(\,\cdot\,,\,\cdot\,)$, $B(\,\cdot\,,\,\cdot\,) : X \times X \to \mathbb{C}$ are measurable such that*

$$
\begin{aligned}
& A(\,\cdot\,,x), B(x,\,\cdot\,) \in L^2(X;d\mu) \ \text{ for a.e. } x \in X, \\
& \int_X d\mu(y)\, \|A(\,\cdot\,,y)\|_{L^2(X;d\mu)} \|B(y,\,\cdot\,)\|_{L^2(X;d\mu)} < \infty.
\end{aligned} \qquad (2.41)
$$

Then there exists a trace class operator $AB : L^2(X; d\mu) \to L^2(X; d\mu)$ with integral kernel

$$AB(x, y) = \int_X d\mu(t)\, A(x, t)B(t, y), \tag{2.42}$$

such that

$$\|AB\|_{\mathcal{B}_1(L^2(X;d\mu))} \leqslant \|A(\,\cdot\,, y)\|_{L^2(X;d\mu)} \|B(y,\,\cdot\,)\|_{L^2(X;d\mu)}. \tag{2.43}$$

Proof. Introducing $g(y) := \|B(y,\,\cdot\,)\|_{L^2(X;d\mu)}$, $h(y) := \|A(\,\cdot\,, y)\|_{L^2(X;d\mu)}$, and employing the (unusual) convention $g(x)^{-1} = 0$ if $g(x) = 0$, we denote by M_f the maximally defined operator of multiplication by f in the space $L^2(X; d\mu)$. Then

$$AB = AM_{h^{-1}} M_{(hg)^{1/2}} M_{(hg)^{1/2}} M_{g^{-1}} B, \tag{2.44}$$

and $AM_{h^{-1}} M_{(hg)^{1/2}}$ and $M_{(hg)^{1/2}} M_{g^{-1}} B$ are seen to be Hilbert–Schmidt operators. For instance,

$$\begin{aligned}\|AM_{h^{-1}} M_{(hg)^{1/2}}\|^2_{\mathcal{B}_2(L^2(X;d\mu))} &= \int_X d\mu(x) \int_X d\mu(y) \left|A(x, y)h(y)^{-1}(hg)(x)^{1/2}\right|^2 \\ &= \int_X d\mu(x) \int_X d\mu(y) \left|A(x, y)h(y)^{-1/2}(g)(x)^{1/2}\right|^2 \\ &= \int_X d\mu(y)\, g(y)h(y) < \infty. \end{aligned} \tag{2.45}$$

□

One easily verifies that inequality (2.43) becomes an equality for rank-one operators A, B in $L^2(\mathbb{R}^n)$ generated by Lebesgue-a.e. nonnegative functions.

While Theorem 2.7 provides a positive result for a certain class of integral operators, it is limited in scope to the trace class. Therefore, Theorem 2.7 is not well-suited for application to (1.2), owing to the condition $p > n/(2\gamma)$ in (1.2). To circumvent these difficulties, we shall employ interpolation methods in Section 3 to prove (1.2). In particular, we will make use of the following trace ideal interpolation result, see, for instance, [14, Theorem III.13.1], [31, Theorem 0.2.6] (see also [13], [15, Theorem III.5.1]) in the proof of (1.2) (cf. Theorem 3.1).

Theorem 2.8. *Let $p_j \in [1, \infty) \cup \{\infty\}$, $\Sigma = \{\zeta \in \mathbb{C} \,|\, \mathrm{Re}(\zeta) \in (\xi_1, \xi_2)\}$, $\xi_j \in \mathbb{R}$, $\xi_1 < \xi_2$, $j = 1, 2$. Suppose that $A(\zeta) \in \mathcal{B}(\mathcal{H})$, $\zeta \in \overline{\Sigma}$ and that $A(\,\cdot\,)$ is analytic on Σ, continuous up to $\partial\Sigma$, and that $\|A(\,\cdot\,)\|_{\mathcal{B}(\mathcal{H})}$ is bounded on $\overline{\Sigma}$. Assume that for some $C_j \in (0, \infty)$,*

$$\sup_{\eta\in\mathbb{R}} \|A(\xi_j + i\eta)\|_{\mathcal{B}_{p_j}(\mathcal{H})} \leqslant C_j, \quad j = 1, 2. \tag{2.46}$$

Then

$$A(\zeta) \in \mathcal{B}_{p(\mathrm{Re}(\zeta))}(\mathcal{H}), \quad \frac{1}{p(\mathrm{Re}(\zeta))} = \frac{1}{p_1} + \frac{\mathrm{Re}(\zeta) - \xi_1}{\xi_2 - \xi_1}\left[\frac{1}{p_2} - \frac{1}{p_1}\right], \quad \zeta \in \overline{\Sigma}, \tag{2.47}$$

and

$$\|A(\zeta)\|_{\mathcal{B}_{p(\mathrm{Re}(\zeta))}(\mathcal{H})} \leqslant C_1^{(\xi_2 - \mathrm{Re}(\zeta))/(\xi_2-\xi_1)} C_2^{(\mathrm{Re}(\zeta)-\xi_1)/(\xi_2-\xi_1)}, \quad \zeta \in \overline{\Sigma}. \tag{2.48}$$

In case $p_j = \infty$, $\mathcal{B}_\infty(\mathcal{H})$ can be replaced by $\mathcal{B}(\mathcal{H})$.

In the next section, we shall employ Theorem 2.8 to interpolate between the $\mathcal{B}(L^2(\mathbb{R}^n))$ and $\mathcal{B}_p(L^2(\mathbb{R}^n))$ properties for a family of integral operators $T_{\gamma,\delta}$ in $L^2(\mathbb{R}^n)$, $n \geqslant 2$, with kernels bounded in absolute value according to (1.1), for appropriate values of the parameters γ, δ.

3. Interpolation and Trace Ideal Properties of a Class of Integral Operators

In this section we combine Theorems 2.1, 2.2, 2.8, and an interpolation procedure to prove Theorem 3.1 below. The latter asserts a trace ideal containment for integral operators in $L^2(\mathbb{R}^n)$, $n \geqslant 2$, with kernels bounded in absolute value by a constant times a function of the form $\langle x\rangle^{-\delta}|x-y|^{2\gamma-n}\langle y\rangle^{-\delta}$, $x, y \in \mathbb{R}^n$, $x \neq y$, for appropriate values of the parameters γ, δ. Theorem 3.4 then provides an application of Theorem 3.1 to the case of n-dimensional massless Dirac-type operators.

A combination of Theorems 2.1, 2.2, and 2.8 yields the following general result.

Theorem 3.1. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $0 < 2\gamma < n$, $\delta > \gamma$, and suppose that $T_{\gamma,\delta}$ is an integral operator in $L^2(\mathbb{R}^n)$ whose integral kernel $T_{\gamma,\delta}(\,\cdot\,,\,\cdot\,)$ satisfies the estimate*

$$|T_{\gamma,\delta}(x,y)| \leqslant C\langle x\rangle^{-\delta}|x-y|^{2\gamma-n}\langle y\rangle^{-\delta}, \quad x, y \in \mathbb{R}^n,\ x \neq y \tag{3.1}$$

for some $C \in (0,\infty)$. Then,

$$T_{\gamma,\delta} \in \mathcal{B}_p\big(L^2(\mathbb{R}^n)\big), \quad p > n/(2\gamma),\ p \geqslant 2, \tag{3.2}$$

and

$$\begin{aligned}\|T_{\gamma,\delta}\|_{\mathcal{B}_{n/(2\gamma-\varepsilon)}(L^2(\mathbb{R}^n))} \leqslant \sup_{\eta\in\mathbb{R}} &\big[\|T_{\gamma,\delta}(-2\gamma+\varepsilon+i\eta)\|_{\mathcal{B}(L^2(\mathbb{R}^n))}\big]^{2[-2\gamma+(n/2)+\varepsilon]/n} \\ \times \sup_{\eta\in\mathbb{R}} &\big[\|T_{\gamma,\delta}(-2\gamma+(n/2)+\varepsilon+i\eta)\|_{\mathcal{B}_2(L^2(\mathbb{R}^n))}\big]^{2(2\gamma-\varepsilon)/n}\end{aligned} \tag{3.3}$$

for $0 < \varepsilon$ sufficiently small.

Proof. Following the idea behind Yafaev's proof of [31, Lemma 0.13.4], we introduce the analytic family of integral operators $T_{\gamma,\delta}(\,\cdot\,)$ in $L^2(\mathbb{R}^n)$ generated by the integral kernel

$$T_{\gamma,\delta}(\zeta;x,y) = T_{\gamma,\delta}(x,y)\,\langle x\rangle^{-(\zeta/2)}|x-y|^{\zeta}\langle y\rangle^{-(\zeta/2)}, \quad x, y \in \mathbb{R}^n,\ x \neq y, \tag{3.4}$$

noting $T_{\gamma,\delta}(0) = T_{\gamma,\delta}$. By Theorems 2.2 and A.2 (i) (for $N = 1$),

$$T_{\gamma,\delta}(\zeta) \in \mathcal{B}\big(L^2(\mathbb{R}^n)\big), \quad 0 < \mathrm{Re}(\zeta) + 2\gamma < n,\ \delta \geqslant \gamma. \tag{3.5}$$

To check the Hilbert–Schmidt property of $T_{\gamma,\delta}(\,\cdot\,)$ one estimates for the square of $|T_{\gamma,\delta}(\,\cdot\,;\,\cdot\,,\,\cdot\,)|$,

$$\begin{aligned}|T_{\gamma,\delta}(\zeta;x,y)|^2 \leqslant \langle x\rangle^{-2\delta-\mathrm{Re}(\zeta)}|x-y|^{2\mathrm{Re}(\zeta)+4\gamma-2n}\langle x\rangle^{-2\delta-\mathrm{Re}(\zeta)},& \\ x, y \in \mathbb{R}^n,\ x \neq y,&\end{aligned} \tag{3.6}$$

and hence one can apply Theorem 2.1 upon identifying $\lambda = 2n - 4\gamma - 2\mathrm{Re}(z)$, $r = s = n/[\mathrm{Re}(\zeta) + 2\gamma]$, and $f = h = \langle\,\cdot\,\rangle^{-[2\delta+\mathrm{Re}(\zeta)]}$, to verify that $0 < \lambda < n$ translates into $n/2 < \mathrm{Re}(\zeta) + 2\gamma < n$, and $f \in L^r(\mathbb{R}^n)$ holds with $r \in (1,2)$ if $\delta > \gamma$. Hence,

$$T_{\gamma,\delta}(\zeta) \in \mathcal{B}_2\big(L^2(\mathbb{R}^n)\big), \quad n/2 < \mathrm{Re}(\zeta) + 2\gamma < n,\ \delta > \gamma. \tag{3.7}$$

It remains to interpolate between the $\mathcal{B}\big(L^2(\mathbb{R}^n)\big)$ and $\mathcal{B}_2\big(L^2(\mathbb{R}^n)\big)$ properties, employing Theorem 2.8 as follows. Choosing $0 < \varepsilon$ sufficiently small, one identifies $\xi_1 = -2\gamma + \varepsilon$, $\xi_2 = -2\gamma + (n/2) + \varepsilon$, $p_1 = \infty$, $p_2 = 2$, and hence obtains

$$p(\mathrm{Re}(\zeta)) = n/[\mathrm{Re}(\zeta) + 2\gamma - \varepsilon], \tag{3.8}$$

in particular, $p(0) > n/(2\gamma)$ (and of course, $p(0) \geqslant 2$). Since ε may be taken arbitrarily small, (3.2) follows from (3.8) and (3.3) is a direct consequence of (2.48). $\square$

While subordination in general only applies to $\mathcal{B}_p$-ideals with p even (see the discussion in [27, p. 24 and Addendum E]), the use of complex interpolation in Theorem 3.1 (and the focus on bounded and Hilbert–Schmidt operators) permits one to avoid this restriction.

Theorem 3.1 represents the principal result of this paper and to the best of our knowledge it appears to be new.

The singularity structure on the diagonal of the integral kernels $K_{c,d}$ introduced in (2.2) naturally matches the one of multi-dimensional Schrödinger and Dirac-type operators as we will indicate next.

As a brief preparation we first record the asymptotic behavior of Hankel functions of the first kind with index $\nu \geqslant 0$ (cf. e.g., [1, Sect. 9.1]), $H_\nu^{(1)}(\,\cdot\,)$, as the latter are crucial in the context of constant coefficient (i.e., free, or non-interacting) Schrödinger and Dirac-type operators, a natural first step in studying Schrödinger and Dirac-type operators with nontrivial interaction terms (i.e., potentials). Employing, for instance, [1, p. 360, 364], one obtains

$$H_0^{(1)}(\zeta) \underset{\zeta\to 0}{=} (2i/\pi)\ln(\zeta) + O\big(|\ln(\zeta)||\zeta|^2\big), \tag{3.9}$$

$$H_\nu^{(1)}(\zeta) \underset{\zeta\to 0}{=} -(i/\pi)2^\nu\Gamma(\nu)\zeta^{-\nu} + \begin{cases} O\big(|\zeta|^{\min(\nu,-\nu+2)}\big), & \nu \notin \mathbb{N}, \\ O\big(|\ln(\zeta)||\zeta|^\nu\big) + O\big(\zeta^{-\nu+2}\big), & \nu \in \mathbb{N}, \end{cases} \quad \mathrm{Re}(\nu) > 0, \tag{3.10}$$

$$H_\nu^{(1)}(\zeta) \underset{\zeta\to\infty}{=} (2/\pi)^{1/2}\zeta^{-1/2}e^{i\zeta-(\nu\pi/2)-(\pi/4)}, \quad \nu \geqslant 0, \ \mathrm{Im}(\zeta) \geqslant 0. \tag{3.11}$$

Starting with the Laplacian in $L^2(\mathbb{R}^n)$,

$$h_0 = -\Delta, \quad \mathrm{dom}(h_0) = H^2(\mathbb{R}^n), \tag{3.12}$$

the Green's function of h_0, denoted by $g_0(z;\,\cdot\,,\,\cdot\,)$, is then of the form,

$$\begin{aligned} g_0(z;x,y) &:= (h_0 - zI)^{-1}(x,y) \\ &= \begin{cases} (i/4)\big(2\pi z^{-1/2}|x-y|\big)^{(2-n)/2} H^{(1)}_{(n-2)/2}\big(z^{1/2}|x-y|\big), & n \geqslant 2, \ z \in \mathbb{C}\backslash\{0\}, \\ \dfrac{1}{(n-2)\omega_{n-1}}|x-y|^{2-n}, & n \geqslant 3, \ z = 0, \end{cases} \\ &\qquad z \in \mathbb{C}\backslash[0,\infty), \ \mathrm{Im}\big(z^{1/2}\big) > 0, \ x,y \in \mathbb{R}^n, \ x \neq y, \end{aligned} \tag{3.13}$$

where $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ ($\Gamma(\,\cdot\,)$ the Gamma function, cf., e.g., [1, Sect. 6.1]) represents the area of the unit sphere S^{n-1} in $\mathbb{R}^n$.

As $z \to 0$, $g_0(z;\,\cdot\,,\,\cdot\,)$ is continuous on the off-diagonal for $n \geqslant 3$,

$$\begin{aligned} \lim_{z\to 0} g_0(z;x,y) = g_0(0;x,y) = \frac{1}{(n-2)\omega_{n-1}}|x-y|^{2-n}, \\ x,y\in\mathbb{R}^n,\ x\neq y,\ n\in\mathbb{N},\ n\geqslant 3, \end{aligned} \tag{3.14}$$

but blows up for $n = 2$ as

$$\begin{aligned} g_0(z;x,y) \underset{z\to 0}{=} -\frac{1}{2\pi}\ln\big(z^{1/2}|x-y|/2\big)\big[1+O\big(z|x-y|^2\big)\big] + \frac{1}{2\pi}\psi(1) \\ + O\big(|z||x-y|^2\big), \quad x,y\in\mathbb{R}^2,\ x\neq y. \end{aligned} \tag{3.15}$$

Here $\psi(w) = \Gamma'(w)/\Gamma(w)$ denotes the digamma function (cf., e.g., [1, Sect. 6.3]). This briefly illustrates the relevance of the diagonal singularity structure $|x-y|^{(c+d)-n}$ in $K_{c,d}$ in (2.2).

To describe an application to massless Dirac operators we need additional preparations. To rigorously define the free massless n-dimensional Dirac operators to be studied in the sequel, we now introduce the following set of basic hypotheses assumed for the remainder of this section.

Hypothesis 3.2. *Let $n \in \mathbb{N}$, $n \geqslant 2$.*
(i) Set $N = 2^{\lfloor (n+1)/2\rfloor}$ and let α_j, $1 \leqslant j \leqslant n$, $\alpha_{n+1} := \beta$, denote $n+1$ anti-commuting Hermitian $N \times N$ matrices with squares equal to I_N, that is,

$$\alpha_j^* = \alpha_j, \quad \alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{j,k}I_N, \quad 1 \leqslant j,k \leqslant n+1. \tag{3.16}$$

Here I_N denotes the $N \times N$ identity matrix.
(ii) Introduce in $[L^2(\mathbb{R}^n)]^N$ the free massless Dirac operator

$$H_0 = \alpha\cdot(-i\nabla) = \sum_{j=1}^{n}\alpha_j(-i\partial_j), \quad \operatorname{dom}(H_0) = [W^{1,2}(\mathbb{R}^n)]^N, \tag{3.17}$$

where $\partial_j = \partial/\partial x_j$, $1 \leqslant j \leqslant n$.
(iii) Next, consider the self-adjoint matrix-valued potential $V = \{V_{\ell,m}\}_{1\leqslant \ell,m\leqslant N}$ satisfying for some fixed $\rho > 1$, $C \in (0,\infty)$,

$$V \in \big[L^\infty(\mathbb{R}^n)\big]^{N\times N}, \quad |V_{\ell,m}(x)| \leqslant C\langle x\rangle^{-\rho} \text{ for a.e. } x\in\mathbb{R}^n,\ 1\leqslant \ell,m\leqslant N. \tag{3.18}$$

Under these assumptions on V, the massless Dirac operator H in $[L^2(\mathbb{R}^n)]^N$ is defined via

$$H = H_0 + V, \quad \operatorname{dom}(H) = \operatorname{dom}(H_0) = [W^{1,2}(\mathbb{R}^n)]^N. \tag{3.19}$$

Here we employed the short-hand notation

$$[L^2(\mathbb{R}^n)]^N = L^2(\mathbb{R}^n;\mathbb{C}^N), \quad [W^{1,2}(\mathbb{R}^n)]^N = W^{1,2}(\mathbb{R}^n;\mathbb{C}^N), \text{ etc.} \tag{3.20}$$

Then H_0 and H are self-adjoint in $[L^2(\mathbb{R}^n)]^N$, with essential spectrum covering the entire real line,

$$\sigma_{ess}(H) = \sigma_{ess}(H_0) = \sigma(H_0) = \mathbb{R}, \tag{3.21}$$

a consequence of relative compactness of V with respect to H_0. In addition,

$$\sigma_{ac}(H_0) = \mathbb{R}, \quad \sigma_p(H_0) = \sigma_{sc}(H_0) = \emptyset. \tag{3.22}$$

With the exception of the comment following (3.25) and one more in connection with spectral shift functions in Remark 3.5, we will now drop the self-adjointness hypothesis on the $N \times N$ matrix V and still define a closed operator H in $[L^2(\mathbb{R}^n)]^N$ as in (3.19).

Turning to the the Green's matrix of the massless free Dirac operator H_0 we assume

$$z \in \mathbb{C}_+, \ x, y \in \mathbb{R}^n, \ x \neq y, \ n \in \mathbb{N}, \ n \geqslant 2, \tag{3.23}$$

and compute for the Green's function $G_0(z; \cdot\,, \cdot\,)$ of H_0,

$$\begin{aligned} &G_0(z; x, y) := (H_0 - zI)^{-1}(x, y) \\ &\quad = i4^{-1}(2\pi)^{(2-n)/2}|x-y|^{2-n} z\,[z|x-y|]^{(n-2)/2} H^{(1)}_{(n-2)/2}(z|x-y|) I_N \\ &\qquad - 4^{-1}(2\pi)^{(2-n)/2}|x-y|^{1-n}[z|x-y|]^{n/2} H^{(1)}_{n/2}(z|x-y|)\,\alpha \cdot \frac{(x-y)}{|x-y|}. \end{aligned} \tag{3.24}$$

The Green's function $G_0(z; \cdot\,, \cdot\,)$ of H_0 continuously extends to $z \in \overline{\mathbb{C}_+}$. In addition, in the massless case $m = 0$, the limit $z \to 0$ exists,

$$\begin{aligned} &\lim_{\substack{z \to 0, \\ z \in \overline{\mathbb{C}_+} \backslash \{0\}}} G_0(z; x, y) := G_0(0; x, y) \\ &\quad = i2^{-1}\pi^{-n/2}\Gamma(n/2)\,\alpha \cdot \frac{(x-y)}{|x-y|^n}, \quad x, y \in \mathbb{R}^n, \, x \neq y, \ n \in \mathbb{N}, \, n \geqslant 2, \end{aligned} \tag{3.25}$$

and no blow up occurs for all $n \in \mathbb{N}$, $n \geqslant 2$. This observation is consistent with the sufficient condition for the Dirac operator $H = H_0 + V$ (in dimensions $n \in \mathbb{N}$, $n \geqslant 2$), with V an appropriate self-adjoint $N \times N$ matrix-valued potential, having no eigenvalues, as derived in [19, Theorems 2.1, 2.3].

Returning to our analysis of the resolvent of H_0, the asymptotic behavior (3.9)–(3.11) implies for some $c_n \in (0, \infty)$,

$$\|G_0(0; x, y)\|_{\mathcal{B}(\mathbb{C}^N)} \leqslant c_n |x-y|^{1-n}, \quad x, y \in \mathbb{R}^n, \, x \neq y, \ n \in \mathbb{N}, \ n \geqslant 2, \tag{3.26}$$

and for given $R \geqslant 1$,

$$\begin{aligned} \|G_0(z; x, y)\|_{\mathcal{B}(\mathbb{C}^N)} \leqslant c_{n,R}(z) e^{-\operatorname{Im}(z)|x-y|} &\begin{cases} |x-y|^{1-n}, & |x-y| \leqslant 1, \ x \neq y, \\ 1, & 1 \leqslant |x-y| \leqslant R, \\ |x-y|^{(1-n)/2}, & |x-y| \geqslant R, \end{cases} \\ &z \in \overline{\mathbb{C}_+}, \ x, y \in \mathbb{R}^n, \, x \neq y, \ n \in \mathbb{N}, \ n \geqslant 2, \end{aligned} \tag{3.27}$$

for some $c_{n,R}(\,\cdot\,) \in (0, \infty)$ continuous and locally bounded on $\overline{\mathbb{C}_+}$.

For future purposes we now rewrite $G_0(z; \cdot\,, \cdot\,)$ as follows:

$$\begin{aligned} G_0(z; x, y) &= i4^{-1}(2\pi)^{(2-n)/2}|x-y|^{2-n} z\,[z|x-y|]^{(n-2)/2} H^{(1)}_{(n-2)/2}(z|x-y|) I_N \\ &\quad - 4^{-1}(2\pi)^{(2-n)/2}|x-y|^{1-n}[z|x-y|]^{n/2} H^{(1)}_{n/2}(z|x-y|)\,\alpha \cdot \frac{(x-y)}{|x-y|} \\ &= |x-y|^{1-n} f_n(z, x-y), \\ &\qquad z \in \overline{\mathbb{C}_+}, \ x, y \in \mathbb{R}^n, \, x \neq y, \, n \in \mathbb{N}, \, n \geqslant 2, \end{aligned} \tag{3.28}$$

where f_n is continuous and locally bounded on $\overline{\mathbb{C}_+} \times \mathbb{R}^n$, in addition,

$$\begin{aligned} \|f_n(z, x)\|_{\mathcal{B}(\mathbb{C}^N)} \leqslant c_n(z) e^{-\operatorname{Im}(z)|x|} &\begin{cases} 1, & 0 \leqslant |x| \leqslant 1, \\ |x|^{(n-1)/2}, & |x| \geqslant 1, \end{cases} \\ &z \in \overline{\mathbb{C}_+}, \ x, y \in \mathbb{R}^n, \end{aligned} \tag{3.29}$$

for some constant $c_n(\,\cdot\,) \in (0,\infty)$ continuous and locally bounded on $\overline{\mathbb{C}_+}$. In particular, decomposing $G_0(z;\,\cdot\,,\,\cdot\,)$ into

$$\begin{aligned} G_0(z;x,y) &= G_0(z;x,y)\chi_{[0,1]}(|x-y|) + G_0(z;x,y)\chi_{[1,\infty)}(|x-y|) \\ &:= G_{0,<}(z;x-y) + G_{0,>}(z;x-y), \qquad (3.30) \\ &\quad z \in \overline{\mathbb{C}_+},\ x,y \in \mathbb{R}^n,\ x \neq y,\ n \in \mathbb{N},\ n \geqslant 2, \end{aligned}$$

one verifies that

$$|G_{0,>}(z;x-y)_{j,k}| \leqslant \begin{cases} C_n|x-y|^{-(n-1)}, & z = 0, \\ C_n(z)|x-y|^{-(n-1)/2}, & z \in \overline{\mathbb{C}_+}, \end{cases} \qquad (3.31)$$
$$x,y \in \mathbb{R}^n,\ |x-y| \geqslant 1,\ 1 \leqslant j,k \leqslant N,$$

for some constants $C_n, C_n(\,\cdot\,) \in (0,\infty)$, in particular,

$$G_{0,>}(z;\,\cdot\,) \in [L^\infty(\mathbb{R}^n)]^{N\times N}, \quad z \in \overline{\mathbb{C}_+}, \qquad (3.32)$$

and that $G_{0,>}(\,\cdot\,;\,\cdot\,)$ is continuous on $\overline{\mathbb{C}_+} \times \mathbb{R}^n$.

Starting our analysis of integral operators connected to the resolvent of H_0 we first note that Theorem 2.2 implies the following fact.

Theorem 3.3. *Let $n \in \mathbb{N}$, $n \geqslant 2$. Then the integral operator $R_0(\delta)$ in $[L^2(\mathbb{R}^n)]^N$ with integral kernel $R_0(\delta;\,\cdot\,,\,\cdot\,)$ bounded entrywise by*

$$|R_0(\delta;\,\cdot\,,\,\cdot\,)_{j,k}| \leqslant C\langle\,\cdot\,\rangle^{-\delta}|G_0(0;\,\cdot\,,\,\cdot\,)_{j,k}|\langle\,\cdot\,\rangle^{-\delta}, \quad \delta \geqslant 1/2,\ 1 \leqslant j,k \leqslant N, \qquad (3.33)$$

for some $C \in (0,\infty)$, is bounded,

$$R_0(\delta) \in \mathcal{B}\big([L^2(\mathbb{R}^n)]^N\big). \qquad (3.34)$$

In a similar fashion, the integral operator $R_0(z,\delta)$ in $[L^2(\mathbb{R}^n)]^N$, with integral kernel $R_0(z,\delta;\,\cdot\,,\,\cdot\,)$ bounded entrywise by

$$\begin{aligned} &|R_0(z,\delta;\,\cdot\,,\,\cdot\,)_{j,k}| \leqslant C\langle\,\cdot\,\rangle^{-\delta}|G_0(z;\,\cdot\,,\,\cdot\,)_{j,k}|\langle\,\cdot\,\rangle^{-\delta}, \\ &\qquad \delta \geqslant (n+1)//4,\ z \in \overline{\mathbb{C}_+},\ 1 \leqslant j,k \leqslant N, \end{aligned} \qquad (3.35)$$

for some $C \in (0,\infty)$, is bounded,

$$R_0(z,\delta) \in \mathcal{B}\big([L^2(\mathbb{R}^n)]^N\big), \quad z \in \overline{\mathbb{C}_+}. \qquad (3.36)$$

Proof. The inclusion (3.34) is an immediate consequence of (3.25) and hence the estimate $|G_0(0;x,y)_{j,k}| \leqslant C|x-y|^{1-n}$, $x,y \in \mathbb{R}^n$, $x \neq y$, $1 \leqslant j,k \leqslant N$, Theorem 2.2, choosing $c = d = 1/2$ in (2.2), and an application of Theorem A.2 (i) and Remark A.3.

To prove the inclusion (3.36) we employ the estimates (3.9)–(3.11) (cf. also (3.27)) to obtain

$$\begin{aligned} |G_0(z;x,y)_{j,k}| &\leqslant C(z)|x-y|^{1-n}\chi_{[0,1]}(|x-y|) \\ &\quad + D(z)|x-y|^{(1-n)/2}\chi_{[1,\infty)}(|x-y|), \qquad (3.37) \\ &\quad z \in \overline{\mathbb{C}_+},\ x,y \in \mathbb{R}^n,\ x \neq y,\ 1 \leqslant j,k \leqslant N, \end{aligned}$$

for some $C, D(z) \in (0,\infty)$, and apply Theorems 2.2 and A.2 (i) (cf. also Remark A.3) to both terms on the right-hand sides of (3.37). The part $0 \leqslant |x-y| \leqslant 1$ leads to $\delta \geqslant 1/2$, whereas the part $|x-y| \geqslant 1$ yields $\delta \geqslant (n+1)/4$, implying (3.36). □

Combining Theorems 2.1, 2.2, 2.8, and 3.1 then yields the second principal result of this section, an application to massless Dirac-type operators.

Theorem 3.4. *Let $n \in \mathbb{N}$, $n \geqslant 2$. Then the integral operator $R_0(\delta)$ in $[L^2(\mathbb{R}^n)]^N$ with integral kernel $R_0(\delta; \cdot\,, \cdot\,)$ permitting the entrywise bound*

$$|R_0(\delta; \cdot\,, \cdot\,)_{j,k}| \leqslant C\langle \cdot \rangle^{-\delta}|G_0(0; \cdot\,, \cdot\,)_{j,k}|\langle \cdot \rangle^{-\delta}, \quad \delta > 1/2,\ 1 \leqslant j,k \leqslant N, \tag{3.38}$$

for some $C \in (0,\infty)$, satisfies

$$R_0(\delta) \in \mathcal{B}_p\big([L^2(\mathbb{R}^n)]^N\big), \quad p > n. \tag{3.39}$$

In a similar fashion, the integral operator $R_0(z,\delta)$ in $[L^2(\mathbb{R}^n)]^N$ with integral kernel $R_0(z,\delta; \cdot\,, \cdot\,)$ permitting the entrywise bound

$$\begin{split} |R_0(z,\delta; \cdot\,, \cdot\,)_{j,k}| \leqslant C\langle \cdot \rangle^{-\delta}|G_0(z; \cdot\,, \cdot\,)_{j,k}|\langle \cdot \rangle^{-\delta}, \\ z \in \overline{\mathbb{C}_+},\ \delta > (n+1)/4,\ 1 \leqslant j,k \leqslant N, \end{split} \tag{3.40}$$

for some $C \in (0,\infty)$, satisfies

$$R_0(z,\delta) \in \mathcal{B}_p\big([L^2(\mathbb{R}^n)]^N\big), \quad p > n,\ z \in \overline{\mathbb{C}_+}. \tag{3.41}$$

Proof. We will apply the fact (A.5).

The inclusion (3.39) is immediate from (3.25) (employing the elementary estimate $|G_0(0;x,y)_{j,k}| \leqslant C|x-y|^{1-n}$, $x,y \in \mathbb{R}^n$, $x \neq y$, $1 \leqslant j,k \leqslant N$) and Theorem 3.1 (with $\gamma = 1/2$).

To prove the inclusion (3.41) we again employ the estimate (3.37). An application of Theorem 3.1 to both terms in (3.37), then yields for the part where $0 \leqslant |x-y| \leqslant 1$ that $\gamma = 1/2$ and hence $\delta > 1/2$ and $p > n$. Similarly, for the part where $|x-y| \geqslant 1$ one infers $\gamma = (n+1)/4$ and hence $\delta > (n+1)/4$ and $p > 2n/(n+1)$, $p \geqslant 2$, and thus one concludes $\delta > (n+1)/4$ and $p > n$. □

Remark 3.5. To put Theorem 3.4 a bit into perspective we note that inclusions of the type (3.39), even in the far weaker situation with $\mathcal{B}_p\big([L^2(\mathbb{R}^n)]^N\big)$ replaced by $\mathcal{B}\big([L^2(\mathbb{R}^n)]^N\big)$, imply a global limiting absorption principle with strong spectral implications (such as, the absence of any singular spectrum) for the underlying Dirac-type operators, H_0 and $H = H_0 + c\,V$, for sufficiently small coupling constants $c \in \mathbb{C}$. (For details in this limiting absorption context context we refer to [4], [5], [25, Sects. XIII.7, XIII.8], [30, Ch. 4], [31, Chs. 1, 2, 6] and the detailed bibliography cited therein). The actual $\mathcal{B}_p\big([L^2(\mathbb{R}^n)]^N\big)$ result in Theorem 3.4 permits one to go a step further and derive continuity properties of the spectral shift function (cf., e.g., [30, Ch. 8], [31, Ch. 9]) between the pair of self-adjoint operators (H, H_0) (here we again assume the $N \times N$ matrix-valued potential V to be self-adjoint), which in turn permits a discussion of the Witten index of class of non-Fredholm model operators as discussed in [3]–[9], [12], [24], with additional material in preparation. ◇

We conclude this section by noting once more that massless Dirac operators, particularly, in two dimensions, are known to be of relevance in applications to graphene. This fact, and particularly the prominent role massless Dirac-type operators play in connection with the Witten index of certain classes of non-Fredholm operators, explains our interest in them.

Appendix A. Some Remarks on Block Matrix Operators

In this appendix we collect some useful (and well-known) material on pointwise domination of linear operators in connection with boundedness, compactness, and the Hilbert–Schmidt property, with particular emphasis on the block matrix operator situation (required in the context of Dirac-type operators).

Definition A.1. *Let $(M;\mathcal{M};\mu)$ be a σ-finite, separable measure space, μ a nonnegative, measure with $0<\mu(M)\leqslant\infty$, and consider the linear operators A,B defined on $L^2(M;d\mu)$. Then B pointwise dominates A*

$$\text{if for all } f\in L^2(M;d\mu),\ |(Af)(\,\cdot\,)|\leqslant (B|f|)(\,\cdot\,)\ \mu\text{-a.e. on } M. \tag{A.1}$$

For a linear block operator matrix $T=\{T_{j,k}\}_{1\leqslant j,k\leqslant N}$, $N\in\mathbb{N}$, in the Hilbert space $[L^2(M;d\mu)]^N$ (where $[L^2(M;d\mu)]^N = L^2(M;d\mu;\mathbb{C}^N)$, we recall that $T\in\mathcal{B}_2\big([L^2(M;d\mu)]^N\big)$ if and only if $T_{j,k}\in\mathcal{B}_2\big(L^2(M;d\mu)\big)$, $1\leqslant j,k\leqslant N$. Moreover, we recall that (cf. e.g., [2, Theorem 11.3.6])

$$\begin{aligned}
\|T\|^2_{\mathcal{B}_2(L^2(M;d\mu)^N)} &= \int_{M\times M} d\mu(x)\,d\mu(y)\,\|T(x,y)\|^2_{\mathcal{B}_2(\mathbb{C}^N)}\\
&= \int_{M\times M} d\mu(x)\,d\mu(y)\,\sum_{j,k=1}^N |T_{j,k}(x,y)|^2\\
&= \sum_{j,k=1}^N \int_{M\times M} d\mu(x)\,d\mu(y)\,|T_{j,k}(x,y)|^2\\
&= \sum_{j,k=1}^N \|T_{j,k}\|^2_{\mathcal{B}_2(L^2(M;d\mu))},
\end{aligned} \tag{A.2}$$

where, in obvious notation, $T(\,\cdot\,,\,\cdot\,)$ denotes the $N\times N$ matrix-valued integral kernel of T in $[L^2(M;d\mu)]^N$, and $T_{j,k}(\,\cdot\,,\,\cdot\,)$ represents the integral kernel of $T_{j,k}$ in $L^2(M;d\mu)$, , $1\leqslant j,k\leqslant N$.

In addition, employing the fact that for any $N\times N$ matrix $D\in\mathbb{C}^{N\times N}$,

$$\|D\|_{\mathcal{B}(\mathbb{C}^N)}\leqslant\|D\|_{\mathcal{B}_2(\mathbb{C}^N)}\leqslant N^{1/2}\|D\|_{\mathcal{B}(\mathbb{C}^N)}, \tag{A.3}$$

one also obtains

$$\|T\|^2_{\mathcal{B}_2(L^2(M;d\mu)^N)}\leqslant N\int_{M\times M} d\mu(x)\,d\mu(y)\,\|T(x,y)\|^2_{\mathcal{B}(\mathbb{C}^N)}. \tag{A.4}$$

More generally, for $\mathcal{H}$ a complex separable Hilbert space and $T=\{T_{j,k}\}_{1\leqslant j,k\leqslant N}$, $N\in\mathbb{N}$, a block operator matrix in $\mathcal{H}^N$, one confirms that

$$\begin{aligned}
&T\in\mathcal{B}\big(\mathcal{H}^N\big)\ \big(\text{resp., } T\in\mathcal{B}_p\big(\mathcal{H}^N\big),\ p\in[1,\infty)\cup\{\infty\}\big)\\
&\quad\text{if and only if}\\
&\text{for each } 1\leqslant j,k\leqslant N,\ T_{j,k}\in\mathcal{B}\big(\mathcal{H}^N\big)\ \big(\text{resp., } T_{j,k}\in\mathcal{B}_p\big(\mathcal{H}^N\big),\ p\in[1,\infty)\cup\{\infty\}\big).
\end{aligned} \tag{A.5}$$

In other words, for membership of T in $\mathcal{B}\big(\mathcal{H}^N\big)$ or $\mathcal{B}_p\big(\mathcal{H}^N\big)$, $p\in[1,\infty)\cup\{\infty\}$, it suffices to focus on each of its matrix elements $T_{j,k}$, $1\leqslant j,k\leqslant N$. (For necessity of the last line in (A.5) it suffices to multiply T from the left and right by $N\times N$ diagonal matrices with $I_{\mathcal{H}}$ on the jth and kth position, respectively, and zeros otherwise, to isolate $T_{j,k}$ and appeal to the ideal property. For sufficiency, it suffices to write T as a sum of N^2 terms with $T_{j,k}$ at the j,kth position and zeros otherwise.)

The next result is useful in connection with Section 3.

Theorem A.2. *Let $N\in\mathbb{N}$ and suppose that T_1,T_2 are linear $N\times N$ block operator matrices defined on $[L^2(M;d\mu)]^N$, such that for each $1\leqslant j,k\leqslant N$, $T_{2,j,k}$ pointwise* *dominates $T_{1,j,k}$. Then the following items (i)–(iii) hold:*

(i) If $T_2 \in \mathcal{B}\big([L^2(M;d\mu)]^N\big)$ then $T_1 \in \mathcal{B}\big([L^2(M;d\mu)]^N\big)$ and

$$\|T_1\|_{\mathcal{B}([L^2(M;d\mu)]^N)} \leqslant \|T_2\|_{\mathcal{B}([L^2(M;d\mu)]^N)}. \tag{A.6}$$

(ii) If $T_2 \in \mathcal{B}_\infty\big([L^2(M;d\mu)]^N\big)$ then $T_1 \in \mathcal{B}_\infty\big([L^2(M;d\mu)]^N\big)$ and

$$\|T_1\|_{\mathcal{B}([L^2(M;d\mu)]^N)} \leqslant \|T_2\|_{\mathcal{B}([L^2(M;d\mu)]^N)}. \tag{A.7}$$

(iii) If $T_2 \in \mathcal{B}_2\big([L^2(M;d\mu)]^N\big)$ then $T_1 \in \mathcal{B}_2\big([L^2(M;d\mu)]^N\big)$ and

$$\|T_1\|_{\mathcal{B}_2([L^2(M;d\mu)]^N)} \leqslant \|T_2\|_{\mathcal{B}_2([L^2(M;d\mu)]^N)}. \tag{A.8}$$

Proof. For item *(ii)* we refer to [11] and [23] (see also [21]) combined with (A.5) as we will not use it in this paper. While the proofs of items *(i)* and *(iii)* are obviously well-known, we briefly recall them here as we will be using these facts in Section 3. Starting with item *(i)*, we introduce the notation $f = (f_1, \dots, f_N) \in [L^2(M;d\mu)]^N$ and $|f| = (|f_1|, \dots, |f_N|) \in [L^2(M;d\mu)]^N$ and compute,

$$\begin{aligned}
\|T_1 f\|^2_{[L^2(M;d\mu)]^N} &= \sum_{j=1}^N \|(T_1 f)_j\|^2_{L^2(M;d\mu)} = \sum_{j=1}^N ((T_1 f)_j, (T_1 f)_j)_{L^2(M;d\mu)} \\
&= \sum_{j=1}^N \bigg| \sum_{k,\ell=1}^N (T_{1,j,k} f_k, T_{1,j,\ell} f_\ell)_{L^2(M;d\mu)} \bigg| \\
&\leqslant \sum_{j=1}^N \sum_{k,\ell=1}^N |(T_{1,j,k} f_k, T_{1,j,\ell} f_\ell)_{L^2(M;d\mu)}| \\
&\leqslant \sum_{j=1}^N \sum_{k,\ell=1}^N (|T_{1,j,k} f_k|, |T_{1,j,\ell} f_\ell|)_{L^2(M;d\mu)} \\
&\leqslant \sum_{j=1}^N \sum_{k,\ell=1}^N (T_{2,j,k} |f_k|, T_{2,j,\ell} |f_\ell|)_{L^2(M;d\mu)} \\
&= \sum_{j=1}^N ((T_2 |f|)_j, (T_2 |f|)_j)_{L^2(M;d\mu)} = \|T_2 |f|\|^2_{[L^2(M;d\mu)]^N} \\
&\leqslant \|T_2\|^2_{\mathcal{B}(L^2(M;d\mu)^N)} \||f|\|^2_{[L^2(M;d\mu)]^N} \\
&= \|T_2\|^2_{\mathcal{B}(L^2(M;d\mu)^N)} \|f\|^2_{[L^2(M;d\mu)]^N},
\end{aligned} \tag{A.9}$$

implying item *(i)*. For item *(iii)* we recall from [27, Theorem 2.13] that $T_{1,j,k} \in \mathcal{B}_2\big(L^2(M;d\mu)\big)$, $1 \leqslant j,k \leqslant N$, and $\|T_{1,j,k}\|_{\mathcal{B}_2(L^2(M;d\mu))} \leqslant \|T_{2,j,k}\|_{\mathcal{B}_2(L^2(M;d\mu))}$, $1 \leqslant j,k \leqslant N$, and hence by (A.2),

$$\begin{aligned}
\|T_1\|^2_{\mathcal{B}_2([L^2(M;d\mu)]^N)} &= \sum_{j,k=1}^N \|T_{1,j,k}\|^2_{\mathcal{B}_2(L^2(M;d\mu))} \leqslant \sum_{j,k=1}^N \|T_{2,j,k}\|^2_{\mathcal{B}_2(L^2(M;d\mu))} \\
&= \|T_2\|^2_{\mathcal{B}_2([L^2(M;d\mu)]^N)}.
\end{aligned} \tag{A.10}$$

$\square$

Remark A.3. We note that the subordination assumption $|(Af)(\,\cdot\,)| \leqslant (B|f|)(\,\cdot\,)$ μ-a.e. on M, if A and B are integral operators in $\mathcal{H}$ with integral kernels $A(\,\cdot\,,\,\cdot\,)$ and

$B(\,\cdot\,,\,\cdot\,)$, respectively, is implied by the condition $|A(\,\cdot\,,\,\cdot\,)| \leqslant B(\,\cdot\,,\,\cdot\,)$ $\mu\otimes\mu$-a.e. on $M \times M$ since

$$\begin{aligned} |(Af)(x)| &= \left| \int_M d\mu(y)\, A(x,y) f(y)) \right| \leqslant \int_M d\mu(y)\, |A(x,y)||f(y)| \\ &\leqslant \int_M d\mu(y)\, B(x,y)|f(y)| = (B|f|)(x)| \text{ for a.e. } x \in M. \end{aligned} \tag{A.11}$$

◇

Next, we state the following result.

Lemma A.4. *Let $n \in \mathbb{N}$ and suppose that $K : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ satisfies*

$$0 \leqslant K(x,y) \leqslant \prod_{j=1}^n K_j(x_j,y_j), \quad x = (x_j)_{j=1}^n, y = (y_j)_{j=1}^n \in \mathbb{R}^n, \tag{A.12}$$

for functions $K_j : \mathbb{R}\times\mathbb{R} \to [0,\infty)$. If $K_j(\,\cdot\,,\,\cdot\,)$ is the kernel of a bounded integral operator $K_j \in \mathcal{B}(L^2(\mathbb{R}))$ for each $1 \leqslant j \leqslant n$, then $K(\,\cdot\,,\,\cdot\,)$ is the kernel of a bounded integral operator $K \in \mathcal{B}(L^2(\mathbb{R}^n))$, and

$$\|K\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \leqslant \prod_{j=1}^n \|K_j\|_{\mathcal{B}(L^2(\mathbb{R}))}. \tag{A.13}$$

Proof. We proceed by induction on n. The claim is evident in the case $n = 1$. Let $n \in \mathbb{N}$, and suppose the claim is true for $n-1 \in \mathbb{N}$. In order to establish the claim for n, we compute for $f \in L^2(\mathbb{R}^n)$:

$$\begin{aligned} &\int_{\mathbb{R}^n} d^n x \left| \int_{\mathbb{R}^n} d^n y\, K(x,y) f(y) \right|^2 \\ &\quad\leqslant \int_{\mathbb{R}} dx_1 \cdots \int_{\mathbb{R}} dx_n \left(\int_{\mathbb{R}} dy_1 \cdots \int_{\mathbb{R}} dy_n \prod_{j=1}^n K(x_j,y_j) |f(y_1,\dots,y_{n-1},y_n)| \right)^2 \\ &\quad\leqslant \|K_1\|^2_{\mathcal{B}(L^2(\mathbb{R}))} \cdots \|K_{n-1}\|^2_{\mathcal{B}(L^2(\mathbb{R}))} \\ &\qquad \times \int_{\mathbb{R}} dx_n \left\| \int_{\mathbb{R}} dy_n\, K_n(x_n,y_n) |f(y_1,\dots,y_{n-1},y_n)| \right\|^2_{L^2(\mathbb{R}^{n-1}; dy_1\cdots dy_{n-1})}, \end{aligned} \tag{A.14}$$

and

$$\begin{aligned} &\int_{\mathbb{R}} dx_n \left\| \int_{\mathbb{R}} dy_n\, K_n(x_n,y_n) |f(y_1,\dots,y_n)| \right\|^2_{L^2(\mathbb{R}^{n-1}; dy_1\cdots dy_{n-1})} \\ &\quad= \int_{\mathbb{R}} dx_n \left[\int_{\mathbb{R}} dy_1 \cdots \int_{\mathbb{R}} dy_{n-1} \left(\int_{\mathbb{R}} dy_n\, K_n(x_n,y_n) |f(y_1,\dots,y_{n-1},y_n)| \right)^2 \right] \\ &\quad= \int_{\mathbb{R}} dy_1 \cdots \int_{\mathbb{R}} dy_{n-1} \left[\int_{\mathbb{R}} dx_n \left(\int_{\mathbb{R}} dy_n\, K_n(x_n,y_n) |f(y_1,\dots,y_{n-1},y_n)| \right)^2 \right] \\ &\quad\leqslant \|K_n\|^2_{\mathcal{B}(L^2(\mathbb{R}))} \int_{\mathbb{R}} dy_1 \cdots \int_{\mathbb{R}} dy_{n-1} \left(\int_{\mathbb{R}} dy_n\, |f(y_1,\dots,y_{n-1},y_n)|^2 \right) \\ &\quad= \|K_n\|^2_{\mathcal{B}(L^2(\mathbb{R}))} \int_{\mathbb{R}^n} dy_1 \cdots dy_n\, |f(y_1,\dots,y_{n-1},y_n)|^2 \\ &\quad= \|K_n\|^2_{\mathcal{B}(L^2(\mathbb{R}))} \|f\|^2_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{A.15}$$

To obtain the inequality in (A.15), we used the boundedness property of K_n in the form

$$\int_{\mathbb{R}} dx_n \bigg(\int_{\mathbb{R}} dy_n \, K_n(x_n,y_n)|f(y_1,\dots,y_{n-1},y_n)|\bigg)^2 \tag{A.16}$$

$$\leqslant \|K_n\|^2_{\mathcal{B}(L^2(\mathbb{R}))} \int_{\mathbb{R}} dy_n \, |f(y_1,\dots,y_{n-1},y_n)|^2 \text{ for a.e. } (y_j)_{j=1}^{n-1} \in \mathbb{R}^{n-1}. \tag{A.17}$$

The claim and the estimate in (A.13) now follow upon combining (A.14) and (A.15). □

We conclude with one more fact from [17, Theorem 319]:

Lemma A.5. *Let $p \in (1,\infty)$. If $K : \mathbb{R} \times \mathbb{R} \to [0,\infty)$ is homogeneous of degree (-1) and the (necessarily identical) quantities*

$$\int_0^\infty ds \, K(s,1)\, s^{-1/p'} \quad \text{and} \quad \int_0^\infty dt\, K(1,t)\, t^{-1/p} \tag{A.18}$$

are equal to some number $C \in (0,\infty)$, then the integral operator K with kernel $K(\,\cdot\,,\,\cdot\,)$ belongs to $\mathcal{B}(L^p((0,\infty)))$ and

$$\|K\|_{\mathcal{B}(L^p((0,\infty)))} \leqslant C. \tag{A.19}$$

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