

ON ABSENCE OF THRESHOLD RESONANCES FOR SCHRÖDINGER AND DIRAC OPERATORS

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Dedicated with great pleasure to Gisèle Goldstein

ABSTRACT. Using a unified approach employing a homogeneous Lippmann-Schwinger-type equation satisfied by resonance functions and basic facts on Riesz potentials, we discuss the absence of threshold resonances for Dirac and Schrödinger operators with sufficiently short-range interactions in general space dimensions.

More specifically, assuming a sufficient power law decay of potentials, we derive the absence of zero-energy resonances for massless Dirac operators in space dimensions $n \geq 3$, the absence of resonances at $\pm m$ for massive Dirac operators (with mass $m > 0$) in dimensions $n \geq 5$, and recall the well-known case of absence of zero-energy resonances for Schrödinger operators in dimension $n \geq 5$.

1. INTRODUCTION

Happy Birthday, Gisèle, we sincerely hope that our modest contribution to threshold resonances of Schrödinger and Dirac operators will create some joy.

The principal purpose of this paper is a systematic investigation of threshold resonances, more precisely, the absence of the latter, for massless and massive Dirac operators, and for Schrödinger operators in general space dimensions $n \in \mathbb{N}$, $n \geq 2$, given sufficiently fast decreasing interaction potentials at infinity (i.e., short-range interactions).

In the case that $(-\infty, E_0)$ and/or (E_1, E_2) , $E_j \in \mathbb{R}$, $j = 0, 1, 2$, are essential (resp., absolutely continuous) spectral gaps of a self-adjoint unbounded operator A in some complex, separable Hilbert space \mathcal{H} , then typically, the numbers E_0 and/or the numbers E_1, E_2 are called threshold energies of A . The intuition behind the term “threshold” being that if the coupling constant of an appropriate interaction potential is varied, then eigenvalues will eventually “emerge” out of the continuous spectrum of the operator A and enter the spectral gap $(-\infty, E_0)$ and/or (E_0, E_1) . The precise phenomenon behind this intuition is somewhat involved, depending, in particular, on the (possibly singular) behavior of the resolvent of A at the points E_j , $j = 0, 1, 2$. In the concrete case of Dirac and Schrödinger operators with sufficient short-range interactions at hand, this behavior is well understood (see, e.g., [56], [57], [59] and the literature cited therein) and $E_0 = 0$ is then a threshold

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for Schrödinger operators h , with $\sigma_{ess}(h) = [0, \infty)$ (cf. (2.16)–(2.18)), whereas $\pm m$ are thresholds for massive Dirac operators $H(m)$ corresponding to mass $m > 0$, with $\sigma_{ess}(H(m)) = (-\infty, -m] \cup [m, \infty)$ (cf. (2.9)–(2.11)). The case of massless Dirac operators H is more intricate as $\sigma_{ess}(H) = \mathbb{R}$ (cf. (2.4), (2.6)), and hence H exhibits no spectral gap. However, since the potential coefficients tend to zero at infinity, 0 is still a threshold point of H that is known to possibly support an eigenvalue and/or a resonance.

Threshold resonances for Dirac and Schrödinger operators then are associated with distributional solutions ψ of $h\psi = 0$, or $\Psi_{\pm m}$ of $H(m)\Psi_{\pm m} = \pm m\Psi_{\pm m}$, respectively, distributional solutions Ψ of $H\Psi = 0$, that do not belong to the domains of $h, H(m)$, respectively H , but these functions are “close” to belonging to the respective operator domains in the sense that they (actually, their components) belong to $L^p(\mathbb{R}^n)$ for $p \in (p_0, \infty) \cup \{\infty\}$ for appropriate $p_0 > 2$. More precisely, we then recall in our principal Section 3 that Schrödinger operators have no zero-energy resonances in dimensions $n \geq 5$ (a well-known result, cf. Remark 3.5 (iii)), that massless Dirac operators have no zero-energy resonances in dimension $n \geq 3$ (the case $n = 3$ was well-known, cf. Remark 3.9 (iii)), and that massive Dirac operators have no resonances at $\pm m$ in dimensions $n \geq 5$. We prove these results in a unified manner employing a homogeneous Lippmann–Schwinger-type equation satisfied by threshold resonance functions and the use of basic properties of Riesz potentials. While our power law decay assumptions of the potentials at infinity are not optimal, optimality was not the main motivation for writing this paper. Instead, we wanted to present a simple, yet unified, approach to the question of absence of threshold resonances with the added bonus that our results in the massless case for $n \geq 4$ and in the massive case appear to be new. Since we explicitly permit matrix-valued potentials in the (massless and massive) Dirac case, we note that electric and magnetic potentials are included in our treatment (cf. Remark 2.2).

We also emphasize that threshold resonances and/or threshold eigenvalues of course profoundly influence the (singularity) behavior of resolvents at threshold energies, the threshold behavior of (fixed energy, or on-shell) scattering matrices, as well as dispersive (i.e., large time) estimates for wave functions. These issues have received an abundance of attention since the late 1970’s in connection with Schrödinger operators, and very recently especially in the context of (massless and massive) Dirac operators (the case of massless Dirac operators having applications to graphene). While a complete bibliography in this connection is clearly beyond the scope of our paper, we refer to [2]–[11], [12, Ch. 4], [13], [15]–[21], [28]–[43], [45], [46], [50]–[54], [58]–[62], [65]–[73], [76]–[78], and the literature therein to demonstrate some of the interest generated by this circle of ideas. We also mention the relevance of threshold states in connection with the Witten index for non-Fredholm operators, see, for instance, [22]–[26].

In Section 2 we present the necessary background for Dirac and Schrödinger operators, including a description of Green’s functions (resp., matrices) and their threshold asymptotics in the free case, that is, in the absence of interaction potentials in terms of appropriate Hankel functions. Appendix A discusses the failure of our technique to produce essential boundedness (of components) of threshold resonances and eigenfunctions for space dimensions sufficiently large.

We conclude this introduction with some comments on the notation employed in this paper: If \mathcal{H} denotes a separable, complex Hilbert space, $I_{\mathcal{H}}$ represents the

identity operator in \mathcal{H} , but we simplify this to I_N in the finite-dimensional case \mathbb{C}^N , $N \in \mathbb{N}$.

If T is a linear operator mapping (a subspace of) a Hilbert space into another, then $\text{dom}(T)$ and $\text{ker}(T)$ denote the domain and kernel (i.e., null space) of T . The spectrum, point spectrum (the set of eigenvalues), and the essential spectrum of a closed linear operator in \mathcal{H} will be denoted by $\sigma(\cdot)$, $\sigma_p(\cdot)$, and $\sigma_{ess}(\cdot)$, respectively. Similarly, the absolutely continuous and singularly continuous spectrum of a self-adjoint operator in \mathcal{H} are denoted by $\sigma_{ac}(\cdot)$ and $\sigma_{sc}(\cdot)$.

The Banach space of bounded linear operators on a separable complex Hilbert space \mathcal{H} is denoted by $\mathcal{B}(\mathcal{H})$.

If $p \in [1, \infty) \cup \{\infty\}$, then $p' \in [1, \infty) \cup \{\infty\}$ denotes its conjugate index, that is, $p' := (1 - 1/p)^{-1}$. If Lebesgue measure is understood, we simply write $L^p(M)$, $M \subseteq \mathbb{R}^n$ measurable, $n \in \mathbb{N}$, instead of the more elaborate notation $L^p(M; d^n x)$. $\lfloor \cdot \rfloor$ denotes the floor function on \mathbb{R} , that is, $\lfloor x \rfloor$ characterizes the largest integer less than or equal to $x \in \mathbb{R}$. Finally, if $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $n \in \mathbb{N}$, then we abbreviate $\langle x \rangle := (1 + |x|^2)^{1/2}$.

2. SOME BACKGROUND MATERIAL

This preparatory section is primarily devoted to various results on Green's functions for the free Schrödinger operator (i.e., minus the Laplacian) and free (massive and massless) Dirac operators (i.e., in the absence of interaction potentials) in dimensions $n \in \mathbb{N}$, $n \geq 2$.

To rigorously define the case of free massless n -dimensional Dirac operators, $n \geq 2$, we now introduce the following set of basic hypotheses assumed for the remainder of this manuscript.

Hypothesis 2.1. *Let $n \in \mathbb{N}$, $n \geq 2$.*

(i) *Set $N = 2^{\lfloor (n+1)/2 \rfloor}$ and let α_j , $1 \leq j \leq n$, $\alpha_{n+1} := \beta$, denote $n + 1$ anti-commuting Hermitian $N \times N$ matrices with squares equal to I_N , that is,*

$$\alpha_j^* = \alpha_j, \quad \alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{j,k} I_N, \quad 1 \leq j, k \leq n + 1. \quad (2.1)$$

(ii) *Introduce in $[L^2(\mathbb{R}^n)]^N$ the free massless Dirac operator*

$$H_0 = \alpha \cdot (-i\nabla) = \sum_{j=1}^n \alpha_j (-i\partial_j), \quad \text{dom}(H_0) = [W^{1,2}(\mathbb{R}^n)]^N, \quad (2.2)$$

where $\partial_j = \partial/\partial x_j$, $1 \leq j \leq n$.

(iii) *Next, consider the self-adjoint matrix-valued potential $V = \{V_{\ell,m}\}_{1 \leq \ell, m \leq N}$ satisfying for some fixed $\rho \in (1, \infty)$, $C \in (0, \infty)$,*

$$V \in [L^\infty(\mathbb{R}^n)]^{N \times N}, \quad |V_{\ell,m}(x)| \leq C \langle x \rangle^{-\rho} \text{ for a.e. } x \in \mathbb{R}^n, \quad 1 \leq \ell, m \leq N. \quad (2.3)$$

Under these assumptions on V , the massless Dirac operator H in $[L^2(\mathbb{R}^n)]^N$ is defined via

$$H = H_0 + V, \quad \text{dom}(H) = \text{dom}(H_0) = [W^{1,2}(\mathbb{R}^n)]^N. \quad (2.4)$$

We recall that

$$[L^2(\mathbb{R}^n)]^N = L^2(\mathbb{R}^n; \mathbb{C}^N), \quad [W^{1,2}(\mathbb{R}^n)]^N = W^{1,2}(\mathbb{R}^n; \mathbb{C}^N), \text{ etc.} \quad (2.5)$$

Then H_0 and H are self-adjoint in $[L^2(\mathbb{R}^n)]^N$, with essential spectrum covering the entire real line,

$$\sigma_{ess}(H) = \sigma_{ess}(H_0) = \sigma(H_0) = \mathbb{R}, \quad (2.6)$$

a consequence of relative compactness of V with respect to H_0 . In addition,

$$\sigma_{ac}(H_0) = \mathbb{R}, \quad \sigma_p(H_0) = \sigma_{sc}(H_0) = \emptyset. \quad (2.7)$$

We also recall that the massive free Dirac operator in $[L^2(\mathbb{R}^n)]^N$ associated with the mass parameter $m > 0$ is of the form (with $\beta = \alpha_{n+1}$)

$$H_0(m) = H_0 + m\beta, \quad \text{dom}(H_0(m)) = [W^{1,2}(\mathbb{R}^n)]^N, \quad m > 0, \quad (2.8)$$

and the corresponding interacting massive Dirac operator in $[L^2(\mathbb{R}^n)]^N$ is given by

$$H(m) = H_0(m) + V = H_0 + m\beta + V, \quad \text{dom}(H(m)) = [W^{1,2}(\mathbb{R}^n)]^N, \quad m > 0. \quad (2.9)$$

In this case,

$$\sigma_{ess}(H(m)) = \sigma_{ess}(H_0(m)) = \sigma(H_0(m)) = (-\infty, -m] \cup [m, \infty), \quad m > 0, \quad (2.10)$$

and

$$\sigma_{ac}(H_0(m)) = (-\infty, -m] \cup [m, \infty), \quad \sigma_p(H_0(m)) = \sigma_{sc}(H_0(m)) = \emptyset, \quad m > 0. \quad (2.11)$$

In the special one-dimensional case $n = 1$, one can choose for α_1 either a real constant or one of the three Pauli matrices. Similarly, in the massive case, β would typically be a second Pauli matrix (different from α_1). For simplicity we confine ourselves to $n \in \mathbb{N}$, $n \geq 2$, in the following.

Employing the relations (2.1), one observes that

$$H_0^2 = -I_N \Delta, \quad \text{dom}(H_0^2) = [W^{2,2}(\mathbb{R}^n)]^N. \quad (2.12)$$

Remark 2.2. Since we permit a (sufficiently decaying) matrix-valued potential V in H , this includes, in particular, the case of electromagnetic interactions introduced via minimal coupling, that is, V describes also special cases of the form,

$$H(v, A) := \alpha \cdot (-i\nabla - A) + vI_N = H_0 + [vI_N - \alpha \cdot A], \quad \text{dom}(H(v, A)) = [W^{1,2}(\mathbb{R}^n)]^N, \quad (2.13)$$

where (v, A) represent the electromagnetic potentials on \mathbb{R}^n , with $v : \mathbb{R}^n \rightarrow \mathbb{R}$, $v \in L^\infty(\mathbb{R}^n)$, $A = (A_1, \dots, A_n)$, $A_j : \mathbb{R}^n \rightarrow \mathbb{R}$, $A_j \in L^\infty(\mathbb{R}^n)$, $1 \leq j \leq n$, and for some fixed $\rho > 1$, $C \in (0, \infty)$,

$$|v(x)| + |A_j(x)| \leq C \langle x \rangle^{-\rho}, \quad x \in \mathbb{R}^n, \quad 1 \leq j \leq n. \quad (2.14)$$

The analogous remark applies of course to the massive case with H_0 replaced by $H_0(m)$, $m > 0$. \diamond

We continue with the self-adjoint Laplacian in $L^2(\mathbb{R}^n)$,

$$h_0 = -\Delta, \quad \text{dom}(h_0) = W^{2,2}(\mathbb{R}^n), \quad (2.15)$$

and the interacting Schrödinger operator in $L^2(\mathbb{R}^n)$ given by

$$h = h_0 + v, \quad \text{dom}(h) = W^{2,2}(\mathbb{R}^n), \quad (2.16)$$

where $v \in L^\infty(\mathbb{R}^n)$, v real-valued, satisfies the estimate (2.14) (for simplicity). In this case,

$$\sigma_{ess}(h) = \sigma_{ess}(h_0) = \sigma(h_0) = [0, \infty), \quad (2.17)$$

and

$$\sigma_{ac}(h_0) = [0, \infty), \quad \sigma_p(h_0) = \sigma_{sc}(h_0) = \emptyset. \quad (2.18)$$

The Green's function of h_0 , denoted by $g_0(z; \cdot, \cdot)$, is of the form,

$$\begin{aligned} g_0(z; x, y) &:= (h_0 - zI)^{-1}(x, y) \\ &= \begin{cases} (i/2)z^{-1/2}e^{iz^{1/2}|x-y|}, & n = 1, z \in \mathbb{C} \setminus \{0\}, \\ (i/4)(2\pi z^{-1/2}|x-y|)^{(2-n)/2} H_{(n-2)/2}^{(1)}(z^{1/2}|x-y|), & n \geq 2, z \in \mathbb{C} \setminus \{0\}, \end{cases} \\ &\quad \text{Im}(z^{1/2}) > 0, x, y \in \mathbb{R}^n, x \neq y, \end{aligned} \quad (2.19)$$

and for $z = 0, n \geq 3$,

$$\begin{aligned} g_0(0; x, y) &= \frac{1}{(n-2)\omega_{n-1}}|x-y|^{2-n} \\ &= 4^{-1}\pi^{-n/2}\Gamma((n-2)/2)|x-y|^{2-n}, \quad n \geq 3, x, y \in \mathbb{R}^n, x \neq y. \end{aligned} \quad (2.20)$$

Here $H_\nu^{(1)}(\cdot)$ denotes the Hankel function of the first kind with index $\nu \geq 0$ (cf. [1, Sect. 9.1]) and $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ ($\Gamma(\cdot)$ the Gamma function, cf. [1, Sect. 6.1]) represents the area of the unit sphere S^{n-1} in \mathbb{R}^n .

As $z \rightarrow 0$, $g_0(z; \cdot, \cdot)$ is continuous on the off-diagonal for $n \geq 3$,

$$\begin{aligned} \lim_{\substack{z \rightarrow 0 \\ z \in \mathbb{C} \setminus \{0\}}} g_0(z; x, y) &= g_0(0; x, y) = \frac{1}{(n-2)\omega_{n-1}}|x-y|^{2-n}, \\ &\quad x, y \in \mathbb{R}^n, x \neq y, n \in \mathbb{N}, n \geq 3, \end{aligned} \quad (2.21)$$

but blows up for $n = 1$ as

$$g_0(z; x, y) \underset{\substack{z \rightarrow 0 \\ z \in \mathbb{C} \setminus \{0\}}}{=} (i/2)z^{-1/2} - 2^{-1}|x-y| + O(z^{1/2}|x-y|^2), \quad x, y \in \mathbb{R}, \quad (2.22)$$

and for $n = 2$ as

$$\begin{aligned} g_0(z; x, y) &\underset{\substack{z \rightarrow 0 \\ z \in \mathbb{C} \setminus \{0\}}}{=} -\frac{1}{2\pi} \ln(z^{1/2}|x-y|/2) [1 + O(z|x-y|^2)] + \frac{1}{2\pi} \psi(1) \\ &\quad + O(|z||x-y|^2), \quad x, y \in \mathbb{R}^2, x \neq y. \end{aligned} \quad (2.23)$$

Here $\psi(w) = \Gamma'(w)/\Gamma(w)$ denotes the digamma function (cf. [1, Sect. 6.3]).

In connection with the free massive Dirac operator $H_0(m) = H_0 + m\beta$, $m > 0$, one computes,

$$\begin{aligned} (H_0(m) - zI)^{-1} &= (H_0(m) + zI)(H_0(m)^2 - z^2I)^{-1} \\ &= (-i\alpha \cdot \nabla + m\beta + zI)(h_0 - (z^2 - m^2)I)^{-1}I_N, \end{aligned} \quad (2.24)$$

employing

$$H_0(m)^2 = (h_0 + m^2I)I_N. \quad (2.25)$$

Assuming

$$m > 0, z \in \mathbb{C} \setminus (\mathbb{R} \setminus [-m, m]), \text{Im}(z^2 - m^2)^{1/2} > 0, x, y \in \mathbb{R}^n, x \neq y, n \in \mathbb{N}, n \geq 2, \quad (2.26)$$

and exploiting (2.24), one thus obtains for the Green's function $G_0(m, z; \cdot, \cdot)$ of $H_0(m)$,

$$\begin{aligned} G_0(m, z; x, y) &:= (H_0(m) - zI)^{-1}(x, y) \\ &= i4^{-1}(2\pi)^{(2-n)/2}|x-y|^{2-n}(m\beta + zI_N) \\ &\quad \times [(z^2 - m^2)^{1/2}|x-y|]^{(n-2)/2} H_{(n-2)/2}^{(1)}((z^2 - m^2)^{1/2}|x-y|) \end{aligned}$$

$$\begin{aligned}
& -4^{-1}(2\pi)^{(2-n)/2}|x-y|^{1-n}\alpha \cdot \frac{(x-y)}{|x-y|} \\
& \times [(z^2-m^2)^{1/2}|x-y|]^{n/2}H_{n/2}^{(1)}((z^2-m^2)^{1/2}|x-y|). \tag{2.27}
\end{aligned}$$

Here we employed the identity ([1, p. 361]),

$$[H_\nu^{(1)}(\zeta)]' = -H_{\nu+1}^{(1)}(\zeta) + \nu\zeta^{-1}H_\nu^{(1)}(\zeta), \quad \nu, \zeta \in \mathbb{C}. \tag{2.28}$$

We also recall the asymptotic behavior (cf. [1, p. 360], [53, p. 723–724])

$$H_0^{(1)}(\zeta) \underset{\substack{\zeta \rightarrow 0 \\ \zeta \in \mathbb{C} \setminus \{0\}}}{=} (2i/\pi)\ln(\zeta) + O(|\ln(\zeta)||\zeta|^2), \tag{2.29}$$

$$\begin{aligned}
H_\nu^{(1)}(\zeta) \underset{\substack{\zeta \rightarrow 0 \\ \zeta \in \mathbb{C} \setminus \{0\}}}{=} & -(i/\pi)2^\nu\Gamma(\nu)\zeta^{-\nu} + \begin{cases} O(|\zeta|^{\min(\nu, -\nu+2)}), & \nu \notin \mathbb{N}, \\ O(|\ln(\zeta)||\zeta|^\nu) + O(\zeta^{-\nu+2}), & \nu \in \mathbb{N}, \end{cases} \tag{2.30} \\
& \text{Re}(\nu) > 0,
\end{aligned}$$

$$H_\nu^{(1)}(\zeta) \underset{\zeta \rightarrow \infty}{=} (2/\pi)^{1/2}\zeta^{-1/2}e^{i[\zeta - (\nu\pi/2) - (\pi/4)]}, \quad \nu \geq 0, \text{Im}(\zeta) \geq 0. \tag{2.31}$$

Equations (2.29), (2.30) reveal the facts (still assuming (2.26)),

$$\lim_{\substack{z \rightarrow \pm m \\ z \in \mathbb{C} \setminus \{\pm m\}}} G_0(m, z; x, y) = 4^{-1}\pi^{-n/2}\Gamma((n-2)/2)|x-y|^{2-n}(m\beta \pm mI_N) \tag{2.32}$$

$$\begin{aligned}
& + i2^{-1}\pi^{-n/2}\Gamma(n/2)\alpha \cdot \frac{(x-y)}{|x-y|^n}, \quad m > 0, x, y \in \mathbb{R}^n, x \neq y, n \in \mathbb{N}, n \geq 3, \\
G_0(m, z; x, y) \underset{\substack{z \rightarrow \pm m \\ z \in \mathbb{C} \setminus \{\pm m\}}}{=} & -(4\pi)^{-1}\ln(z^2-m^2)(m\beta \pm mI_2) \\
& - (2\pi)^{-1}\ln(|x-y|)(m\beta \pm mI_2) + i(2\pi)^{-1}\alpha \cdot \frac{(x-y)}{|x-y|^2} \\
& + O((z^2-m^2)\ln(z^2-m^2)), \quad m > 0, x, y \in \mathbb{R}^2, x \neq y. \tag{2.33}
\end{aligned}$$

(Here the remainder term $O((z^2-m^2)\ln(z^2-m^2))$ depends on $x, y \in \mathbb{R}^2$, but this is of no concern at this point.) In particular, $G_0(m, z; \cdot, \cdot)$ blows up logarithmically as $z \rightarrow \pm m$ in dimensions $n = 2$, just as $g_0(z, \cdot, \cdot)$ does as $z \rightarrow 0$.

By contrast, the massless case is quite different and assuming

$$z \in \mathbb{C}_+, x, y \in \mathbb{R}^n, x \neq y, n \in \mathbb{N}, n \geq 2, \tag{2.34}$$

one computes in the case $m = 0$ for the Green's function $G_0(z; \cdot, \cdot)$ of H_0 ,

$$\begin{aligned}
G_0(z; x, y) & := (H_0 - zI)^{-1}(x, y) \\
& = i4^{-1}(2\pi)^{(2-n)/2}|x-y|^{2-n}z[z|x-y|]^{(n-2)/2}H_{(n-2)/2}^{(1)}(z|x-y|)I_N \\
& \quad - 4^{-1}(2\pi)^{(2-n)/2}|x-y|^{1-n}[z|x-y|]^{n/2}H_{n/2}^{(1)}(z|x-y|)\alpha \cdot \frac{(x-y)}{|x-y|}. \tag{2.35}
\end{aligned}$$

The Green's function $G_0(z; \cdot, \cdot)$ of H_0 continuously extends to $z \in \overline{\mathbb{C}_+}$. In addition, in the massless case $m = 0$, the limit $z \rightarrow 0$ exists¹,

$$\begin{aligned} \lim_{\substack{z \rightarrow 0, \\ z \in \overline{\mathbb{C}_+} \setminus \{0\}}} G_0(z; x, y) &:= G_0(0 + i0; x, y) \\ &= i2^{-1} \pi^{-n/2} \Gamma(n/2) \alpha \cdot \frac{(x-y)}{|x-y|^n}, \quad x, y \in \mathbb{R}^n, x \neq y, n \in \mathbb{N}, n \geq 2, \end{aligned} \tag{2.36}$$

and no blow up occurs for all $n \in \mathbb{N}$, $n \geq 2$. This observation is consistent with the sufficient condition for the Dirac operator $H = H_0 + V$ (in dimensions $n \in \mathbb{N}$, $n \geq 2$), with V an appropriate self-adjoint $N \times N$ matrix-valued potential, having no eigenvalues, as derived in [54, Theorems 2.1, 2.3].

The following remark is primarily of a heuristic nature; at this point it just serves as a motivation for our detailed discussion of absence of threshold resonances for Schrödinger and Dirac operators in Section 3.

Remark 2.3. The asymptotic behavior, for some $d_n \in (0, \infty)$,

$$\|G_0(0 + i0; x, y)\|_{\mathcal{B}(\mathbb{C}^N)} \underset{\substack{z \rightarrow 0, \\ z \in \overline{\mathbb{C}_+} \setminus \{0\}}}{=} d_n |x - y|^{1-n}, \quad x, y \in \mathbb{R}^n, x \neq y, n \in \mathbb{N}, n \geq 2, \tag{2.37}$$

combined with the homogeneous Lippmann–Schwinger-type integral equation in (3.79), formally “implies” the absence of zero-energy resonances (cf. Section 3 for a detailed discussion) of H for $n \in \mathbb{N}$, $n \geq 3$, for sufficiently fast decaying short-range potentials V at infinity, as $|\cdot|^{1-n}$ lies in $L^2(\mathbb{R}^n; d^n x)$ near infinity if and only if $n \geq 3$. This is consistent with observations in [7], [12, Sect. 4.4], [13], [16], [67], [68], [78] for $n = 3$ (see also Remark 3.9 (iii)). This should be contrasted with the behavior of Schrödinger operators where

$$\begin{aligned} \lim_{\substack{z \rightarrow 0 \\ z \in \overline{\mathbb{C}} \setminus \{0\}}} g_0(z; x, y) &= g_0(0; x, y) = \frac{1}{(n-2)\omega_{n-1}} |x-y|^{2-n}, \\ &x, y \in \mathbb{R}^n, x \neq y, n \in \mathbb{N}, n \geq 3, \end{aligned} \tag{2.38}$$

“implies” the absence of zero-energy resonances of $h = h_0 + v$ for $n \in \mathbb{N}$, $n \geq 5$, again for sufficiently fast decaying short-range potentials v at infinity, as $|\cdot|^{2-n}$ lies in $L^2(\mathbb{R}^n; d^n x)$ near infinity if and only if $n \geq 5$, as observed in [50].

Since H_0 has no spectral gap, $\sigma(H_0) = \mathbb{R}$, but h_0 has the half-line $(-\infty, 0)$ in its resolvent set, a comparison of h_0 with the massive free Dirac operator $H_0(m) = H_0 + m\beta$, $m > 0$, with spectral gap $(-m, m)$, replacing the energy $z = 0$ by $z = \pm m$, is quite natural and then exhibits a similar logarithmic blowup behavior as $z \rightarrow 0$ in dimensions $n = 2$. Finally, the leading asymptotic behavior $|\cdot|^{2-n}$ of the free massive Green's matrix $G_0(m, \pm m; \cdot, \cdot)$ in (2.32) near infinity then “implies” the absence of threshold resonances of $H(m) = H_0(m) + V$ for $n \in \mathbb{N}$, $n \geq 5$, as in the case of Schrödinger operators h .

Section 3 is devoted to a rigorous treatment of the formal statements in this remark. \diamond

Next, we also recall some basic facts on L^p -properties of Riesz potentials (see, e.g., [74, Sect. V.1]):

¹Our choice of notation $0 + i0$ in $G_0(0 + i0; x, y)$ indicates that the limit $\lim_{z \rightarrow 0}$ is performed in the closed upper half-plane $\overline{\mathbb{C}_+}$.

Theorem 2.4. *Let $n \in \mathbb{N}$, $\alpha \in (0, n)$, and introduce the Riesz potential operator $\mathcal{R}_{\alpha, n}$ as follows:*

$$\begin{aligned} (\mathcal{R}_{\alpha, n} f)(x) &= ((-\Delta)^{-\alpha/2} f)(x) = \gamma(\alpha, n)^{-1} \int_{\mathbb{R}^n} d^n y |x - y|^{\alpha-n} f(y), \\ \gamma(\alpha, n) &= \pi^{n/2} 2^\alpha \Gamma(\alpha/2) / \Gamma((n - \alpha)/2), \end{aligned} \quad (2.39)$$

for appropriate functions f (see below).

(i) *Let $p \in [1, \infty)$ and $f \in L^p(\mathbb{R}^n)$. Then the integral $(\mathcal{R}_{\alpha, n} f)(x)$ converges for (Lebesgue) a.e. $x \in \mathbb{R}^n$.*

(ii) *Let $1 < p < q < \infty$, $q^{-1} = p^{-1} - \alpha n^{-1}$, and $f \in L^p(\mathbb{R}^n)$. Then there exists $C_{p, q, \alpha, n} \in (0, \infty)$ such that*

$$\|\mathcal{R}_{\alpha, n} f\|_{L^q(\mathbb{R}^n)} \leq C_{p, q, \alpha, n} \|f\|_{L^p(\mathbb{R}^n)}. \quad (2.40)$$

We also note the β function-type integral (cf. [74, p. 118]),

$$\begin{aligned} \int_{\mathbb{R}^n} d^n y |e_k - y|^{\alpha-n} |y|^{\beta-n} &= \gamma(\alpha, n) \gamma(\beta, n) / \gamma(\alpha + \beta, n), \\ 0 < \alpha < n, \quad 0 < \beta < n, \quad \alpha + \beta < n, \\ e_k &= (0, \dots, \underbrace{1}_k, \dots, 0), \quad 1 \leq k \leq n, \end{aligned} \quad (2.41)$$

and more generally, the Riesz composition formula (see [27, Sects. 3.1, 3.2]),

$$\begin{aligned} \int_{\mathbb{R}^n} d^n y |x - y|^{\alpha-n} |y - w|^{\beta-n} &= [\gamma(\alpha, n) \gamma(\beta, n) / \gamma(\alpha + \beta, n)] |x - w|^{\alpha+\beta-n}, \\ 0 < \alpha < n, \quad 0 < \beta < n, \quad 0 < \alpha + \beta < n, \quad x, w \in \mathbb{R}^n. \end{aligned} \quad (2.42)$$

For later use in Section 3, we recall the following estimate taken from [36, Lemma 6.3].

Lemma 2.5. *Let $n \in \mathbb{N}$ and $x_1, x_2 \in \mathbb{R}^n$. If $k, \ell \in [0, n)$, $\varepsilon, \beta \in (0, \infty)$, with $k + \ell + \beta \geq n$, and $k + \ell \neq n$, then*

$$\begin{aligned} \int_{\mathbb{R}^n} d^n y |x_1 - y|^{-k} \langle y \rangle^{-\beta-\varepsilon} |y - x_2|^{-\ell} \\ \leq C_{n, k, \ell, \beta, \varepsilon} \cdot \begin{cases} |x_1 - x_2|^{-\max\{0, k+\ell-n\}}, & |x_1 - x_2| \leq 1, \\ |x_1 - x_2|^{-\min\{k, \ell, k+\ell+\beta-n\}}, & |x_1 - x_2| \geq 1, \end{cases} \end{aligned} \quad (2.43)$$

where $C_{n, k, \ell, \beta, \varepsilon} \in (0, \infty)$ is an x_1, x_2 -independent constant.

We conclude this section by recalling the following interesting results of McOwen [61] and Nirenberg–Walker [64], which provide necessary and sufficient conditions for the boundedness of certain classes of integral operators in $L^p(\mathbb{R}^n)$:

Theorem 2.6. *Let $n \in \mathbb{N}$, $c, d \in \mathbb{R}$, $c + d > 0$, $p \in (1, \infty)$, and $p' = p/(p - 1)$. The following items (i) and (ii) hold.*

(i) *If*

$$K_{c, d}(x, y) = |x|^{-c} |x - y|^{(c+d)-n} |y|^{-d}, \quad x, y \in \mathbb{R}^n, \quad x \neq y, \quad (2.44)$$

then the integral operator $K_{c, d}$ in $L^p(\mathbb{R}^n)$ with integral kernel $K_{c, d}(\cdot, \cdot)$ in (2.44) is bounded if and only if $c < n/p$ and $d < n/p'$.

(ii) *If*

$$\tilde{K}_{c, d}(x, y) = (1 + |x|)^{-c} |x - y|^{(c+d)-n} (1 + |y|)^{-d}, \quad x, y \in \mathbb{R}^n, \quad x \neq y, \quad (2.45)$$

then the integral operator $\tilde{K}_{c,d}$ in $L^p(\mathbb{R}^n)$ with integral kernel $\tilde{K}_{c,d}(\cdot, \cdot)$ in (2.45) is bounded if and only if $c < n/p$ and $d < n/p'$.

3. NONEXISTENCE OF THRESHOLD RESONANCES

In this section we prove our principal results on nonexistence of threshold resonances in three cases: First, in the case of Schrödinger operators in dimension $n \geq 3$; second, in the case of massless Dirac operators in dimensions $n \geq 2$; and third, in the case of massive Dirac operators in dimensions $n \geq 3$.

Schrödinger Operators in \mathbb{R}^n , $n \geq 3$. We begin with the case of Schrödinger operators h in dimension $n \geq 3$ as defined in (2.15), and start by making the following assumptions on the potential v .

Hypothesis 3.1. *Let $n \in \mathbb{N}$, $n \geq 3$. Assume the a.e. real-valued potential v satisfies for some fixed $C \in (0, \infty)$,*

$$v \in L^\infty(\mathbb{R}^n), \quad |v(x)| \leq C\langle x \rangle^{-4} \text{ for a.e. } x \in \mathbb{R}^n. \quad (3.1)$$

In addition, we suppose that

$$\begin{aligned} v &= v_1 v_2 = |v|^{1/2} u_v |v|^{1/2}, \\ \text{where } v_1 &= |v|^{1/2}, \quad v_2 = u_v |v|^{1/2}, \quad u_v = \text{sgn}(v). \end{aligned} \quad (3.2)$$

Here we abbreviated

$$\text{sgn}(v(x)) = \begin{cases} 1, & v(x) \geq 0, \\ -1, & v(x) < 0, \end{cases} \text{ for a.e. } x \in \mathbb{R}^n. \quad (3.3)$$

We continue with the threshold behavior, that is, the $z = 0$ behavior, of h :

Definition 3.2. *Assume Hypothesis 3.1.*

(i) *The point 0 is called a zero-energy eigenvalue of h if $h\psi = 0$ has a distributional solution ψ satisfying*

$$\psi \in \text{dom}(h) = W^{2,2}(\mathbb{R}^n) \quad (3.4)$$

(equivalently, $\ker(h) \supsetneq \{0\}$).

(ii) *The point 0 is called a zero-energy (or threshold) resonance of h if*

$$\ker \left([I_{L^2(\mathbb{R}^n)} + v_2(h_0 + 0I_{L^2(\mathbb{R}^n)})^{-1}v_1] \right) \supsetneq \{0\} \quad (3.5)$$

and if there exists $0 \neq \phi \in \ker \left([I_{L^2(\mathbb{R}^n)} + v_2(h_0 + 0I_{L^2(\mathbb{R}^n)}^{-1}v_1] \right)$ such that ψ defined by

$$\begin{aligned} \psi(x) &= -((h_0 + 0I_{L^2(\mathbb{R}^n)})^{-1}v_1\phi)(x) \\ &= [(n-2)\omega_{n-1}]^{-1} \int_{\mathbb{R}^n} d^n y |x-y|^{2-n} v_1(y)\phi(y) \end{aligned} \quad (3.6)$$

(for a.e. $x \in \mathbb{R}^n$, $n \geq 3$) is a distributional solution of $h\psi = 0$ satisfying

$$\psi \notin L^2(\mathbb{R}^n). \quad (3.7)$$

(iii) *0 is called a regular point for h if it is neither a zero-energy eigenvalue nor a zero-energy resonance of h .*

Additional properties of ψ are isolated in Theorem 3.3.

While the point 0 being regular for h is the generic situation, zero-energy eigenvalues and/or resonances are exceptional cases.

Next, we introduce the following convenient abbreviation (for $x, y \in \mathbb{R}^n$, $x \neq y$):

$$\begin{aligned} r_{0,0}(x-y) &= \lim_{\substack{z \rightarrow 0 \\ z \in \mathbb{C} \setminus \{0\}}} g(z; x, y) = g_0(0; x, y) = [(n-2)\omega_{n-1}]^{-1}|x-y|^{2-n} \\ &= 4^{-1}\pi^{-n/2}\Gamma((n-2)/2)|x-y|^{2-n}, \quad n \geq 3, \end{aligned} \quad (3.8)$$

and note that

$$-\Delta_x r_{0,0}(x-y) = -\Delta_x g_0(0; x, y) = \delta(x-y), \quad x, y \in \mathbb{R}^n, \quad x \neq y, \quad n \geq 3, \quad (3.9)$$

in the sense of distributions.

In the remainder of this section we will frequently apply [44, Prop. 6.10], that is, the fact that $0 < p < q < r \leq \infty$ implies $L^p \cap L^r \subset L^q$.

Theorem 3.3. *Assume Hypothesis 3.1.*

(i) *If $n = 3, 4$, there are precisely four possible cases:*

Case (I): 0 is regular for h .

Case (II): 0 is a (necessarily simple²) resonance of h . In this case, the resonance function ψ satisfies

$$\psi \in L^q(\mathbb{R}^3), \quad q \in (3, \infty) \cup \{\infty\}, \quad (3.10)$$

$$\psi \in L^q(\mathbb{R}^4), \quad q \in (2, \infty) \cup \{\infty\}, \quad (3.11)$$

$$\nabla\psi \in [L^2(\mathbb{R}^n)]^n, \quad \Delta\psi \in L^q(\mathbb{R}^n), \quad q \in [2, \infty) \cup \{\infty\}, \quad n = 3, 4, \quad (3.12)$$

$$\psi \notin L^2(\mathbb{R}^n), \quad n = 3, 4. \quad (3.13)$$

Case (III): 0 is a (possibly degenerate) eigenvalue of h . In this case, the corresponding eigenfunctions $\psi \in \text{dom}(h) = W^{2,2}(\mathbb{R}^n)$, $n \in \mathbb{N}$, $n \geq 3$, of $h\psi = 0$ also satisfy

$$\psi, \Delta\psi \in L^q(\mathbb{R}^n), \quad q \in [2, \infty) \cup \{\infty\}, \quad n = 3, 4, \quad (3.14)$$

$$\nabla\psi \in [L^2(\mathbb{R}^n)]^n, \quad n = 3, 4. \quad (3.15)$$

Case (IV): A possible mixture of Cases (II) and (III).

(ii) *If $n \in \mathbb{N}$, $n \geq 5$, there are precisely two possible cases:*

Case (I): 0 is regular for h .

Case (II): 0 is a (possibly degenerate) eigenvalue of h . In this case, the corresponding eigenfunctions $\psi \in \text{dom}(h) = W^{2,2}(\mathbb{R}^n)$ of $h\psi = 0$ also satisfy

$$\psi, \Delta\psi \in L^q(\mathbb{R}^n), \quad q \in \begin{cases} (n/(n-2), \infty) \cup \{\infty\}, & 5 \leq n \leq 7, \\ (2n/(n+4), 2n/(n-4)), & n \geq 8, \end{cases} \quad (3.16)$$

$$\nabla\psi \in [L^q(\mathbb{R}^n)]^n, \quad q \in \begin{cases} (5/3, \infty), & n = 5, \\ (3/2, \infty), & n = 6, \\ (7/5, 14), & n = 7, \\ (2n/(n+4), 2n/(n-6)), & n \geq 8. \end{cases} \quad (3.17)$$

²One can show that if $n = 3, 4$, there is at most one resonance function in Case (II), see [8], [51], [52].

In particular, there are no zero-energy resonances of h in dimension $n \geq 5$.

(iii) The point 0 is regular for h if and only if

$$\ker \left([I_{L^2(\mathbb{R}^n)} + \overline{v_2(h_0 + 0I_{L^2(\mathbb{R}^n)})^{-1}v_1}] \right) = \{0\}. \quad (3.18)$$

Proof. Since $g_0(0; x, y)$, $x \neq y$, exists for all $n \geq 3$ (cf. (2.20)), the Birman–Schwinger eigenvalue equation

$$[I_{L^2(\mathbb{R}^n)} + \overline{v_2(h_0 - 0I_{L^2(\mathbb{R}^n)})^{-1}v_1}] \phi_0 = 0, \quad 0 \neq \phi_0 \in L^2(\mathbb{R}^n), \quad (3.19)$$

gives rise to a distributional zero-energy solution $\psi_0 \in L^1_{\text{loc}}(\mathbb{R}^n)$ of $h\psi_0 = 0$ in terms of ϕ_0 of the form (for a.e. $x \in \mathbb{R}^n$, $n \geq 3$),

$$\begin{aligned} \psi_0(x) &= -((h_0 - 0I_{L^2(\mathbb{R}^n)})^{-1}v_1\phi_0)(x) \\ &= -[r_{0,0} * (v_1\phi_0)](x) \end{aligned} \quad (3.20)$$

$$= -4^{-1}\pi^{-n/2}\Gamma((n-2)/2) \int_{\mathbb{R}^n} d^n y |x-y|^{2-n} v_1(y) \phi_0(y), \quad (3.21)$$

$$\phi_0(x) = (v_2\psi_0)(x). \quad (3.22)$$

In particular, one concludes that $\psi_0 \neq 0$.

We note from the outset that $\Delta\psi = v\psi$ with $v \in L^\infty(\mathbb{R}^n)$ yields $\Delta\psi \in L^r(\mathbb{R}^n)$ whenever $\psi \in L^r(\mathbb{R}^n)$ for some $r \geq 1$. Moreover, since also $\Delta\psi = v_1\phi$, with $v_1 \in L^\infty(\mathbb{R}^n)$ and $\phi \in L^2(\mathbb{R}^n)$, this yields $\Delta\psi \in L^2(\mathbb{R}^n)$, and, if in addition ψ is an eigenfunction of H , then of course $\psi \in W^{1,2}(\mathbb{R}^n)$.

Thus, one estimates, with $|v_1(y)| \leq c\langle y \rangle^{-2}$ and some constant $d_n \in (0, \infty)$,

$$\begin{aligned} |\psi_0(x)| &\leq d_n \int_{\mathbb{R}^n} d^n y |x-y|^{2-n} \langle y \rangle^{-2} |\phi_0(y)| \\ &= d_n \mathcal{R}_{2,n}(\langle \cdot \rangle^{-2} |\phi_0(\cdot)|)(x) \end{aligned} \quad (3.23)$$

$$= d_n \mathcal{R}_{2,n}(\langle \cdot \rangle^{-2} |v_2(\cdot)| |\psi_0(\cdot)|)(x), \quad x \in \mathbb{R}^n. \quad (3.24)$$

An application of Theorem 2.6 (ii) with $c = 0$, $d = 2$, $p = p' = 2$, and the inequality $2 < n/2$, combined with $\phi_0 \in L^2(\mathbb{R}^n)$, then yield

$$\psi_0 \in L^2(\mathbb{R}^n), \quad n \geq 5. \quad (3.25)$$

To prove that actually $\psi_0 \in \text{dom}(h) = W^{2,2}(\mathbb{R}^n)$, it suffices to argue as follows:

$$\Delta\psi_0 = v\psi_0 \in L^2(\mathbb{R}^n), \quad n \geq 5, \quad (3.26)$$

in the sense of distributions since $v \in L^\infty(\mathbb{R}^n)$ and $\psi_0 \in L^2(\mathbb{R}^n)$. Thus, also $\nabla\psi_0 \in [L^2(\mathbb{R}^n)]^n$, and hence,

$$\psi_0 \in W^{2,2}(\mathbb{R}^n), \quad n \geq 5. \quad (3.27)$$

Moreover, employing (3.20), one obtains for $n \geq 3$ that

$$-(\Delta\psi_0)(x) = \Delta_x(r_{0,0} * v_1\phi_0)(x) = -v_1(x)\phi_0(x), \quad (3.28)$$

and hence

$$(\Delta\psi_0) \in L^2(\mathbb{R}^n), \quad n \geq 3, \quad (3.29)$$

since $v_1 \in L^\infty(\mathbb{R}^n)$ and $\phi_0 \in L^2(\mathbb{R}^n)$. In addition, (3.22) yields for a.e. $x \in \mathbb{R}^n$, $n \geq 3$,

$$(\nabla\psi_0)(x) = 4^{-1}\pi^{-n/2}(2-n)\Gamma((n-2)/2) \int_{\mathbb{R}^n} d^n y |x-y|^{1-n} \frac{(x-y)}{|x-y|} v_1(y) \phi_0(y), \quad (3.30)$$

implying

$$\begin{aligned} |(\nabla\psi_0)(x)| &\leq \tilde{c}_n \int_{\mathbb{R}^n} d^n y |x-y|^{1-n} |v_1(y)| |\phi_0(y)| \\ &\leq \hat{c}_n \int_{\mathbb{R}^n} d^n y |x-y|^{1-n} \langle y \rangle^{-2} |\phi_0(y)| \end{aligned} \quad (3.31)$$

for some $\tilde{c}_n, \hat{c}_n \in (0, \infty)$. An application of Theorem 2.6 (ii) with $c = 0$, $d = 1$, $p = p' = 2$, and the condition $1 < n/2$ then implies

$$(\nabla\psi_0) \in [L^2(\mathbb{R}^n)]^n, \quad n \geq 3. \quad (3.32)$$

Returning to the estimate (3.23), and invoking the Riesz potential $\mathcal{R}_{2,n}$ (cf. Theorem 2.4), one obtains (for some constant $\tilde{d}_n \in (0, \infty)$)

$$\begin{aligned} |\psi_0(x)| &\leq d_n \int_{\mathbb{R}^n} d^n y |x-y|^{2-n} \langle y \rangle^{-2} |\phi_0(y)| \\ &\leq \tilde{d}_n \mathcal{R}_{2,n}(\langle y \rangle^{-2} |\phi_0(\cdot)|)(x), \quad x \in \mathbb{R}^n, \end{aligned} \quad (3.33)$$

and hence (2.40) implies (for some constant $\tilde{C}_{p,q,n} \in (0, \infty)$)

$$\begin{aligned} \|\psi_0\|_{L^q(\mathbb{R}^n)} &\leq \tilde{d}_n \|\mathcal{R}_{2,n}(\langle y \rangle^{-2} |\phi_0(\cdot)|)\|_{L^q(\mathbb{R}^n)} \\ &\leq \tilde{C}_{p,q,n} \|\langle y \rangle^{-2} |\phi_0(\cdot)|\|_{L^p(\mathbb{R}^n)} \\ &\leq \tilde{C}_{p,q,n} \|\langle y \rangle^{-2}\|_{L^s(\mathbb{R}^n)} \|\phi_0\|_{L^2(\mathbb{R}^n)}, \end{aligned} \quad (3.34)$$

$$1 < p < q < \infty, \quad p^{-1} = q^{-1} + 2n^{-1}, \quad s = 2qn[2n + 4q - qn]^{-1} \geq 1.$$

In particular,

$$p = qn/(n + 2q), \quad 2n + 4q - qn > 0. \quad (3.35)$$

(a) The case $n = 3$: Then $p = 3q/(3 + 2q)$ and $p > 1$ requires $q \in (3, \infty)$, hence $p \in (1, 3/2)$, and $s = 6q/(q + 6) > 2$. Thus, (3.34) and $\|\langle \cdot \rangle^{-2}\|_{L^s(\mathbb{R}^3)} < \infty$ imply

$$\psi_0 \in L^q(\mathbb{R}^3), \quad q \in (3, \infty). \quad (3.36)$$

To prove that a resonance function (resp., eigenfunction) satisfies $\psi \in L^\infty(\mathbb{R}^3)$, one applies (3.22) and the fact that $|v_1(\cdot)| \leq c\langle \cdot \rangle^{-2}$ for some constant $c \in (0, \infty)$ in (3.21) to obtain

$$|\psi(x)| \leq \tilde{d}_3 \int_{\mathbb{R}^3} d^3 y |x-y|^{-1} \langle y \rangle^{-4} |\psi(y)|, \quad x \in \mathbb{R}^3, \quad (3.37)$$

for an appropriate x -independent constant $\tilde{d}_3 \in (0, \infty)$. By Hölder's inequality (with conjugate exponents $p = 5/4$ and $p' = 5$),

$$|\psi(x)| \leq \tilde{d}_3 \left(\int_{\mathbb{R}^3} d^3 y |x-y|^{-5/4} \langle y \rangle^{-5} \right)^{4/5} \left(\int_{\mathbb{R}^3} d^3 y |\psi(y)|^5 \right)^{1/5}, \quad x \in \mathbb{R}^3. \quad (3.38)$$

The second integral on the right-hand side in (3.38) is finite since a resonance function (resp., eigenfunction) satisfies $\psi \in L^5(\mathbb{R}^3)$. An application of Lemma 2.5 with $x_1 = x$, $k = 5/4$, $\ell = 0$, $\beta = 4$, and $\varepsilon = 1$ then implies

$$|\psi(x)| \leq \tilde{d}_3 C_{3,5/4,0,4,1}^{4/5} \|\psi\|_{L^5(\mathbb{R}^3)}, \quad x \in \mathbb{R}^3, \quad (3.39)$$

and hence it follows that $\psi \in L^\infty(\mathbb{R}^3)$.

(b) The case $n = 4$: Then $p = 4q/(4 + 2q)$ and $p > 1$ requires $q \in (2, \infty)$, hence $p \in (1, 2)$, and $s = q > 2$. Thus, (3.34) and $\|(1 + |\cdot|)^{-2}\|_{L^s(\mathbb{R}^4)} < \infty$ implies

$$\psi_0 \in L^q(\mathbb{R}^4), \quad q \in (2, \infty). \quad (3.40)$$

To prove that a resonance function (resp., eigenfunction) satisfies $\psi \in L^\infty(\mathbb{R}^4)$, one applies (3.22) and the fact that $|v_1(\cdot)| \leq c\langle \cdot \rangle^{-2}$ for some constant $c \in (0, \infty)$ in (3.21) to obtain

$$|\psi(x)| \leq \tilde{d}_4 \int_{\mathbb{R}^4} d^4y |x - y|^{-2} \langle y \rangle^{-4} |\psi(y)|, \quad x \in \mathbb{R}^4, \quad (3.41)$$

for an appropriate x -independent constant $\tilde{d}_4 \in (0, \infty)$. By Hölder's inequality (with conjugate exponents $p = 3/2$ and $p' = 3$),

$$|\psi(x)| \leq \tilde{d}_4 \left(\int_{\mathbb{R}^4} d^4y |x - y|^{-3} \langle y \rangle^{-6} \right)^{2/3} \left(\int_{\mathbb{R}^4} d^4y |\psi(y)|^3 \right)^{1/3}, \quad x \in \mathbb{R}^4. \quad (3.42)$$

The second integral on the right-hand side in (3.42) is finite since a resonance function (resp., eigenfunction) satisfies $\psi \in L^3(\mathbb{R}^4)$. An application of Lemma 2.5 with $x_1 = x$, $k = 3$, $\ell = 0$, $\beta = 5$, and $\varepsilon = 1$ then implies

$$|\psi(x)| \leq \tilde{d}_4 C_{4,3,0,5,1}^{2/3} \|\psi\|_{L^3(\mathbb{R}^4)}, \quad x \in \mathbb{R}^4, \quad (3.43)$$

and it follows that $\psi \in L^\infty(\mathbb{R}^4)$.

(c) The case $n \geq 5$: Returning to (3.21), we employ the fact $\phi_0 = v_2\psi_0$ and hence obtain for some constants $\tilde{D}_n, D_n \in (0, \infty)$,

$$\begin{aligned} |\psi_0(x)| &\leq \tilde{D}_n \int_{\mathbb{R}^n} d^n y |x - y|^{2-n} \langle y \rangle^{-4} |\psi_0(y)| \\ &= D_n \mathcal{R}_{2,n}(\langle \cdot \rangle^{-4} |\psi_0(\cdot)|)(x), \quad x \in \mathbb{R}^n, \end{aligned} \quad (3.44)$$

and hence by Theorem 2.4 (ii)

$$\begin{aligned} \|\psi_0\|_{L^q(\mathbb{R}^n)} &\leq D_n \|\mathcal{R}_{2,n}(\langle \cdot \rangle^{-4} |\psi_0(\cdot)|)\|_{L^q(\mathbb{R}^n)} \\ &\leq \widehat{C}_{p,q,n} \|\langle \cdot \rangle^{-4} |\psi_0(\cdot)|\|_{L^p(\mathbb{R}^n)} \\ &\leq \widehat{C}_{p,q,n} \|\langle \cdot \rangle^{-4}\|_{L^s(\mathbb{R}^n)} \|\psi_0\|_{L^2(\mathbb{R}^n)} \end{aligned} \quad (3.45)$$

for some $\widehat{C}_{p,q,n} \in (0, \infty)$. Then $p = qn/(n + 2q) > 1$ yields $q > n/(n - 2)$. Since $s = 2qn/(2n + 4q - qn)$ and one needs $s \geq 1$ and hence $2n + 4q - qn > 0$, this implies $q < 2n/(n - 4)$. Thus, $q \in (n/(n - 2), 2n/(n - 4))$ and hence $s > 2$. The condition $\|\langle \cdot \rangle^{-4}\|_{L^s(\mathbb{R}^n)} < \infty$ then implies $q > 2n/(n + 4)$. Altogether one obtains

$$q \in \begin{cases} (n/(n - 2), 2n/(n - 4)), & 5 \leq n \leq 7, \\ (2n/(n + 4), 2n/(n - 4)), & n \geq 8, \end{cases} \quad (3.46)$$

and thus,

$$\psi_0 \in L^q(\mathbb{R}^n), \quad q \in \begin{cases} (n/(n - 2), 2n/(n - 4)), & 5 \leq n \leq 7, \\ (2n/(n + 4), 2n/(n - 4)), & n \geq 8. \end{cases} \quad (3.47)$$

Since $\Delta\psi_0 = v\psi_0$ and $v \in L^\infty(\mathbb{R}^n)$, (3.47) also implies

$$\Delta\psi_0 \in L^q(\mathbb{R}^n), \quad q \in \begin{cases} (n/(n - 2), 2n/(n - 4)), & 5 \leq n \leq 7, \\ (2n/(n + 4), 2n/(n - 4)), & n \geq 8, \end{cases} \quad (3.48)$$

and hence (3.16).

Employing $\phi_0 = v_2\psi_0$ in (3.31) yields

$$\begin{aligned} |(\nabla\psi_0)(x)| &\leq \tilde{C}_n \int_{\mathbb{R}^n} d^n y |x-y|^{1-n} \langle y \rangle^{-4} |\psi_0(y)| \\ &= C_n \mathcal{R}_{1,n}(\langle \cdot \rangle^{-4} |\psi_0(\cdot)|)(x), \quad x \in \mathbb{R}^n, \end{aligned} \quad (3.49)$$

for some $\tilde{C}_n, C_n \in (0, \infty)$. Thus, Theorem 2.4 (ii) yields

$$|\nabla\psi_0| \in L^r(\mathbb{R}^n), \quad r = qn/(n-q), \quad q < n, \quad (3.50)$$

with q given as in (3.47). Working out the details yields

$$|\nabla\psi_0| \in L^r(\mathbb{R}^n), \quad r \in \begin{cases} (5/2, \infty), & n = 5, \\ (2, \infty), & n = 6, \\ (7/4, 14), & n = 7, \\ (2n/(n-2), 2n/(n-6)), & n \geq 8. \end{cases} \quad (3.51)$$

On the other hand, applying Theorem 2.6 (ii) to the first line in (3.49) with $c = 0$, $d = 1$, $1 < n/q' = n(q-1)/q$, and hence $q > n/(n-1)$ yields

$$|\nabla\psi_0| \in L^q(\mathbb{R}^n), \quad q \in \begin{cases} (n/(n-2), 2n/(n-4)), & 5 \leq n \leq 7, \\ (2n/(n+4), 2n/(n-4)), & n \geq 8. \end{cases} \quad (3.52)$$

A comparison of (3.51) and (3.52) implies (3.17).

To prove essential boundedness of eigenfunctions ψ in dimensions $n \in \{5, 6, 7\}$, one notes that a combination of (3.21), (3.22), and the fact that $|v_1(\cdot)| \leq c\langle \cdot \rangle^{-2}$ for some constant $c \in (0, \infty)$, yields

$$|\psi(x)| \leq \tilde{d}_n \int_{\mathbb{R}^n} d^n y |x-y|^{2-n} \langle y \rangle^{-4} |\psi(y)|, \quad x \in \mathbb{R}^n, \quad n \geq 5, \quad (3.53)$$

for an appropriate x -independent constant $\tilde{d}_n \in (0, \infty)$. To prove that the right-hand side in (3.53) is essentially bounded, we make use of (3.47) and consider the cases $n \in \{5, 6, 7\}$ next. By (3.47), $\psi \in L^q(\mathbb{R}^n)$ for all

$$q \in (n/(n-2), 2n/(n-4)). \quad (3.54)$$

Note that q satisfies (3.54) if and only if its conjugate exponent $q' = q/(q-1)$ satisfies

$$q' \in (2n/(n+4), n/2). \quad (3.55)$$

Choose $q'_0 \in (2n/(n+4), n/(n-2)) \subset (2n/(n+4), n/2)$. Then $q_0 := q'_0/(q'_0 - 1) \in (n/(n-2), 2n/(n-4))$ and $\psi \in L^{q_0}(\mathbb{R}^n)$. Applying Hölder's inequality with conjugate exponents q'_0 and q_0 on the right-hand side of (3.53), one obtains

$$|\psi(x)| \leq \tilde{d}_n \left(\int_{\mathbb{R}^n} d^n y |x-y|^{(2-n)q'_0} \langle y \rangle^{-4q'_0} \right)^{1/q'_0} \left(\int_{\mathbb{R}^n} d^n y |\psi(y)|^{q_0} \right)^{1/q_0}, \quad x \in \mathbb{R}^n. \quad (3.56)$$

The second integral on the right-hand side in (3.56) is finite since $\psi \in L^{q_0}(\mathbb{R}^n)$. To estimate the first integral on the right-hand side in (3.56), one applies Lemma 2.5 with the choices $x_1 = x$, $k = k_0 := (n-2)q'_0$, $\ell = 0$, $\beta = \beta_0 := 4q'_0 - [4n/(n+4)]$, and $\varepsilon = \varepsilon_0 := 4n/(n+4)$, to obtain

$$|\psi(x)| \leq \tilde{d}_n C_{n, k_0, 0, \beta_0, \varepsilon_0}^{1/q'_0} \|\psi\|_{L^{q_0}(\mathbb{R}^n)}, \quad x \in \mathbb{R}^n. \quad (3.57)$$

One notes that the hypotheses of Lemma 2.5 are satisfied since $k = (n-2)q'_0 < (n-2) \cdot n/(n-2) = n$, while

$$\beta = 4q'_0 - 4n/(n+4) \geq 8n/(n+4) - 4n/(n+4) > 0, \quad \langle \cdot \rangle^{-4q'_0} = \langle \cdot \rangle^{-\beta_0 - 4n/(n+4)}, \quad (3.58)$$

and

$$k + \ell + \beta = nq'_0 + 2q'_0 - 4n/(n+4) \geq nq'_0 + 4n/(n+4) - 4n/(n+4) = nq'_0 > n. \quad (3.59)$$

The inequality in (3.57) implies $\psi \in L^\infty(\mathbb{R}^n)$ for $5 \leq n \leq 7$. In Remark A.1 we will illustrate why the same line of reasoning fails for $n \geq 8$.

Finally, we will prove that if $\ker(h) \not\supseteq \{0\}$ then one necessarily also obtains that $\ker\left(\left[I_{L^2(\mathbb{R}^n)} + \overline{v_2(h_0 + 0I_{L^2(\mathbb{R}^n))^{-1}v_1}\right]\right) \not\supseteq \{0\}$. Indeed, if $0 \neq \psi_0 \in \ker(h)$, then $\phi_0 := v_2\psi_0 = u_v v_1\psi_0 \in L^2(\mathbb{R}^n)$ and hence $v_1\phi_0 \in L^2(\mathbb{R}^n)$. Then, $h\psi_0 = 0$ yields $\Delta\psi_0 = v\psi_0 = v_1v_2\psi_0 = v_1\phi_0$.

Thus, an application of (3.9) yields for all $n \geq 3$,

$$\begin{aligned} & -\Delta[\psi_0 + (h_0 + 0I_{L^2(\mathbb{R}^n)})^{-1}v_1\phi_0](x) \\ &= -[\Delta\psi_0](x) - \Delta_x[r_{0,0} * (v_1\phi_0)](x) \\ &= -v(x)\psi_0(x) + v_1(x)\phi_0(x) \\ &= -v(x)\psi_0(x) + v(x)\psi_0(x) = 0. \end{aligned}$$

Consequently,

$$-\Delta[\psi_0 + (h_0 + 0I_{L^2(\mathbb{R}^n)})^{-1}v_1\phi_0] = 0. \quad (3.60)$$

Since $\psi_0 \in [L^2(\mathbb{R}^n)]^N$, and by exactly the same arguments employed in (3.23)–(3.25) also $(h_0 + 0I_{L^2(\mathbb{R}^n)})^{-1}(v_1\phi_0) \in L^2(\mathbb{R}^n)$, one concludes that $-\Delta\tilde{\psi}_0 = 0$ in the sense of distributions, where

$$\tilde{\psi}_0 = [\psi_0 + (h_0 + 0I_{L^2(\mathbb{R}^n)})^{-1}v_1\phi_0] \in L^2(\mathbb{R}^n). \quad (3.61)$$

Thus, also $|\nabla\tilde{\psi}_0| \in L^2(\mathbb{R}^n)$ implying $\tilde{\psi}_0 \in W^{2,2}(\mathbb{R}^n)$. But then $\tilde{\psi}_0 = 0$ since $\ker(h_0) = \{0\}$. Hence,

$$\psi_0 = -(h_0 + 0I_{L^2(\mathbb{R}^n)})^{-1}v_1\phi_0, \quad (3.62)$$

implying, $\phi_0 \neq 0$, and

$$\begin{aligned} 0 &= v_2\tilde{\psi}_0 = v_2\psi_0 + v_2(h_0 + 0I_{L^2(\mathbb{R}^n)})^{-1}v_1\phi_0 \\ &= [I_{L^2(\mathbb{R}^n)} + \overline{v_2(h_0 + 0I_{L^2(\mathbb{R}^n))^{-1}v_1}]\phi_0, \end{aligned} \quad (3.63)$$

that is,

$$0 \neq \phi_0 \in \ker\left(\left[I_{L^2(\mathbb{R}^n)} + \overline{v_2(h_0 + 0I_{L^2(\mathbb{R}^n))^{-1}v_1}\right]\right). \quad (3.64)$$

This concludes the proof. \square

Remark 3.4. Employing $\Delta\psi = v\psi$ with $v \in L^r(\mathbb{R}^n)$ for $r > n/4$ and $\psi \in L^q(\mathbb{R}^n)$ for $q \in (3, \infty) \cup \{\infty\}$, respectively, $\Delta\psi = v_1\phi$, with $v_1 \in L^s(\mathbb{R}^n)$ for $s > n/2$ and $\phi \in L^2(\mathbb{R}^n)$, an application of Hölder's inequality yields additional $L^p(\mathbb{R}^n)$ -properties of $\Delta\psi$, but we omit further details at this point. \diamond

Remark 3.5. (i) For basics on the Birman–Schwinger principle in an abstract context, especially, if the energy parameter in the Birman–Schwinger operator belongs to the resolvent set of the unperturbed operator, we refer to [48] (cf. also [14], [47]) and the extensive literature cited therein. In the concrete case of Schrödinger operators, relations (3.78), (3.80) are discussed at length in [8], [19], [20], [31], [37],

[41], [42], [46], [50]–[53], [62], [76] (see also the list of references quoted therein).

(ii) In physical notation (see, e.g., [63, footnote 3 on p. 300] for details), the zero-energy resonances in Cases (II) and (IV) for $n = 3, 4$, are s -wave resonances (i.e., corresponding to angular momentum zero) in the case where V is spherically symmetric (see also the discussion in [59]).

(iii) As mentioned in Remark 2.3, the absence of zero-energy resonances is well-known in dimensions $n \geq 5$, see [50].

(iv) For discussions of the threshold behavior of resolvents of Schrödinger operators in dimensions $n = 1, 2$ (the cases not studied in this paper) we refer to [17]–[21], [37], [53], [59], [62], [76]. The cases $n \geq 3$ are treated in [8], [31], [35], [41], [42], [46], [50]–[53], [59], [62], [77]. \diamond

Massless Dirac Operators in \mathbb{R}^n , $n \geq 2$. Next, we turn to the case of massless Dirac operators H in dimension $n \geq 2$ as defined in (2.4), and start by making the following assumptions on the matrix-valued potential V .

Hypothesis 3.6. *Let $n \in \mathbb{N}$, $n \geq 2$. Assume the a.e. self-adjoint matrix-valued potential $V = \{V_{\ell,m}\}_{1 \leq \ell, m \leq N}$ satisfies for some $C \in (0, \infty)$,*

$$\begin{aligned} V &\in [L^\infty(\mathbb{R}^n)]^{N \times N}, \\ |V_{\ell,m}(x)| &\leq C \langle x \rangle^{-2} \text{ for a.e. } x \in \mathbb{R}^n, 1 \leq \ell, m \leq N. \end{aligned} \quad (3.65)$$

In addition, alluding to the polar decomposition of $V(\cdot)$ (i.e., $V(\cdot) = U_V(\cdot)|V(\cdot)|$) in the following symmetrized form (cf. [49]), we suppose that

$$V = V_1^* V_2 = |V|^{1/2} U_V |V|^{1/2}, \text{ where } V_1 = V_1^* = |V|^{1/2}, \quad V_2 = U_V |V|^{1/2}. \quad (3.66)$$

We continue with the threshold behavior, that is, the $z = 0$ behavior, of H :

Definition 3.7. *Assume Hypothesis 3.6.*

(i) *The point 0 is called a zero-energy eigenvalue of H if $H\Psi = 0$ has a distributional solution Ψ satisfying*

$$\Psi \in \text{dom}(H) = [W^{1,2}(\mathbb{R}^n)]^N \quad (3.67)$$

(equivalently, $\ker(H) \supsetneq \{0\}$).

(ii) *The point 0 is called a zero-energy (or threshold) resonance of H if*

$$\ker \left([I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0 - (0 + i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}] \right) \supsetneq \{0\}, \quad (3.68)$$

and if there exists $0 \neq \Phi \in \ker \left([I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0 - (0 + i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^}] \right)$ such that Ψ defined by*

$$\begin{aligned} \Psi(x) &= -((H_0 - (0 + i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*\Phi)(x) \\ &= -i2^{-1}\pi^{-n/2}\Gamma(n/2) \int_{\mathbb{R}^n} d^n y |x - y|^{-n} [\alpha(x - y)] V_1(y)^* \Phi(y) \end{aligned} \quad (3.69)$$

(for a.e. $x \in \mathbb{R}^n$, $n \geq 2$) is a distributional solution of $H\Psi = 0$ satisfying

$$\Psi \notin [L^2(\mathbb{R}^n)]^N. \quad (3.70)$$

(iii) *0 is called a regular point for H if it is neither a zero-energy eigenvalue nor a zero-energy resonance of H .*

Additional properties of Ψ are isolated in Theorem 3.8.

While the point 0 being regular for H is the generic situation, zero-energy eigenvalues and/or resonances are exceptional cases.

Next, we introduce the following convenient abbreviation (for $x, y \in \mathbb{R}^n$, $x \neq y$):

$$\begin{aligned} R_{0,0}(x-y) &= \lim_{\substack{z \rightarrow 0 \\ z \in \mathbb{C}_+ \setminus \{0\}}} G_0(z; x, y) = G_0(0 + i0; x, y) \\ &= i2^{-1}\pi^{-n/2}\Gamma(n/2)\alpha \cdot \frac{(x-y)}{|x-y|^n} \\ &= \begin{cases} (2\pi)^{-1}i\alpha[\nabla_x \ln(|x-y|)], & n=2, \\ -i\alpha[\nabla_x g_0(0; x, y)], & n \geq 3. \end{cases} \end{aligned} \quad (3.71)$$

Theorem 3.8. *Assume Hypothesis 3.6.*

(i) *If $n=2$, there are precisely four possible cases:*

Case (I): 0 is regular for H .

Case (II): 0 is a (possibly degenerate³) resonance of H . In this case, the resonance functions Ψ satisfy

$$\begin{aligned} \Psi &\in [L^q(\mathbb{R}^2)]^2, \quad q \in (2, \infty) \cup \{\infty\}, \quad \nabla \Psi \in [L^2(\mathbb{R}^2)]^{2 \times 2}, \\ \Psi &\notin [L^2(\mathbb{R}^2)]^2. \end{aligned} \quad (3.72)$$

Case (III): 0 is a (possibly degenerate) eigenvalue of H . In this case, the corresponding eigenfunctions $\Psi \in \text{dom}(H) = [W^{1,2}(\mathbb{R}^2)]^2$ of $H\Psi = 0$ also satisfy

$$\Psi \in [L^q(\mathbb{R}^2)]^2, \quad q \in [2, \infty) \cup \{\infty\}. \quad (3.73)$$

Case (IV): A possible mixture of Cases (II) and (III).

(ii) *If $n \in \mathbb{N}$, $n \geq 3$, there are precisely two possible cases:*

Case (I): 0 is regular for H .

Case (II): 0 is a (possibly degenerate) eigenvalue of H . In this case, the corresponding eigenfunctions $\Psi \in \text{dom}(H) = [W^{1,2}(\mathbb{R}^n)]^N$ of $H\Psi = 0$ also satisfy

$$\Psi \in [L^q(\mathbb{R}^n)]^N, \quad q \in \begin{cases} (3/2, \infty) \cup \{\infty\}, & n=3, \\ (4/3, 4), & n=4, \\ (2n/(n+2), 2n/(n-2)), & n \geq 5. \end{cases} \quad (3.74)$$

In particular, there are no zero-energy resonances of H in dimension $n \geq 3$.

(iii) *The point 0 is regular for H if and only if*

$$\ker \left([I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}] \right) = \{0\}. \quad (3.75)$$

Proof. Since $G_0(0+i0; x, y)$, $x \neq y$, exists for all $n \geq 2$ (cf. (2.36)), the Birman-Schwinger eigenvalue equation

$$[I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}] \Phi_0 = 0, \quad 0 \neq \Phi_0 \in [L^2(\mathbb{R}^n)]^N, \quad (3.76)$$

gives rise to a distributional zero-energy solution $\Psi_0 \in [L^1_{\text{loc}}(\mathbb{R}^n)]^N$ of $H\Psi_0 = 0$ in terms of Φ_0 of the form (for a.e. $x \in \mathbb{R}^n$, $n \geq 2$),

$$\Psi_0(x) = -((H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*\Phi_0)(x)$$

³One can show that if $n=2$, the degeneracy in Case (II) is at most two, see [33].

$$= -[R_{0,0} * (V_1^* \Phi_0)](x) \quad (3.77)$$

$$= -i2^{-1}\pi^{-n/2}\Gamma(n/2) \int_{\mathbb{R}^n} d^n y |x-y|^{-n} [\alpha(x-y)] V_1(y)^* \Phi_0(y), \quad (3.78)$$

$$= -i2^{-1}\pi^{-n/2}\Gamma(n/2) \int_{\mathbb{R}^n} d^n y |x-y|^{-n} [\alpha(x-y)] V_1(y)^* V_2(y) \Psi_0(y), \quad (3.79)$$

$$\Phi_0(x) = (V_2 \Psi_0)(x). \quad (3.80)$$

In particular, one concludes that $\Psi_0 \neq 0$. Thus, one estimates, with $\|V_1(\cdot)\|_{\mathbb{C}^{N \times N}} \leq c\langle \cdot \rangle^{-1}$ and some constant $d_n \in (0, \infty)$,

$$\begin{aligned} \|\Psi_0(x)\|_{\mathbb{C}^N} &\leq d_n \int_{\mathbb{R}^n} d^n y |x-y|^{1-n} \langle y \rangle^{-1} \|\Phi_0(y)\|_{\mathbb{C}^N} \\ &= d_n \mathcal{R}_{1,n}(\langle \cdot \rangle^{-1} \|\Phi_0(\cdot)\|_{\mathbb{C}^N})(x), \quad x \in \mathbb{R}^n. \end{aligned} \quad (3.81)$$

An application of Theorem 2.6 (ii) with $c = 0$, $d = 1$, $p = p' = 2$, and the inequality $1 < n/2$, combined with $\|\Phi_0(\cdot)\|_{\mathbb{C}^N} \in L^2(\mathbb{R}^n)$, then yield

$$\|\Psi_0(\cdot)\|_{\mathbb{C}^N} \in L^2(\mathbb{R}^n) \text{ and hence, } \Psi_0 \in [L^2(\mathbb{R}^n)]^N, \quad n \geq 3. \quad (3.82)$$

To prove that actually $\Psi_0 \in \text{dom}(H) = [W^{1,2}(\mathbb{R}^n)]^N$, it suffices to argue as follows:

$$i\alpha \cdot \nabla \Psi_0 = -V \Psi_0 \in [L^2(\mathbb{R}^n)]^N \quad (3.83)$$

in the sense of distributions since $V \in [L^\infty(\mathbb{R}^n)]^{N \times N}$ and $\Psi_0 \in [L^2(\mathbb{R}^n)]^N$. Given the fact $\text{dom}(H_0) = [W^{1,2}(\mathbb{R}^n)]^N$ (cf. (2.2)), one concludes

$$\Psi_0 \in [W^{1,2}(\mathbb{R}^n)]^N, \quad n \geq 3. \quad (3.84)$$

Returning to the estimate (3.81), and invoking the Riesz potential $\mathcal{R}_{1,n}$ (cf. Theorem 2.4), one obtains (for some constant $\tilde{d}_n \in (0, \infty)$)

$$\begin{aligned} \|\Psi_0(x)\|_{\mathbb{C}^N} &\leq d_n \int_{\mathbb{R}^n} d^n y |x-y|^{1-n} \langle y \rangle^{-1} \|\Phi_0(y)\|_{\mathbb{C}^N} \\ &\leq \tilde{d}_n \mathcal{R}_{1,n}(\langle \cdot \rangle^{-1} \|\Phi_0(\cdot)\|_{\mathbb{C}^N})(x), \quad x \in \mathbb{R}^n, \end{aligned} \quad (3.85)$$

and hence (2.40) implies (for some constant $\tilde{C}_{p,q,n} \in (0, \infty)$)

$$\begin{aligned} \|\Psi_0\|_{[L^q(\mathbb{R}^n)]^N} &\leq \tilde{d}_n \|\mathcal{R}_{1,n}(\langle \cdot \rangle^{-1} \|\Phi_0(\cdot)\|_{\mathbb{C}^N})\|_{L^q(\mathbb{R}^n)} \\ &\leq \tilde{C}_{p,q,n} \|\langle \cdot \rangle^{-1} \|\Phi_0(\cdot)\|_{\mathbb{C}^N}\|_{L^p(\mathbb{R}^n)} \\ &\leq \tilde{C}_{p,q,n} \|\langle \cdot \rangle^{-1}\|_{L^s(\mathbb{R}^n)} \|\|\Phi_0(\cdot)\|_{\mathbb{C}^N}\|_{L^2(\mathbb{R}^n)} \\ &= \tilde{C}_{p,q,n} \|\langle \cdot \rangle^{-1}\|_{L^s(\mathbb{R}^n)} \|\Phi_0\|_{[L^2(\mathbb{R}^n)]^N}, \end{aligned} \quad (3.86)$$

$$1 < p < q < \infty, \quad p^{-1} = q^{-1} + n^{-1}, \quad s = 2qn[2n + 2q - qn]^{-1} \geq 1.$$

In particular,

$$p = qn/(n+q), \quad 2n + 2q - qn > 0. \quad (3.87)$$

(a) The case $n = 2$: Then one can choose $q \in (2, \infty)$, hence $p = 2q/(q+2) \in (1, 2)$, and $s = q > 2$. Thus, (3.86) and $\|\langle \cdot \rangle^{-1}\|_{L^s(\mathbb{R}^2)} < \infty$ imply

$$\Psi_0 \in [L^q(\mathbb{R}^2)]^N, \quad q \in (2, \infty). \quad (3.88)$$

Recalling $R_{0,0}(\cdot)$ in (3.71), this implies

$$\begin{aligned} -i\alpha \cdot \nabla_x R_{0,0}(x-y) &= -\Delta_x g_0(0; x, y) I_N = \delta(x-y) I_N, \\ x, y \in \mathbb{R}^n, \quad x \neq y, \quad n \geq 2, \end{aligned} \quad (3.89)$$

in the sense of distributions. Here we abused notation a bit and denoted also in the case $n = 2$,

$$g_0(0; x, y) = -(2\pi)^{-1} \ln(|x-y|), \quad x, y \in \mathbb{R}^2, \quad x \neq y, \quad n = 2. \quad (3.90)$$

Thus, one obtains

$$\begin{aligned} i\alpha \cdot (\nabla \Psi)(x) &= -i\alpha \cdot \nabla_x [R_{0,0} * (V_1^* \Phi_0)](x) = -i\alpha \cdot \nabla_x [-i(\alpha \cdot \nabla_x g_0 * (V_1^* \Phi_0))](x) \\ &= (-\Delta_x g_0 I_N) * (V_1^* \Phi_0)](x) = (V_1^* \Phi_0)(x) \in [L^2(\mathbb{R}^2)]^2, \end{aligned} \quad (3.91)$$

proving $\nabla \Psi_0 \in [L^2(\mathbb{R}^2)]^{2 \times 2}$, upon employing the fact that $[\alpha \cdot p]^2 = I_N |p|^2$, $p \in \mathbb{R}^n$.

To prove that $\Psi \in [L^\infty(\mathbb{R}^2)]^2$ in (3.72) and (3.73), one applies (3.80) to the inequality in (3.81), and then employs the condition $\|V_2(\cdot)\|_{\mathcal{C}^{2 \times 2}} \leq C \langle \cdot \rangle^{-1}$ for some constant $C \in (0, \infty)$ to obtain

$$\|\Psi_0(x)\|_{\mathcal{C}^2} \leq \tilde{d}_2 \int_{\mathbb{R}^2} d^2 y |x-y|^{-1} \langle y \rangle^{-2} \|\Psi_0(y)\|_{\mathcal{C}^2}, \quad x \in \mathbb{R}^2, \quad (3.92)$$

where $\tilde{d}_2 \in (0, \infty)$ is an appropriate x -independent constant. By Hölder's inequality,

$$\|\Psi_0(x)\|_{\mathcal{C}^2} \leq \tilde{d}_2 \left(\int_{\mathbb{R}^2} d^2 y |x-y|^{-3/2} \langle y \rangle^{-3} \right)^{2/3} \left(\int_{\mathbb{R}^2} d^2 y \|\Psi_0(y)\|_{\mathcal{C}^2}^3 \right)^{1/3}, \quad x \in \mathbb{R}^2. \quad (3.93)$$

The second integral on the right-hand side in (3.93) is finite since $\Psi_0 \in [L^3(\mathbb{R}^2)]^2$. Choosing $x_1 = x$, $k = 3/2$, $\ell = 0$, $\beta = 2$, and $\varepsilon = 1$ in Lemma 2.5, one infers that

$$\int_{\mathbb{R}^2} d^2 y |x-y|^{-3/2} \langle y \rangle^{-3} \leq C_{2,3/2,0,2,1}, \quad x \in \mathbb{R}^2. \quad (3.94)$$

Hence, the containment $\Psi_0 \in [L^\infty(\mathbb{R}^2)]^2$ follows from (3.93) and (3.94).

(b) The case $n \geq 3$: By (3.84) we know that $\Psi_0 \in [W^{1,2}(\mathbb{R}^n)]^N$. Employing the fact that $\Phi_0 = V_2 \Psi_0$ in the first line of (3.85), one obtains

$$\begin{aligned} \|\Psi_0(x)\|_{\mathcal{C}^N} &\leq \tilde{D}_n \int_{\mathbb{R}^n} d^n y |x-y|^{1-n} \langle y \rangle^{-2} \|\Psi_0(y)\|_{\mathcal{C}^N} \\ &= D_n \mathcal{R}_{1,n}(\langle \cdot \rangle^{-2} \|\Psi_0(\cdot)\|_{\mathcal{C}^N})(x), \quad x \in \mathbb{R}^n, \end{aligned} \quad (3.95)$$

for some constants $\tilde{D}_n, D_n \in (0, \infty)$. Thus, as in (3.86), (2.40) implies for $n \geq 3$,

$$\begin{aligned} \|\Psi_0\|_{[L^q(\mathbb{R}^n)]^N} &\leq D_n \|\mathcal{R}_{1,n}(\langle \cdot \rangle^{-2} \|\Psi_0(\cdot)\|_{\mathcal{C}^N})\|_{L^q(\mathbb{R}^n)} \\ &\leq \tilde{D}_{p,q,n} \|\langle \cdot \rangle^{-2} \|\Psi_0(\cdot)\|_{\mathcal{C}^N}\|_{L^p(\mathbb{R}^n)} \\ &\leq \tilde{D}_{p,q,n} \|\langle \cdot \rangle^{-2}\|_{L^s(\mathbb{R}^n)} \|\|\Psi_0(\cdot)\|_{\mathcal{C}^N}\|_{L^2(\mathbb{R}^n)} \\ &= \tilde{D}_{p,q,n} \|\langle \cdot \rangle^{-2}\|_{L^s(\mathbb{R}^n)} \|\Psi_0\|_{[L^2(\mathbb{R}^n)]^N}, \end{aligned} \quad (3.96)$$

$$1 < p < q < \infty, \quad p^{-1} = q^{-1} + n^{-1}, \quad s = 2qn[2n + 2q - qn]^{-1} \geq 1,$$

for some constant $\tilde{D}_{p,q,n} \in (0, \infty)$. In particular, one again has $p = qn/(n+q)$ and $2n + 2q - qn > 0$. The latter condition implies $q < 2n/(n-2)$. The requirement $p > 1$ results in $q > n/(n-1)$, and the condition $s \geq 1$ yields $q \geq 2n/(3n-2)$ which,

however, is superseded by $q > p > 1$. Moreover, the requirement $\|\langle \cdot \rangle^{-2}\|_{L^s(\mathbb{R}^n)} < \infty$ yields $q > 2n/(n+2)$. Putting it all together implies (3.74).

To prove the containment $\Psi \in [L^\infty(\mathbb{R}^3)]^4$ in (3.74), one applies (3.80) to the inequality in (3.81) to obtain

$$\|\Psi_0(x)\|_{\mathbb{C}^4} \leq d_3 \int_{\mathbb{R}^3} d^3y |x-y|^{-2} \langle y \rangle^{-2} \|\Psi_0(y)\|_{\mathbb{C}^4}, \quad x \in \mathbb{R}^3. \quad (3.97)$$

By Hölder's inequality (with conjugate exponents $q' = 27/20$ and $q = 27/7$), one infers that

$$\|\Psi_0(x)\|_{\mathbb{C}^4} \leq d_3 \left(\int_{\mathbb{R}^3} d^3y |x-y|^{-27/10} \langle y \rangle^{-27/10} \right)^{20/27} \left(\int_{\mathbb{R}^3} d^3y \|\Psi_0(y)\|_{\mathbb{C}^4}^{2/77} \right)^{7/27}, \quad x \in \mathbb{R}^3. \quad (3.98)$$

The second integral on the right-hand side in (3.98) is finite since $\Psi_0 \in [L^{27/7}(\mathbb{R}^3)]^4$, and the first integral on the right-hand side in (3.98) may be estimated by taking $x_1 = x$, $k = 27/10$, $\ell = 0$, $\beta = 2$, and $\varepsilon = 7/10$ in Lemma 2.5,

$$\int_{\mathbb{R}^3} d^3y |x-y|^{-\frac{27}{10}} \langle y \rangle^{-\frac{27}{10}} \leq C_{3, \frac{27}{10}, 0, 2, 7/10}, \quad x \in \mathbb{R}^3 \setminus \{0\}. \quad (3.99)$$

Hence, the containment $\Psi_0 \in [L^\infty(\mathbb{R}^3)]^4$ follows from (3.98) and (3.99). We will illustrate in Remark A.2 that the same line of reasoning fails for $n \geq 4$.

Finally (following the proof of [33, Lemma 7.4]), we show that if $\ker(H) \not\supseteq \{0\}$ then also $\ker([I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}]) \not\supseteq \{0\}$. Indeed, if $0 \neq \Psi_0 \in \ker(H)$, then $\Phi_0 := V_2\Psi_0 = U_V V_1\Psi_0 \in [L^2(\mathbb{R}^n)]^N$ and hence $V_1^*\Phi_0 \in [L^2(\mathbb{R}^n)]^N$. Then, $H\Psi_0 = 0$ yields $i\alpha \cdot \nabla \Psi_0 = V\Psi_0 = V_1^*V_2\Psi_0 = V_1^*\Phi_0$.

Thus, applying (3.71), (3.89)–(3.90) once again, one obtains for all $n \geq 2$,

$$\begin{aligned} & -i\alpha \cdot \nabla [\Psi_0 + (H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*\Phi_0](x) \\ &= -i[\alpha \cdot \nabla \Psi_0](x) - i\alpha \cdot \nabla_x [R_{0,0} * (V_1^*\Phi_0)](x) \\ &= -i[\alpha \cdot \nabla \Psi_0](x) - i\alpha \cdot \nabla_x [-i(\alpha \cdot \nabla_x g_0 * (V_1^*\Phi_0)](x) \\ &= -i[\alpha \cdot \nabla \Psi_0](x) + (-\Delta_x g_0 I_N) * (V_1^*\Phi_0)(x) \\ &= -i[\alpha \cdot \nabla \Psi_0](x) + (V_1^*\Phi_0)(x) \\ &= -V(x)\Psi_0(x) + V(x)\Psi_0(x) = 0. \end{aligned} \quad (3.100)$$

Consequently,

$$-i\alpha \cdot \nabla [\Psi_0 + (H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*\Phi_0] = 0, \quad (3.101)$$

implying

$$\Psi_0 + (H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*\Phi_0 = c^\top, \quad (3.102)$$

for some $c \in \mathbb{C}^N$. Since $\Psi_0 \in [L^2(\mathbb{R}^n)]^N$, and by exactly the same arguments employed in (3.81)–(3.82), also $R_{0,0} * (V_1^*\Phi_0) \in [L^2(\mathbb{R}^n)]^N$, one concludes that $c = 0$ and hence

$$\Psi_0 = -(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*\Phi_0. \quad (3.103)$$

Thus, $\Phi_0 \neq 0$, and

$$\begin{aligned} 0 &= V_2\Psi_0 + V_2(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*\Phi_0 \\ &= [I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}] \Phi_0, \end{aligned} \quad (3.104)$$

that is,

$$0 \neq \Phi_0 \in \ker \left([I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0 - (0 + i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}] \right). \quad (3.105)$$

This concludes the proof. \square

Remark 3.9. (i) For basics on the Birman–Schwinger principle in the concrete case of massless Dirac operators see [32], [33].

(ii) In physical notation, the zero-energy resonances in Cases (II) and (IV) for $n = 2$ correspond to eigenvalues $\pm 1/2$ of the spin-orbit operator (cf. the operator S in [54], [55]) when V is spherically symmetric, see the discussion in [33].

(iii) As mentioned in Remark 2.3, the absence of zero-energy resonances is well-known in the three-dimensional case $n = 3$, see [7], [12, Sect. 4.4], [13], [16], [67], [68], [78]. In fact, for $n = 3$ the absence of zero-energy resonances has been shown under the weaker decay $|V_{j,k}| \leq C\langle x \rangle^{-1-\varepsilon}$, $x \in \mathbb{R}^3$, in [7]. The absence of zero-energy resonances for massless Dirac operators in dimensions $n \geq 4$ as contained in Theorem 3.8 (ii) appears to have gone unnoticed in the literature. \diamond

Massive Dirac Operators in \mathbb{R}^n , $n \geq 3$. Finally, we turn to the case of massive Dirac operators $H(m)$, $m > 0$, in dimension $n \geq 3$ as defined in (2.8), and start by making the following assumptions on the matrix-valued potential V .

Hypothesis 3.10. Let $n \in \mathbb{N}$, $n \geq 3$. Assume the a.e. self-adjoint matrix-valued potential $V = \{V_{j,k}\}_{1 \leq j,k \leq N}$ satisfies for some $C \in (0, \infty)$,

$$\begin{aligned} V &\in [L^\infty(\mathbb{R}^n)]^{N \times N}, \\ |V_{j,k}(x)| &\leq C\langle x \rangle^{-4} \text{ for a.e. } x \in \mathbb{R}^n, 1 \leq j, k \leq N. \end{aligned} \quad (3.106)$$

In addition, alluding to the polar decomposition of $V(\cdot)$ (i.e., $V(\cdot) = U_V(\cdot)|V(\cdot)|$) in the following symmetrized form (cf. [49]), we suppose that

$$V = V_1^*V_2 = |V|^{1/2}U_V|V|^{1/2}, \text{ where } V_1 = V_1^* = |V|^{1/2}, \quad V_2 = U_V|V|^{1/2}. \quad (3.107)$$

We continue with the threshold behavior, that is, the $z = \pm m$ behavior, of $H(m)$:

Definition 3.11. Assume Hypothesis 3.6.

(i) The point $\pm m$ is called a threshold eigenvalue of $H(m)$ if $H(m)\Psi = \pm m\Psi$ has a distributional solution Ψ satisfying

$$\Psi \in \text{dom}(H(m)) = [W^{1,2}(\mathbb{R}^n)]^N \quad (3.108)$$

(equivalently, $\ker(H \mp mI_{[L^2(\mathbb{R}^n)]^N}) \not\supseteq \{0\}$).

(ii) The point $\pm m$ is called a threshold resonance of $H(m)$ if

$$\ker \left([I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0(m) - (\pm m + i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}] \right) \not\supseteq \{0\} \quad (3.109)$$

and if there exists

$$0 \neq \Phi_{\pm m} \in \ker \left([I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0(m) - (\pm m + i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}] \right) \quad (3.110)$$

such that $\Psi_{\pm m}$ defined by

$$\begin{aligned} \Psi_{\pm m}(x) &= -((H_0(m) - (\pm m + i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*\Phi)(x) \\ &= - \int_{\mathbb{R}^n} d^n y \left[4^{-1}\pi^{-n/2}\Gamma((n-2)/2)|x-y|^{2-n}(m\beta \pm mI_N) \right. \\ &\quad \left. + i2^{-1}\pi^{-n/2}\Gamma(n/2)\alpha \cdot \frac{(x-y)}{|x-y|^n} \right] V_1(y)^*\Phi_{\pm m}(y) \end{aligned} \quad (3.111)$$

(for a.e. $x \in \mathbb{R}^n$, $n \geq 2$) is a distributional solution of $H(m)\Psi_{\pm m} = \pm m\Psi_{\pm m}$ satisfying

$$\Psi_{\pm m} \notin [L^2(\mathbb{R}^n)]^N. \quad (3.112)$$

(iii) $\pm m$ is called a regular point for $H(m)$ if it is neither a threshold eigenvalue nor a threshold resonance of $H(m)$.

Additional properties of $\Psi_{\pm m}$ are isolated in Theorem 3.8.

While the points $\pm m$ being regular for $H(m)$ is the generic situation, threshold eigenvalues and/or resonances at energies $\pm m$ are exceptional cases.

Next, we introduce the following convenient abbreviation (for $x, y \in \mathbb{R}^n$, $x \neq y$):

$$\begin{aligned} R_{0,\pm m}(x-y) &= \lim_{\substack{z \rightarrow \pm m \\ z \in \overline{\mathbb{C}_+} \setminus \{\pm m\}}} G_0(m, z; x, y) = G_0(m, \pm m + i0; x, y) \\ &= 4^{-1} \pi^{-n/2} \Gamma((n-2)/2) |x-y|^{2-n} (m\beta \pm mI_N) \\ &\quad + i 2^{-1} \pi^{-n/2} \Gamma(n/2) \alpha \cdot \frac{(x-y)}{|x-y|^n} \\ &= g_0(0; x, y) (m\beta \pm mI_N) + G_0(0 + i0; x, y) \\ &= r_{0,0}(x-y) (m\beta \pm mI_N) + R_{0,0}(x-y), \quad n \in \mathbb{N}, n \geq 3. \end{aligned} \quad (3.113)$$

Theorem 3.12. *Assume Hypothesis 3.10.*

(i) If $n = 3, 4$, there are precisely four possible cases for each of $\pm m$:

Case (I): $\pm m$ is regular for $H(m)$.

Case (II): $\pm m$ is a (possibly degenerate) resonance of $H(m)$. In this case the resonance functions $\Psi_{\pm m}$ satisfy

$$\begin{aligned} \Psi_{\pm m} &\in [L^q(\mathbb{R}^n)]^N, \quad q \in \begin{cases} (3, \infty) \cup \{\infty\}, & n = 3, \\ (2, 4), & n = 4, \end{cases} \\ \nabla \Psi_{\pm m} &\in [L^2(\mathbb{R}^n)]^{N \times n}, \quad \Psi_{\pm m} \notin [L^2(\mathbb{R}^n)]^N. \end{aligned} \quad (3.114)$$

Case (III): $\pm m$ is a (possibly degenerate) eigenvalue of $H(m)$. In this case the corresponding eigenfunctions $\Psi_{\pm m} \in \text{dom}(H(m)) = [W^{1,2}(\mathbb{R}^n)]^N$ of $H(m)\Psi_{\pm m} = \pm m\Psi_{\pm m}$ also satisfy

$$\Psi_{\pm m} \in [L^q(\mathbb{R}^n)]^N, \quad q \in \begin{cases} [2, \infty) \cup \{\infty\}, & n = 3, \\ [2, 4), & n = 4. \end{cases} \quad (3.115)$$

Case (IV): A possible mixture of Cases (II) and (III).

(ii) If $n \in \mathbb{N}$, $n \geq 5$, there are precisely two possible cases:

Case (I): $\pm m$ is regular for $H(m)$.

Case (II): $\pm m$ is a (possibly degenerate) eigenvalue of $H(m)$. In this case, the corresponding eigenfunctions $\Psi_{\pm m} \in \text{dom}(H(m)) = [W^{1,2}(\mathbb{R}^n)]^N$ of $H\Psi_{\pm m} = \pm m\Psi_{\pm m}$ also satisfy

$$\Psi_{\pm m} \in [L^q(\mathbb{R}^n)]^N, \quad q \in \begin{cases} (n/(n-1), 2n/(n-2)), & 5 \leq n \leq 8, \\ (2n/(n+6), 2n/(n-4)), & n \geq 9. \end{cases} \quad (3.116)$$

In particular, there are no resonances at energies $\pm m$ of $H(m)$ in dimension $n \geq 5$.

(iii) The point $\pm m$ is regular for $H(m)$ if and only if

$$\ker \left([I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0(m) - (\pm m + i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}] \right) = \{0\}. \quad (3.117)$$

Proof. Since $G_0(m, \pm m + i0; x, y)$, $x \neq y$, exists for all $n \geq 3$ (cf. (3.113)), the Birman-Schwinger eigenvalue equation

$$\begin{aligned} 0 &= [I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0(m) - (\pm m + i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}] \Phi_{\pm m}, \\ 0 &\neq \Phi_{\pm m} \in [L^2(\mathbb{R}^n)]^N, \end{aligned} \quad (3.118)$$

gives rise to a distributional solution $\Psi_{\pm m} \in [L^1_{\text{loc}}(\mathbb{R}^n)]^N$ of the equation

$$H(m)\Psi_{\pm m} = \pm m\Psi_{\pm m} \quad (3.119)$$

in terms of $\Phi_{\pm m}$ of the form (for a.e. $x \in \mathbb{R}^n$, $n \geq 3$),

$$\begin{aligned} \Psi_{\pm m}(x) &= -((H_0(m) - (\pm m + i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*\Phi_0)(x) \\ &= -[R_{0, \pm m} * (V_1^*\Phi_{\pm m})](x) \end{aligned} \quad (3.120)$$

$$= - \int_{\mathbb{R}^n} d^n y \left[4^{-1} \pi^{-n/2} \Gamma((n-2)/2) |x-y|^{2-n} (m\beta \pm mI_N) \right. \quad (3.121)$$

$$\left. + i2^{-1} \pi^{-n/2} \Gamma(n/2) \alpha \cdot \frac{(x-y)}{|x-y|^n} \right] V_1(y)^* \Phi(y), \quad (3.122)$$

$$\Phi_{\pm m}(x) = (V_2\Psi_{\pm m})(x). \quad (3.123)$$

In particular, one concludes that $\Psi_{\pm m} \neq 0$. Using the inequality $\|V_1(\cdot)\|_{\mathbb{C}^{N \times N}} \leq c\langle \cdot \rangle^{-2}$, one obtains the following estimate for some $d_n, \tilde{d}_n \in (0, \infty)$ (for a.e. $x \in \mathbb{R}^n$, $n \geq 3$):

$$\|\Psi_{\pm m}(x)\|_{\mathbb{C}^N} \leq d_n \left\{ \int_{\mathbb{R}^n} d^n y |x-y|^{2-n} \langle y \rangle^{-2} \|\Phi_{\pm m}(y)\|_{\mathbb{C}^N} \right. \quad (3.124)$$

$$\left. + \int_{\mathbb{R}^n} d^n y |x-y|^{1-n} \langle y \rangle^{-2} \|\Phi_{\pm m}(y)\|_{\mathbb{C}^N} \right\}$$

$$\leq \tilde{d}_n \{ \mathcal{R}_{1,n}(\langle \cdot \rangle^{-2} \|\Phi_{\pm m}(\cdot)\|_{\mathbb{C}^N})(x) + \mathcal{R}_{2,n}(\langle \cdot \rangle^{-2} \|\Phi_{\pm m}(\cdot)\|_{\mathbb{C}^N})(x) \}. \quad (3.125)$$

The first term on the right-hand side in (3.125) may be estimated as follows:

$$\begin{aligned} \mathcal{R}_{1,n}(\langle \cdot \rangle^{-2} \|\Phi_{\pm m}(\cdot)\|_{\mathbb{C}^N})(x) &= \gamma(1, n)^{-1} \int_{\mathbb{R}^n} d^n y |x-y|^{1-n} \langle y \rangle^{-2} \|\Phi_{\pm m}(y)\|_{\mathbb{C}^N} \\ &\leq \gamma(1, n)^{-1} \int_{\mathbb{R}^n} d^n y |x-y|^{1-n} \langle y \rangle^{-1} \|\Phi_{\pm m}(y)\|_{\mathbb{C}^N}, \quad x \in \mathbb{R}^n. \end{aligned} \quad (3.126)$$

An application of Theorem 2.6 (ii) with $c = 0$, $d = 1$, $p = p' = 2$, and the inequality $1 < n/2$, combined with $\|\Phi_{\pm m}(\cdot)\|_{\mathbb{C}^N} \in L^2(\mathbb{R}^n)$, then yield

$$\mathcal{R}_{1,n}(\langle \cdot \rangle^{-2} \|\Phi_{\pm m}(\cdot)\|_{\mathbb{C}^N}) \in L^2(\mathbb{R}^n), \quad n \geq 3. \quad (3.127)$$

Similarly, the second term on the right-hand side in (3.125) may be estimated as follows:

$$\begin{aligned} \mathcal{R}_{2,n}(\langle \cdot \rangle^{-2} \|\Phi_{\pm m}(\cdot)\|_{\mathbb{C}^N})(x) &= \gamma(2, n)^{-1} \int_{\mathbb{R}^n} d^n y |x-y|^{2-n} \langle y \rangle^{-2} \|\Phi_{\pm m}(y)\|_{\mathbb{C}^N}, \\ &x \in \mathbb{R}^n. \end{aligned} \quad (3.128)$$

An application of Theorem 2.6 (ii) with $c = 0$, $d = 2$, $p = p' = 2$, and the inequality $2 < n/2$, combined with $\|\Phi_{\pm m}(\cdot)\|_{\mathbb{C}^N} \in L^2(\mathbb{R}^n)$, then yield

$$\mathcal{R}_{2,n}(\langle \cdot \rangle^{-2} \|\Phi_{\pm m}(\cdot)\|_{\mathbb{C}^N}) \in L^2(\mathbb{R}^n), \quad n \geq 5. \quad (3.129)$$

Thus, (3.125), (3.129), and (3.127) imply

$$\|\Psi_{\pm m}(\cdot)\|_{\mathbb{C}^N} \in L^2(\mathbb{R}^n) \text{ and hence, } \Psi_{\pm m} \in [L^2(\mathbb{R}^n)]^N, \quad n \geq 5. \quad (3.130)$$

To prove that actually $\Psi_{\pm m} \in \text{dom}(H(m)) = [W^{1,2}(\mathbb{R}^n)]^N$ for $n \geq 5$, it suffices to argue as follows:

$$i\alpha \cdot \nabla \Psi_{\pm m} = -V\Psi_{\pm m} \pm m\Psi_{\pm m} \in [L^2(\mathbb{R}^n)]^N \quad (3.131)$$

in the sense of distributions since $V \in [L^\infty(\mathbb{R}^n)]^{N \times N}$ and $\Psi_{\pm m} \in [L^2(\mathbb{R}^n)]^N$. Given the fact $\text{dom}(H_0(m)) = [W^{1,2}(\mathbb{R}^n)]^N$ (cf. (2.8)), one concludes that

$$\Psi_{\pm m} \in [W^{1,2}(\mathbb{R}^n)]^N, \quad n \geq 5. \quad (3.132)$$

Returning to (3.125), one applies (2.40) to estimate (for some constant $C_{p,q,n}^{(1)} \in (0, \infty)$)

$$\begin{aligned} \|\mathcal{R}_{1,n}(\langle \cdot \rangle^{-2} \|\Phi_{\pm m}(\cdot)\|_{\mathbb{C}^N})\|_{L^q(\mathbb{R}^n)} &\leq C_{p,q,n}^{(1)} \|\langle \cdot \rangle^{-2} \|\Phi_{\pm m}(\cdot)\|_{\mathbb{C}^N}\|_{L^p(\mathbb{R}^n)} \\ &\leq C_{p,q,n}^{(1)} \|\langle \cdot \rangle^{-2}\|_{L^s(\mathbb{R}^n)} \|\|\Phi_{\pm m}(\cdot)\|_{\mathbb{C}^N}\|_{L^2(\mathbb{R}^n)} \\ &= C_{p,q,n}^{(1)} \|\langle \cdot \rangle^{-2}\|_{L^s(\mathbb{R}^n)} \|\Phi_{\pm m}\|_{[L^2(\mathbb{R}^n)]^N}, \end{aligned} \quad (3.133)$$

$$1 < p < q < \infty, \quad p^{-1} = q^{-1} + n^{-1}, \quad s = 2qn[2n + 2q - qn]^{-1} \geq 1,$$

so that, in particular,

$$p = qn/(n + q), \quad 2n + 2q - qn > 0. \quad (3.134)$$

Similarly, one applies (2.40) to estimate (for some constant $C_{p,q,n}^{(2)} \in (0, \infty)$)

$$\begin{aligned} \|\mathcal{R}_{2,n}(\langle \cdot \rangle^{-2} \|\Phi_{\pm m}(\cdot)\|_{\mathbb{C}^N})\|_{L^q(\mathbb{R}^n)} &\leq C_{p,q,n}^{(2)} \|\langle \cdot \rangle^{-2} \|\Phi_{\pm m}(\cdot)\|_{\mathbb{C}^N}\|_{L^p(\mathbb{R}^n)} \\ &\leq C_{p,q,n}^{(2)} \|\langle \cdot \rangle^{-2}\|_{L^s(\mathbb{R}^n)} \|\|\Phi_{\pm m}(\cdot)\|_{\mathbb{C}^N}\|_{L^2(\mathbb{R}^n)} \\ &= C_{p,q,n}^{(2)} \|\langle \cdot \rangle^{-2}\|_{L^s(\mathbb{R}^n)} \|\Phi_{\pm m}\|_{[L^2(\mathbb{R}^n)]^N}, \end{aligned} \quad (3.135)$$

$$1 < p < q < \infty, \quad p^{-1} = q^{-1} + 2n^{-1}, \quad s = 2qn[2n + 4q - qn]^{-1} \geq 1,$$

so that, in particular,

$$p = qn/(n + 2q), \quad 2n + 4q - qn > 0. \quad (3.136)$$

We note that if $\Psi_{\pm m} \in [L^2(\mathbb{R}^n)]^N$, then an application of (3.123) after the second inequality in (3.133) yields (for some constant $\tilde{C}_{p,q,n}^{(1)} \in (0, \infty)$)

$$\begin{aligned} &\|\mathcal{R}_{1,n}(\langle \cdot \rangle^{-2} \|\Phi_{\pm m}(\cdot)\|_{\mathbb{C}^N})\|_{L^q(\mathbb{R}^n)} \\ &\leq C_{p,q,n}^{(1)} \|\langle \cdot \rangle^{-2} \|\Phi_{\pm m}(\cdot)\|_{\mathbb{C}^N}\|_{L^p(\mathbb{R}^n)} \\ &\leq C_{p,q,n}^{(1)} \|\langle \cdot \rangle^{-2} \|V_2(\cdot)\|_{\mathbb{C}^{N \times N}} \|\Psi_{\pm m}(\cdot)\|_{\mathbb{C}^N}\|_{L^p(\mathbb{R}^n)} \\ &\leq \tilde{C}_{p,q,n}^{(1)} \|\langle \cdot \rangle^{-4} \|\Psi_{\pm m}(\cdot)\|_{\mathbb{C}^N}\|_{L^p(\mathbb{R}^n)} \\ &\leq \tilde{C}_{p,q,n}^{(1)} \|\langle \cdot \rangle^{-4}\|_{L^s(\mathbb{R}^n)} \|\Psi_{\pm m}\|_{[L^2(\mathbb{R}^n)]^N}, \end{aligned} \quad (3.137)$$

under the same conditions in (3.133) and (3.133). Note that $\|V_2(\cdot)\|_{\mathbb{C}^{N \times N}} \leq c\langle \cdot \rangle^{-2}$ has been used in (3.137). Similarly, one obtains (for some constant $\tilde{C}_{p,q,n}^{(2)} \in (0, \infty)$)

$$\|\mathcal{R}_{2,n}(\langle \cdot \rangle^{-2} \|\Phi_{\pm m}(\cdot)\|_{\mathbb{C}^N})\|_{L^q(\mathbb{R}^n)} \leq \tilde{C}_{p,q,n}^{(2)} \|\langle \cdot \rangle^{-4}\|_{L^s(\mathbb{R}^n)} \|\Psi_{\pm m}\|_{[L^2(\mathbb{R}^n)]^N} \quad (3.138)$$

under the same conditions in (3.135) and (3.136).

(a) The case $n = 3$: The first condition in (3.134) implies $p = 3q/(3 + q)$. The condition $p > 1$ requires $q \in (3/2, \infty)$, and the condition $s = 6q/(6 - q) \geq 1$ further requires $q \in (3/2, 6)$. For $q \in (3/2, 6)$, one infers $p \in (1, 2)$ and $s \in (2, \infty)$; the latter ensures $\|\langle \cdot \rangle^{-2}\|_{L^s(\mathbb{R}^3)} < \infty$. Hence, one may take

$$q \in (3/2, 6), \quad p = 3q/(3 + q), \quad s = 6q/(6 - q) \quad (3.139)$$

throughout (3.133) to obtain

$$\mathcal{R}_{1,3}(\langle \cdot \rangle^{-2} \|\Phi_{\pm m}(\cdot)\|_{\mathbb{C}^4}) \in L^q(\mathbb{R}^3), \quad q \in (3/2, 6). \quad (3.140)$$

The first condition in (3.136) implies $p = 3q/(3 + 2q)$. The condition $p > 1$ requires $q \in (3, \infty)$, which implies $s = 6q/(6 + q) \in (2, 6)$, ensuring $\|\langle \cdot \rangle^{-2}\|_{L^s(\mathbb{R}^3)} < \infty$. Hence, one may take

$$q \in (3, \infty), \quad p = 3q/(3 + 2q), \quad s = 6q/(6 + q) \quad (3.141)$$

throughout (3.135) to obtain

$$\mathcal{R}_{2,3}(\langle \cdot \rangle^{-2} \|\Phi_{\pm m}(\cdot)\|_{\mathbb{C}^4}) \in L^q(\mathbb{R}^3), \quad q \in (3, \infty). \quad (3.142)$$

The containments in (3.140) and (3.142), combined with the estimate in (3.125) imply

$$\|\Psi_{\pm m}(\cdot)\|_{\mathbb{C}^4} \in L^q(\mathbb{R}^3) \text{ and hence, } \Psi_{\pm m} \in [L^q(\mathbb{R}^3)]^4, \quad q \in (3, 6). \quad (3.143)$$

To prove that $\nabla \Psi_{\pm m} \in [L^2(\mathbb{R}^3)]^{4 \times 3}$, it suffices to argue as follows. By (3.120), (3.30), and (3.91), one computes

$$\begin{aligned} -i\alpha \cdot [\nabla \Psi_{\pm m}](x) &= i\alpha \cdot 4^{-1} \pi^{-n/2} \Gamma((n-2)/2) \int_{\mathbb{R}^3} d^3 y |x-y|^{1-n} \frac{(x-y)}{|x-y|} \\ &\quad \times V_1^*(y) \Phi_{\pm m}(y) [m\beta \pm mL_4] + (V_1^* \Phi_{\pm m})(x), \quad x \in \mathbb{R}^3. \end{aligned} \quad (3.144)$$

In addition,

$$\begin{aligned} &\left\| \int_{\mathbb{R}^3} d^3 y |x-y|^{1-n} \frac{(x-y)}{|x-y|} V_1^*(y) \Phi_{\pm m}(y) \right\|_{\mathbb{C}^4} \\ &\leq \int_{\mathbb{R}^3} d^3 y |x-y|^{1-n} \|V_1^*(y)\|_{\mathbb{C}^4 \times 4} \|\Phi_{\pm m}(y)\|_{\mathbb{C}^4} \\ &\leq c \int_{\mathbb{R}^3} d^3 y |x-y|^{1-n} \langle y \rangle^{-2} \|\Phi_{\pm m}(y)\|_{\mathbb{C}^4}, \end{aligned} \quad (3.145)$$

and an application of Theorem 2.6 (ii) with $c = 0$, $d = 1$, $p = p' = 2$ then implies

$$\int_{\mathbb{R}^3} d^3 y | \cdot - y |^{1-n} \langle y \rangle^{-2} \|\Phi_{\pm m}(y)\|_{\mathbb{C}^4} \in [L^2(\mathbb{R}^3)]^4. \quad (3.146)$$

Since $V_1^* \Phi_{\pm m} \in [L^3(\mathbb{R}^3)]^4$, the containment in (3.146) and the identity in (3.144) imply $\nabla \Psi_{\pm m} \in [L^2(\mathbb{R}^3)]^{4 \times 3}$.

Finally, if $\Psi_{\pm m} \in [L^2(\mathbb{R}^3)]^4$, then together with (3.143), one obtains (3.115) for $n = 3$.

If $\Psi_{\pm m}$ is a resonance function (resp., eigenfunction), then it has been shown that $\Psi_{\pm m} \in [L^q(\mathbb{R}^3)]^4$ for all $q \in (3, 6)$ (resp., $q \in [2, 6)$). To prove that $\Psi_{\pm m} \in [L^\infty(\mathbb{R}^3)]^4$, one applies (3.123) and the condition $\|V_2(\cdot)\|_{\mathbb{C}^4 \times 4} \leq C \langle \cdot \rangle^{-2}$ for some $C \in (0, \infty)$ in (3.124) to obtain

$$\begin{aligned} \|\Psi_{\pm m}(x)\|_{\mathbb{C}^4} &\leq \tilde{d}_3 \left\{ \int_{\mathbb{R}^3} d^3y |x-y|^{-1} \langle y \rangle^{-4} \|\Psi_{\pm m}(y)\|_{\mathbb{C}^4} \right. \\ &\quad \left. + \int_{\mathbb{R}^3} d^3y |x-y|^{-2} \langle y \rangle^{-4} \|\Psi_{\pm m}(x)\|_{\mathbb{C}^4} \right\}, \quad x \in \mathbb{R}^3. \end{aligned} \quad (3.147)$$

Here $\tilde{d}_3 \in (0, \infty)$ is an appropriate x -independent constant. The first integral on the right-hand side in (3.147) may be estimated with Hölder's inequality (with conjugate exponents $p = 5/4$ and $p' = 5$) as follows:

$$\begin{aligned} &\int_{\mathbb{R}^3} d^3y |x-y|^{-1} \langle y \rangle^{-4} \|\Psi_{\pm m}(y)\|_{\mathbb{C}^4} \\ &\leq \left(\int_{\mathbb{R}^3} d^3y |x-y|^{-5/4} \langle y \rangle^{-5} \right)^{4/5} \left(\int_{\mathbb{R}^3} d^3y \|\Psi_{\pm m}(y)\|_{\mathbb{C}^4}^5 \right)^{1/5}, \quad x \in \mathbb{R}^3. \end{aligned} \quad (3.148)$$

The second integral on the right-hand side in (3.148) is finite since a resonance function (resp., eigenfunction) satisfies $\Psi_{\pm m} \in [L^5(\mathbb{R}^3)]^4$. To estimate the first integral on the right-hand side in (3.148), one applies Lemma 2.5 with the choices $x_1 = x$, $k = 5/4$, $\ell = 0$, $\beta = 4$, and $\varepsilon = 1$, so that

$$\int_{\mathbb{R}^3} d^3y |x-y|^{-5/4} \langle y \rangle^{-5} \leq C_{3,5/4,0,4,1}, \quad x \in \mathbb{R}^3. \quad (3.149)$$

Along similar lines, the second integral on the right-hand side in (3.147) may be estimated with Hölder's inequality (with conjugate exponents $p = 5/4$ and $p' = 5$) as follows:

$$\begin{aligned} &\int_{\mathbb{R}^3} d^3y |x-y|^{-2} \langle y \rangle^{-4} \|\Psi_{\pm m}(y)\|_{\mathbb{C}^4} \\ &\leq \left(\int_{\mathbb{R}^3} d^3y |x-y|^{-5/2} \langle y \rangle^{-5} \right)^{4/5} \left(\int_{\mathbb{R}^3} d^3y \|\Psi_{\pm m}(y)\|_{\mathbb{C}^4}^5 \right)^{1/5}, \quad x \in \mathbb{R}^3. \end{aligned} \quad (3.150)$$

An application of Lemma 2.5 with $x_1 = x$, $k = 5/2$, $\ell = 0$, $\beta = 4$, and $\varepsilon = 1$, yields

$$\int_{\mathbb{R}^3} d^3y |x-y|^{-5/2} \langle y \rangle^{-5} \leq C_{3,5/2,0,4,1}, \quad x \in \mathbb{R}^3. \quad (3.151)$$

Therefore, combining (3.147), (3.148), (3.149), (3.150), and (3.151), one obtains

$$\|\Psi_{\pm m}(x)\|_{\mathbb{C}^4} \leq \tilde{d}_3 \left\{ C_{3,5/4,0,4,1}^{4/5} + C_{3,5/2,0,4,1}^{4/5} \right\} \|\Psi_{\pm m}\|_{[L^5(\mathbb{R}^3)]^4}, \quad x \in \mathbb{R}^3, \quad (3.152)$$

and it follows that $\Psi_{\pm m} \in [L^\infty(\mathbb{R}^3)]^4$, settling the case $n = 3$. We will illustrate in Remark A.3 that the same line of reasoning fails for $n \geq 4$.

(b) The case $n = 4$: The first condition in (3.134) implies $p = 4q/(4+q)$. The condition $p > 1$ requires $q \in (4/3, \infty)$, and the condition $s = 4q/(4-q) \geq 1$ further requires $q \in (4/3, 4)$. For $q \in (4/3, 4)$, one infers $p \in (1, 2)$ and $s \in (2, \infty)$; the latter ensures $\|\langle \cdot \rangle^{-2}\|_{L^s(\mathbb{R}^4)} < \infty$. Hence, one may take

$$q \in (4/3, 4), \quad p = 4q/(4+q), \quad s = 4q/(4-q) \quad (3.153)$$

throughout (3.133) to obtain

$$\mathcal{R}_{1,4}(\langle \cdot \rangle^{-2} \|\Phi_{\pm m}(\cdot)\|_{\mathbb{C}^4}) \in L^q(\mathbb{R}^4), \quad q \in (4/3, 4). \quad (3.154)$$

The first condition in (3.136) implies $p = 2q/(2+q)$. The condition $p > 1$ requires $q \in (2, \infty)$, and the condition on s in (3.135) reduces to $s = q$, so the requirement $s \geq 1$ is satisfied for all $q \in (2, \infty)$. For $q \in (2, \infty)$, $p \in (1, 2)$, and $s \in (2, \infty)$; the latter ensures $\|\langle \cdot \rangle^{-2}\|_{L^s(\mathbb{R}^4)} < \infty$. Hence, one may take

$$q \in (2, \infty), \quad p = 2q/(2+q), \quad s = q \quad (3.155)$$

throughout (3.135) to obtain

$$\mathcal{R}_{2,4}(\langle \cdot \rangle^{-2} \|\Phi_{\pm m}(\cdot)\|_{\mathbb{C}^4}) \in L^q(\mathbb{R}^4), \quad q \in (2, \infty). \quad (3.156)$$

The containments in (3.154) and (3.156), combined with the estimate in (3.125) imply

$$\|\Psi_{\pm m}(\cdot)\|_{\mathbb{C}^4} \in L^q(\mathbb{R}^4) \text{ and hence, } \Psi_{\pm m} \in [L^q(\mathbb{R}^4)]^4, \quad q \in (2, 4). \quad (3.157)$$

One proves $\nabla \Psi_{\pm m} \in [L^2(\mathbb{R}^4)]^{4 \times 4}$ in a manner entirely analogous to (3.144)–(3.146). Finally, if $\Psi_{\pm m} \in [L^2(\mathbb{R}^4)]^4$, then together with (3.157), one obtains (3.115) for $n = 4$. Therefore, the $n = 4$ case is settled.

(c) The case $n \geq 5$: The first condition in (3.134) implies $p = qn/(n+q)$. The condition $p > 1$ requires $q \in (n/(n-1), \infty)$. The condition $2n + 2q - qn > 0$ further requires $q < 2n/(n-2)$, and the condition $s = 2qn/(2n + 2q - qn) \geq 1$ further requires $q \geq 2n/(3n-2)$. Moreover, $\|\langle \cdot \rangle^{-4}\|_{L^s(\mathbb{R}^n)} < \infty$ if and only if $n/4 < s = 2qn/(2n + 2q - qn)$, which requires $q > 2n/(n+6)$. Thus, one may take

$$q \in \begin{cases} (n/(n-1), 2n/(n-2)), & 5 \leq n \leq 8, \\ (2n/(n+6), 2n/(n-2)), & n \geq 9, \end{cases} \quad (3.158)$$

$$p = qn/(n+q), \quad s = 2qn/(2n + 2q - qn)$$

throughout (3.137) to obtain

$$\mathcal{R}_{1,n}(\langle \cdot \rangle^{-2} \|\Phi_{\pm m}(\cdot)\|_{\mathbb{C}^N}) \in L^q(\mathbb{R}^n), \quad q \in \begin{cases} (n/(n-1), 2n/(n-2)), & 5 \leq n \leq 8, \\ (2n/(n+6), 2n/(n-2)), & n \geq 9. \end{cases} \quad (3.159)$$

The first condition in (3.136) implies $p = qn/(n+2q)$. The condition $p > 1$ requires $q \in (2/(n-2), \infty)$. The condition $2n + 4q - qn > 0$ further requires $q < 2n/(n-4)$, and the condition $s = 2qn/(2n + 4q - qn) \geq 1$ further requires $q \geq 2n/(3n-4)$. Moreover, $\|\langle \cdot \rangle^{-4}\|_{L^2(\mathbb{R}^n)} < \infty$ if and only if $n/4 < s = 2qn/(2n + 4q - qn)$, which requires $q > 2n/(n+4)$. Hence, one may take

$$q \in (2n/(n+4), 2n/(n-4)), \quad p = qn/(n+2q), \quad s = 2qn/(2n + 4q - qn) \quad (3.160)$$

throughout (3.135) to obtain

$$\mathcal{R}_{2,n}(\langle \cdot \rangle^{-2} \|\Phi_{\pm m}(\cdot)\|_{\mathbb{C}^N}) \in L^q(\mathbb{R}^n), \quad q \in (2n/(n+8), 2n/(n-4)). \quad (3.161)$$

The containments in (3.159) and (3.161), combined with the estimate in (3.125), imply

$$\|\Psi_{\pm m}(\cdot)\|_{\mathbb{C}^N} \in L^q(\mathbb{R}^n) \text{ and hence, } \Psi_{\pm m} \in [L^q(\mathbb{R}^n)]^N,$$

$$q \in \begin{cases} (n/(n-1), 2n/(n-2)), & 5 \leq n \leq 8, \\ (2n/(n+6), 2n/(n-4)), & n \geq 9. \end{cases} \quad (3.162)$$

Finally (following the proof of [33, Lemma 7.4]), we show that if

$$\ker(H(m) \mp mI_{[L^2(\mathbb{R}^n)]^N}) \supsetneq \{0\}, \quad (3.163)$$

then also

$$\ker\left([I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0(m) - (\pm m + i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}]\right) \supsetneq \{0\}. \quad (3.164)$$

Indeed, if $0 \neq \Psi_{\pm m} \in \ker(H(m) \mp mI_{[L^2(\mathbb{R}^n)]^N})$, then

$$\Phi_{\pm m} := V_2\Psi_{\pm m} = U_V V_1\Psi_{\pm m} \in [L^2(\mathbb{R}^n)]^N, \quad (3.165)$$

and hence $V_1^*\Phi_{\pm m} \in [L^2(\mathbb{R}^n)]^N$. Therefore, $H(m)\Psi_{\pm m} = \pm m\Psi_{\pm m}$ yields

$$\begin{aligned} i\alpha \cdot \nabla \Psi_{\pm m} &= (m\beta \mp mI_N)\Psi_{\pm m} + V\Psi_{\pm m} \\ &= (m\beta \mp mI_N)\Psi_{\pm m} + V_1^*V_2\Psi_{\pm m} \\ &= (m\beta \mp mI_N)\Psi_{\pm m} + V_1^*\Phi_{\pm m}. \end{aligned} \quad (3.166)$$

Hence,

$$\begin{aligned} &(-i\alpha \cdot \nabla + m\beta \mp mI_N)[\Psi_{\pm m} + (H_0(m) - (\mp m + i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*\Phi_{\pm m}](x) \\ &= [(-i\alpha \cdot \nabla + m\beta \mp mI_N)\Psi_{\pm m}](x) \\ &\quad + (-i\alpha \cdot \nabla + m\beta \mp mI_N)[R_{0,\pm m} * (V_1^*\Phi_{\pm m})](x) \\ &= -(V_1^*\Phi_{\pm m})(x) + [((-i\alpha \cdot \nabla + m\beta \mp mI_N)R_{0,\pm m}) * (V_1^*\Phi_{\pm m})](x) \\ &= -(V_1^*\Phi_{\pm m})(x) + (V_1^*\Phi_{\pm m})(x) = 0, \quad x \in \mathbb{R}^n. \end{aligned} \quad (3.167)$$

In addition, $\Psi_{\pm m} \in [L^2(\mathbb{R}^n)]^N$, and the same arguments as those in (3.126)–(3.127) and (3.138)–(3.158) (which now extend to include $n \in \{3, 4\}$, due to the assumption that $\Psi_{\pm m} \in [L^2(\mathbb{R}^n)]^N$) imply $[R_{0,\pm m} * (V_1^*\Phi_{\pm m})] \in [L^2(\mathbb{R}^n)]^N$, so that

$$[\Psi_{\pm m} + (H_0(m) - (\mp m + i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*\Phi_{\pm m}] \in [L^2(\mathbb{R}^n)]^N. \quad (3.168)$$

It follows that

$$\tilde{\Psi}_{\pm m} := [\Psi_{\pm m} + (H_0(m) - (\mp m + i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*\Phi_{\pm m}] = 0; \quad (3.169)$$

otherwise, $\pm m$ is an eigenvalue of $H_0(m)$ and $\tilde{\Psi}_{\pm m}$ is a corresponding eigenfunction. However, this contradicts the fact that the spectrum of $H_0(m)$ is purely absolutely continuous. Hence,

$$\Psi_{\pm m} = -(H_0(m) - (\mp m + i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*\Phi_{\pm m}. \quad (3.170)$$

Thus, $\Phi_{\pm m} \neq 0$, and

$$\begin{aligned} 0 &= V_2\Psi_{\pm m} + V_2(H_0(m) - (\pm m + i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*\Phi_{\pm m} \\ &= [I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0(m) - (\pm m + i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}]\Phi_{\pm m}, \end{aligned} \quad (3.171)$$

that is,

$$0 \neq \Phi_{\pm m} \in \ker\left([I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0(m) - (\pm m + i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}]\right). \quad (3.172)$$

This concludes the proof. \square

Remark 3.13. (i) For basics on the Birman–Schwinger principle in the concrete case of massive Dirac operators, see [32], [38]–[40], [57].

(ii) In physical notation (see, e.g., [54], [55], [58], [75, Sect. 4.6] for details), the threshold resonances at energies $\pm m$ in Cases (II) and (IV) for $n = 3$ correspond to eigenvalues ± 1 of the spin-orbit operator (cf, [54], [55], [75, eq. (4.105), p. 125]) in the case where V is spherically symmetric, see the discussion in [40].

(iii) The absence of threshold resonances for massive Dirac operators in dimensions $n \geq 5$ as contained in Theorem 3.12 (ii) appears to have gone unnoticed in the literature. \diamond

APPENDIX A. SOME REMARKS ON $L^\infty(\mathbb{R}^n)$ -PROPERTIES
OF THRESHOLD EIGENFUNCTIONS

In this appendix we collect some negative results on $L^\infty(\mathbb{R}^n)$ -properties of threshold eigenfunctions. We hope to return to this issue at a later date.

We start with the case of Schrödinger operators:

Remark A.1. We briefly indicate why the line of reasoning used in the proof of $\psi \in L^\infty(\mathbb{R}^n)$ for $3 \leq n \leq 7$ in Theorem 3.3 is bound to fail for $n \geq 8$.

Suppose $n \geq 8$. Then (3.53) continues to hold and by (3.47), $\psi \in L^q(\mathbb{R}^n)$ for all

$$q \in (2n/(n+4), 2n/(n-4)). \quad (\text{A.1})$$

One notes that q satisfies (A.1) if and only if its conjugate exponent $q' := q/(q-1)$ satisfies

$$q' \in (2n/(n+4), 2n/(n-4)). \quad (\text{A.2})$$

Applying Hölder's inequality with conjugate exponents q' and q on the right-hand side in (3.53) yields

$$|\psi(x)| \leq \tilde{d}_n \left(\int_{\mathbb{R}^n} d^n y |x-y|^{(2-n)q'} \langle y \rangle^{-4q'} \right)^{1/q'} \left(\int_{\mathbb{R}^n} d^n y |\psi(y)|^q \right)^{1/q}, \quad x \in \mathbb{R}^n. \quad (\text{A.3})$$

The second integral on the right-hand side in (A.3) is finite for any choice of q' which satisfies (A.2). When applying Lemma 2.5 to estimate the first integral on the right-hand side in (A.3), one must choose $k = (n-2)q'$. The hypotheses in Lemma 2.5 require $k \in [0, n)$. Hence, one must choose q' so that $(n-2)q' < n$, that is, one must choose $q' < n/(n-2)$. However, there is no q' which simultaneously satisfies (A.2) and $q' < n/(n-2)$ since

$$\frac{2n}{n+4} - \frac{n}{n-2} = \frac{n(n-8)}{(n-2)(n+4)} > 0 \text{ implies } \frac{n}{n-2} < \frac{2n}{n+4}. \quad (\text{A.4})$$

Thus, $q' < n/(n-2)$ yields $q' < 2n/(n+4)$, so that q' does not satisfy (A.2). \diamond

Next, we illustrate the case of massless Dirac operators:

Remark A.2. We briefly indicate why the line of reasoning used in the proof of $\Psi \in L^\infty(\mathbb{R}^n)^N$ for $n = 2, 3$ in Theorem 3.8 is bound to fail for $n \geq 4$.

We start with the case $n = 4$. Then the inequality in (3.81) implies, when combined with (3.80),

$$\|\Psi_0(x)\|_{\mathbb{C}^4} \leq \tilde{d}_4 \int_{\mathbb{R}^4} d^4 y |x-y|^{-3} \langle y \rangle^{-2} \|\Psi_0(y)\|_{\mathbb{C}^4}, \quad x \in \mathbb{R}^4. \quad (\text{A.5})$$

In order to invoke the property $\Psi_0 \in [L^q(\mathbb{R}^4)]^4$, $q \in (4/3, 4)$, one applies Hölder's inequality with conjugate indices q' and q on the right-hand side in (A.5) to obtain

$$\|\Psi_0(x)\|_{\mathbb{C}^4} \leq \tilde{d}_4 \left(\int_{\mathbb{R}^4} d^4 y |x-y|^{-3q'} \langle y \rangle^{-2q'} \right)^{1/q'} \left(\int_{\mathbb{R}^4} d^4 y \|\Psi_0(y)\|_{\mathbb{C}^4}^q \right)^{1/q}, \quad x \in \mathbb{R}^4. \quad (\text{A.6})$$

The second integral on the right-hand side in (A.6) is finite for any $q \in (4/3, 4)$. The requirement $q \in (4/3, 4)$ implies

$$q' \in (4/3, 4). \quad (\text{A.7})$$

In order to apply Lemma 2.5 to the first integral on the right-hand side in (A.6), one would choose $k = 3q'$ and $\ell = 0$. The hypotheses (viz., $k \in [0, n)$) of Lemma 2.5 require

$$k = 3q' < 4, \quad (\text{A.8})$$

which is satisfied if and only if $q' < 4/3$. However, the condition $q' < 4/3$ is incompatible with (A.7).

An analogous problem is also encountered for all other dimensions $n \geq 5$: Indeed, for $n \geq 5$, (3.80) and (3.81) combine to yield

$$\|\Psi_0(x)\|_{\mathbb{C}^N} \leq \tilde{d}_n \left(\int_{\mathbb{R}^n} d^n y |x - y|^{(1-n)q'} \langle y \rangle^{-2q'} \right)^{1/q'} \left(\int_{\mathbb{R}^n} d^n y \|\Psi_0(y)\|_{\mathbb{C}^N}^q \right)^{1/q}, \quad x \in \mathbb{R}^n, \quad (\text{A.9})$$

for conjugate exponents q' and q . The second integral on the right-hand side in (A.9) is finite for all $q \in (2n/(n+2), 2n/(n-2))$. For $q \in (2n/(n+2), 2n/(n-2))$, one infers

$$q' \in (2n/(n+2), 2n/(n-2)). \quad (\text{A.10})$$

To apply Lemma 2.5 to the first integral on the right-hand side in (A.9), one must choose $k = (n-1)q'$. The condition $k \in [0, n)$ in Lemma 2.5 requires $(n-1)q' < n$, which is equivalent to

$$q' < \frac{n}{n-1}. \quad (\text{A.11})$$

However, the condition in (A.11) is incompatible with (A.10) in the sense that there is no $q' \in [1, \infty)$ which satisfies both (A.10) and (A.11) simultaneously. \diamond

Finally, we also illustrate the case of massive Dirac operators:

Remark A.3. We briefly indicate why the line of reasoning used in the proof of $\Psi \in L^\infty(\mathbb{R}^3)^4$ in Theorem 3.12 is bound to fail for $n \geq 4$.

For simplicity we focus again on the case $n = 4$. Applying (3.123) and the condition $\|V_2(\cdot)\|_{\mathbb{C}^{4 \times 4}} \leq C\langle \cdot \rangle^{-2}$ for some $C \in (0, \infty)$ in (3.124), one obtains

$$\|\Psi_{\pm m}(x)\|_{\mathbb{C}^4} \leq \tilde{d}_4 \left\{ \int_{\mathbb{R}^4} d^4 y |x - y|^{-2} \langle y \rangle^{-4} \|\Psi_{\pm m}(y)\|_{\mathbb{C}^4} + \int_{\mathbb{R}^4} d^4 y |x - y|^{-3} \langle y \rangle^{-4} \|\Psi_{\pm m}(x)\|_{\mathbb{C}^4} \right\}, \quad x \in \mathbb{R}^4, \quad (\text{A.12})$$

where $\tilde{d}_4 \in (0, \infty)$ is an appropriate x -independent constant.

Applying Hölder's inequality (with conjugate exponents $p = 3/2$ and $p' = 3$) to the first integral on the right-hand side in (A.12), one obtains

$$\begin{aligned} & \int_{\mathbb{R}^4} d^4 y |x - y|^{-2} \langle y \rangle^{-4} \|\Psi_{\pm m}(y)\|_{\mathbb{C}^4} \\ & \leq \left(\int_{\mathbb{R}^4} d^4 y |x - y|^{-3} \langle y \rangle^{-6} \right)^{2/3} \left(\int_{\mathbb{R}^4} d^4 y \|\Psi_{\pm m}(y)\|_{\mathbb{C}^4}^3 \right)^{1/3}, \quad x \in \mathbb{R}^4. \end{aligned} \quad (\text{A.13})$$

The second integral on the right-hand side in (A.13) is finite since a resonance function (resp., eigenfunction) satisfies $\Psi_{\pm m} \in [L^3(\mathbb{R}^4)]^4$. By Lemma 2.5 with $x_1 = x$, $k = 3$, $\ell = 0$, and $\beta = 5$,

$$\int_{\mathbb{R}^4} d^4y |x - y|^{-3} \langle y \rangle^{-6} \leq C_{4,3,0,5,1}, \quad x \in \mathbb{R}^4. \quad (\text{A.14})$$

Applying Hölder's inequality (with conjugate indices q' and q) to the second integral on the right-hand side in (A.12), one obtains

$$\begin{aligned} & \int_{\mathbb{R}^4} d^4y |x - y|^{-3} \langle y \rangle^{-4} \|\Psi_{\pm m}(y)\|_{\mathbb{C}^4} \\ & \leq \left(\int_{\mathbb{R}^4} d^4y |x - y|^{-3q'} \langle y \rangle^{-4q'} \right)^{1/q'} \left(\int_{\mathbb{R}^4} d^4y \|\Psi_{\pm m}(y)\|_{\mathbb{C}^4}^q \right)^{1/q}, \quad x \in \mathbb{R}^4. \end{aligned} \quad (\text{A.15})$$

The second integral on the right-hand side in (A.15) is finite for all $q \in (2, 4)$. The requirement $q \in (2, 4)$ implies

$$q' \in (4/3, 2). \quad (\text{A.16})$$

In order to apply Lemma 2.5 to the first integral on the right-hand side in (A.15), one would choose $k = 3q'$ and $\ell = 0$. The hypotheses (viz., $k \in [0, n)$) of Lemma 2.5 require

$$k = 3q' < 4, \quad (\text{A.17})$$

which is satisfied if and only if $q' < 4/3$. However, the condition $q' < 4/3$ is incompatible with (A.16). \diamond

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REFERENCES

- [1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1972.
- [2] C. Adam, B. Muratori, C. Nash, *Zero modes of the Dirac operator in three dimensions*, Phys. Rev. D (3) **60**, 125001 (1999).
- [3] C. Adam, B. Muratori, C. Nash, *Degeneracy of zero modes of the Dirac operator in three dimensions*, Phys. Lett. B **485**, 314–318 (2000).
- [4] C. Adam, B. Muratori, C. Nash, *Multiple zero modes of the Dirac operator in three dimensions*, Phys. Rev. D (3) **62**, 085026 (2000).
- [5] C. Adam, B. Muratori, C. Nash, *Zero modes in finite range magnetic fields*, Modern Phys. Lett. A **15**, 1577–1581 (2000).
- [6] Y. Aharonov and A. Casher, *Ground state of a spin-1/2 charged particle in a two-dimensional magnetic field*, Phys. Rev. A (3) **19**, 2461–2462 (1979).
- [7] D. Aiba, *Absence of zero resonances of massless Dirac operators*, Hokkaido Math. J. **45**, 263–270 (2016).
- [8] S. Albeverio, F. Gesztesy, and R. Høegh-Krohn, *The low energy expansion in nonrelativistic scattering theory*, Ann. Inst. H. Poincaré **37**, 1–28 (1982).
- [9] A. A. Balinsky and W. D. Evans, *On the zero modes of Pauli operators*, J. Funct. Anal. **179**, 120–135 (2001).
- [10] A. A. Balinsky and W. D. Evans, *On the zero modes of Weyl-Dirac operators and their multiplicity*, Bull. London Math. Soc. **34**, 236–242 (2002).
- [11] A. Balinsky and W. D. Evans, *Zero modes of Pauli and Weyl-Dirac operators*, in *Advances in Differential Equations and Mathematical Physics*, Contemp. Math. **327**, 179 (2003).
- [12] A. Balinsky and W. D. Evans, *Spectral Analysis of Relativistic Operators*, Imperial College Press, London, 2011.

- [13] A. Balinsky, W. D. Evans, and Y. Saito, *Dirac–Sobolev inequalities and estimates for the zero modes of massless Dirac operators*, J. Math. Phys. **49**, 043514 (2008).
- [14] J. Behrndt, F. Gesztesy, H. Holden, and R. Nichols, *Dirichlet-to Neumann maps, abstract Weyl–Titchmarsh M -functions, and a generalized index of unbounded meromorphic operator-valued functions*, J. Diff. Eq. **261**, 3551–3587 (2016).
- [15] R. D. Benguria and H. Van Den Bosch, *A criterion for the existence of zero modes for the Pauli operator with fastly decaying fields* J. Math. Phys. **56**, 052104, (2015).
- [16] H. Blancarte, B. Grebert, R. Weder, *High- and low-energy estimates for the Dirac equation*, J. Math. Phys. **36**, 991–1015 (1995).
- [17] D. Bollé, F. Gesztesy, and S. F. J. Wilk, *New results for scattering on the line*, Phys. Lett. **97A**, 30–34 (1983).
- [18] D. Bollé, F. Gesztesy, and S. F. J. Wilk, *A complete treatment of low-energy scattering in one dimension*, J. Operator Theory **13**, 3–31 (1985).
- [19] D. Bollé, F. Gesztesy, and C. Danneels, *Threshold scattering in two dimensions*, Ann. Inst. H. Poincaré **48**, 175–204 (1988).
- [20] D. Bollé, F. Gesztesy, C. Danneels, and S. F. J. Wilk, *Threshold behavior and Levinson’s theorem for two-dimensional scattering systems: A surprise*, Phys. Rev. Lett. **56**, 900–903 (1986).
- [21] D. Bollé, F. Gesztesy, and M. Klaus, *Scattering theory for one-dimensional systems with $\int dx V(x) = 0$* , J. Math. Anal. Appl. **122**, 496–518 (1987); Errata **130**, 590 (1988).
- [22] A. Carey, F. Gesztesy, H. Grosse, G. Levitina, D. Potapov, F. Sukochev, and D. Zanin, *Trace formulas for a class of non-Fredholm operators: A review*, Rev. Math. Phys. **28**, no. 10, (2016), 1630002 (55 pages).
- [23] A. Carey, F. Gesztesy, G. Levitina, R. Nichols, F. Sukochev, and D. Zanin, *On the limiting absorption principle for massless Dirac operators and properties of spectral shift functions*, in preparation.
- [24] A. Carey, F. Gesztesy, G. Levitina, D. Potapov, F. Sukochev, and D. Zanin, *On index theory for non-Fredholm operators: a $(1+1)$ -dimensional example*, Math. Nachrichten **289**, 575–609 (2016).
- [25] A. Carey, F. Gesztesy, G. Levitina, and F. Sukochev, *On the index of a non-Fredholm model operator*, Operators and Matrices **10**, 881–914 (2016).
- [26] A. Carey, F. Gesztesy, D. Potapov, F. Sukochev, and Y. Tomilov, *On the Witten index in terms of spectral shift functions*, J. Analyse Math. **132**, 1–61 (2017).
- [27] N. Du Plessis, *An Introduction to Potential Theory*, Oliver & Boyd, Edinburgh, 1970.
- [28] D. M. Elton, *New examples of zero modes*, J. Phys. A **33**, 7297–7303 (2000).
- [29] D. M. Elton, *Spectral properties of the equation $(\nabla + ieA) \times u = \pm mu$* , Proc. Roy. Soc. Edinburgh **131 A**, 1065–1089 (2001).
- [30] D. M. Elton, *The local structure of zero mode producing magnetic potentials*, Commun. Math. Phys. **229**, 121–139 (2002).
- [31] M. B. Erdoğan, M. Goldberg, and W. R. Green, *Dispersive estimates for four dimensional Schrödinger and wave equations with obstructions at zero energy*, Commun. PDE **39**, 1936–1964 (2014).
- [32] M. B. Erdoğan, M. Goldberg, and W. R. Green, *Limiting absorption principle and Strichartz estimates for Dirac operators in two and higher dimensions*, Commun. Math. Phys. **367**, 241–263 (2019).
- [33] M. B. Erdoğan, M. Goldberg, and W. R. Green, *The massless Dirac equation in two dimensions: Zero-energy obstructions and dispersive estimates*, ArXiv1807.00219.
- [34] M. B. Erdoğan, M. Goldberg, and W. Schlag, *Strichartz and smoothing estimates for Schrödinger operators with large magnetic potentials in \mathbb{R}^3* , J. Eur. Math. Soc. **10**, 507–531 (2008).
- [35] M. B. Erdoğan, M. Goldberg, and W. Schlag, *Strichartz and smoothing estimates for Schrödinger operators with almost critical magnetic potentials in three and higher dimensions*, Forum Math. **21**, 687–722 (2009).
- [36] M. B. Erdoğan and W. R. Green, *Dispersive estimates for the Schrödinger equation for $C^{\frac{n-3}{2}}$ potentials in odd dimensions*, Int. Math. Res. Notices, 2010, no. 13, 2532–2565.
- [37] M. B. Erdoğan and W. R. Green, *Dispersive estimates for Schrödinger operators in dimension two with obstructions at zero energies*, Trans. Amer. Math. Soc. **365**, 6403–6440 (2013).

- [38] M. B. Erdoğan and W. R. Green, *The Dirac equation in two dimensions: Dispersive estimates and classification of threshold obstructions*, Commun. Math. Phys. **352**, 719–757 (2017).
- [39] M. B. Erdoğan, W. R. Green, and E. Toprak, *Dispersive estimates for massive Dirac operators in dimension two*, J. Diff. Eq. **264**, 5802–5837 (2018).
- [40] M. B. Erdoğan, W. R. Green, and E. Toprak, *Dispersive estimates for Dirac operators in dimension three with obstructions at threshold energies*, Amer. J. Math. (to appear), ArXiv1609.05164.
- [41] M. B. Erdoğan and W. Schlag, *Dispersive estimates for Schrödinger operators in the presence of a resonance and/or an eigenvalue at zero energy in dimension three: I*, Dyn. PDE **1**, 359–379 (2004).
- [42] M. B. Erdoğan and W. Schlag, *Dispersive estimates for Schrödinger operators in the presence of a resonance and/or an eigenvalue at zero energy in dimension three: II*, J. Anal. Math. **99**, 199–248 (2006).
- [43] L. Erdős and J. P. Solovej, *The kernel of Dirac operators on \mathbb{S}^3 and \mathbb{R}^3* , Rev. Math. Phys. **13**, 1247–1280 (2001).
- [44] G. B. Folland, *Real Analysis. Modern Techniques and Their Applications*, 2nd ed., Wiley, New York, 1999.
- [45] J. Fröhlich, E. H. Lieb, and M. Loss, *Stability of Coulomb systems with magnetic fields. I. The one-electron atom*, Commun. Math. Phys. **104**, 251–270 (1986).
- [46] F. Gesztesy and H. Holden, *A unified approach to eigenvalues and resonances of Schrödinger operators using Fredholm determinants*, J. Math. Anal. Appl. **123**, 181–198 (1987).
- [47] F. Gesztesy, H. Holden, and R. Nichols, *On factorizations of analytic operator-valued functions and eigenvalue multiplicity questions*, Integral Eq. Operator Th. **82**, 61–94 (2015); Erratum **85**, 301–302 (2016).
- [48] F. Gesztesy, Y. Latushkin, M. Mitrea, and M. Zinchenko, *Nonselfadjoint operators, infinite determinants, and some applications*, Russ. J. Math. Phys. **12**, 443–471 (2005).
- [49] F. Gesztesy, M. Malamud, M. Mitrea, and S. Naboko, *Generalized polar decompositions for closed operators in Hilbert spaces and some applications*, Integral Eq. Operator Th. **64**, 83–113 (2009).
- [50] A. Jensen, *Spectral properties of Schrödinger operators and time-decay of the wave functions. Results in $L^2(\mathbb{R}^m)$, $m \geq 5$* , Duke Math. J. **47**, 57–80 (1980).
- [51] A. Jensen, *Spectral properties of Schrödinger operators and time-decay of the wave functions. Results in $L^2(\mathbb{R}^4)$* , J. Math. Anal. Appl. **101**, 397–422 (1984).
- [52] A. Jensen and T. Kato, *Spectral properties of Schrödinger operators and time-decay of the wave functions*, Duke Math. J. **46**, 583–611 (1979).
- [53] A. Jensen and G. Nenciu, *A unified approach to resolvent expansions at thresholds*, Rev. Math. Phys. **13**, 717–754 (2001); Erratum **16**, 675–677 (2004).
- [54] H. Kalf, T. Okaji, and O. Yamada, *The Dirac operator with mass $m_0 \geq 0$: Non-existence of zero modes and of threshold eigenvalues*, Doc. Math. **20**, 37–64 (2015); Addendum, Doc. Math. (to appear).
- [55] H. Kalf and O. Yamada, *Essential self-adjointness of n -dimensional Dirac operators with a variable mass term*, J. Math. Phys. **42**, 2667–2676 (2001).
- [56] M. Klaus, *Some applications of the Birman–Schwinger principle*, Helv. Phts. Acta **55**, 49–68 (1982).
- [57] M. Klaus, *On coupling constant thresholds and related eigenvalue properties of Dirac operators*, J. reine angew. Math. **362**, 197–212 (1985).
- [58] M. Klaus, *On the Levinson theorem for Dirac operators*, J. Math. Phys. **31**, 182–190 (1990).
- [59] M. Klaus and B. Simon, *Coupling constant thresholds in nonrelativistic quantum mechanics. I. Short-range two-body case*, Ann. Phys. **130**, 251–281 (1980).
- [60] M. Loss and H.-T. Yau, *Stability of Coulomb systems with magnetic fields. III. Zero energy bound states of the Pauli operator*, Commun. Math. Phys. **104**, 283–290 (1986).
- [61] R. McOwen, *The behavior of the Laplacian on weighted Sobolev spaces*, Commun. Pure Appl. Math. **32**, 783–795 (1979).
- [62] M. Murata, *Asymptotic expansions in time for solutions of Schrödinger-type equations*, J. Funct. Anal. **49**, 10–56 (1982).
- [63] R. G. Newton, *Scattering Theory of Waves and Particles*, 2nd ed., Dover, New York, 2002.
- [64] L. Nirenberg and H. F. Walker, *The null spaces of elliptic partial differential operators on \mathbb{R}^n* , J. Math. Anal. Appl. **42**, 271–301 (1973).

- [65] M. Persson, *Zero modes for the magnetic Pauli operator in even-dimensional Euclidean space*, Lett. Math. Phys. **85**, 111–128 (2008).
- [66] G. Rozenblum and N. Shirokov, *Infiniteness of zero modes for the Pauli operator with singular magnetic field*, J. Funct. Anal. **233**, 135–172 (2006).
- [67] Y. Saitō and T. Umeda, *The zero modes and zero resonances of massless Dirac operators*, Hokkaido Math. J. **37**, 363–388 (2008).
- [68] Y. Saitō and T. Umeda, *The asymptotic limits of zero modes of massless Dirac operators*, Lett. Math. Phys. **83**, 97–106 (2008).
- [69] Y. Saitō and T. Umeda, *Eigenfunctions at the threshold energies of magnetic Dirac operators*, Rev. Math. Phys. **23**, 155–178 (2011).
- [70] Y. Saitō and T. Umeda, *A sequence of zero modes of Weyl–Dirac operators and an associated sequence of solvable polynomials*, in *Spectral theory, Function Spaces and Inequalities. New Techniques and Recent Trends*, B. M. Brown, J. Lang, and I. G. Wood (eds.), Operator Theory: Advances and Applications, Vol. 219, Birkhäuser, Springer, Basel, 2012, pp. 197–209.
- [71] K. M. Schmidt, *Spectral properties of rotationally symmetric massless Dirac operators*, Lett. Math. Phys. **92**, 231–241 (2010).
- [72] K. M. Schmidt and T. Umeda, *Spectral properties of massless Dirac operators with real-valued potentials*, RIMS Kōkyūroku Bessatsu **B45**, 25–30 (2014).
- [73] K. M. Schmidt and T. Umeda, *Schnol’s theorem and spectral properties of massless Dirac operators with scalar potentials*, Lett. Math. Phys. **105**, 1479–1497 (2015).
- [74] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ, 1970.
- [75] B. Thaller, *The Dirac Equation*, Texts and Monographs in Physics, Springer, Berlin, 1992.
- [76] E. Toprak, *A weighted estimate for two dimensional Schrödinger, matrix Schrödinger, and wave equations with resonance of the first kind at zero energy*, J. Spectral Th. **7**, 1235–1284 (2017).
- [77] K. Yajima, *Dispersive estimates for Schrödinger equations with threshold resonance and eigenvalues*, Commun. Math. Phys. **259**, 475–509 (2005).
- [78] Y. Zhong and G. L. Gao, *Some new results about the massless Dirac operator*, J. Math. Phys. **54**, 043510 (2013).

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