WEAK AND VAGUE CONVERGENCE OF SPECTRAL SHIFT FUNCTIONS OF ONE-DIMENSIONAL SCHRÖDINGER OPERATORS WITH COUPLED BOUNDARY CONDITIONS

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Dedicated with great pleasure to Konstantin Makarov on the occasion of his 60th birthday

Abstract. We prove weak and vague convergence results for spectral shift functions associated with self-adjoint one-dimensional Schrödinger operators on intervals of the form $(-\ell, \ell)$ to the full-line spectral shift function in the limit $\ell \to \infty$ for a class of coupled boundary conditions. The boundary conditions considered here include periodic boundary conditions as a special case.

1. Introduction

We consider the limit $\ell \to \infty$ of spectral shift functions corresponding to restrictions of pairs of one-dimensional Schrödinger operators to intervals $(-\ell, \ell)$ with coupled boundary conditions at the endpoints of the form

$$\begin{pmatrix} u(\ell) \\ u'(\ell) \end{pmatrix} = e^{i\phi} \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \begin{pmatrix} u(-\ell) \\ u'(-\ell) \end{pmatrix}, \tag{1.1}$$

where $\phi \in [0, 2\pi)$, $a \in \mathbb{R}\backslash\{0\}$, and $b \in \mathbb{R}$ are fixed, and obtain weak and vague convergence results for the corresponding sequence of spectral shift functions. The class of coupled boundary conditions considered here includes periodic boundary conditions (viz., $\phi = 0$, $a = 1$, and $b = 0$) and, more generally, quasi-periodic boundary conditions (viz., $\phi \in [0, 2\pi)$, $a = 1$, $b = 0$) as special cases.

The infinite volume limit of spectral shift functions of pairs of Schrödinger operators has been studied by many authors (e.g., [3], [4], [7], [12], [13], [17], [8], [9]). Simply stated, the problem is this: consider two Schrödinger operators H_ℓ and $H_\ell^{(0)}$ in the Hilbert space $L^2((-\ell, \ell)^n; d^n x)$ which are self-adjoint realizations of the differential expressions $-\Delta + V$ and $-\Delta$, respectively, with appropriate fixed boundary conditions on the boundary $\partial(-\ell, \ell)^n$ and a measurable function $V : \mathbb{R}^n \to \mathbb{R}$ which decays in some appropriate sense at infinity. If $\xi(\,\cdot\,; H_\ell, H_\ell^{(0)})$ denotes the spectral shift function for the pair $(H_\ell, H_\ell^{(0)})$, $\ell \in \mathbb{N}$, normalized to vanish identically in a neighborhood of $-\infty$, in what manner does the sequence $\{\xi(\,\cdot\,; H_\ell, H_\ell^{(0)})\}_{\ell=1}^\infty$ converge to the normalized spectral shift function $\xi(\,\cdot\,; H, H^{(0)})$ for the pair H and $H^{(0)}$, the self-adjoint realizations of $-\Delta + V$ and $-\Delta$ in $L^2(\mathbb{R}^n; dx)$?

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We refer to [8] for a detailed history of work on the problem up to about 2012, and recall here only those results pertaining to the one-dimensional context, $n = 1$. First, it is known that one cannot expect pointwise convergence of spectral shift functions in the infinite volume limit for the following simple reason. The spectral shift function $\xi(\,\cdot\,; H_\ell, H_\ell^{(0)})$ is the difference of the eigenvalue counting functions for H_ℓ and $H_\ell^{(0)}$ (cf., e.g., the remarks following [25, Theorem 8.7.2]), so it is necessarily integer-valued almost everywhere. On the other hand, $\xi(\,\cdot\,; H, H^{(0)})$, which coincides with the scattering phase for H and $H^{(0)}$ up to a constant multiple (cf., e.g., the remark following [26, Theorem 5.4.3]), is a continuous function of $\lambda > 0$ which goes to zero as $\lambda \to \infty$. Clearly, the sequence of integer valued functions $\xi(\,\cdot\,; H_\ell, H_\ell^{(0)})$ will not converge pointwise to a function that is non-constant and continuous on $(0, \infty)$.

In the context of spectral shift functions, *vague convergence* has proven to be a more suitable mode of convergence. One recalls that a sequence $\{f_\ell\}_{\ell=1}^\infty$ of locally integrable functions on $\mathbb{R}$ is said to converge *vaguely* to the locally integrable function f if for every $g \in C_0(\mathbb{R})$, with $C_0(\mathbb{R})$ denoting the set of all compactly supported continuous functions on $\mathbb{R}$, one has

$$\lim_{\ell\to\infty} \int_{\mathbb{R}} f_\ell(\lambda)\, g(\lambda)\, d\lambda = \int_{\mathbb{R}} f(\lambda)\, g(\lambda)\, d\lambda. \tag{1.2}$$

Borovyk and Makarov ([4], see also [3]) investigated the infinite volume limit problem for spectral shift functions with the half-line $(0, \infty)$ playing the role of the infinite volume and finite intervals of the form $(0, r)$, with $V \in L^1((0, \infty); (1 + x)\, dx)$ and Dirichlet boundary conditions at the endpoints of $(0, r)$. They proved vague convergence of $\xi(\,\cdot\,; H_r, H_r^{(0)})$ to $\xi(\,\cdot\,; H, H^{(0)})$ as the right endpoint r tends to ∞,

$$\lim_{r\to\infty} \int_{\mathbb{R}} \xi(\lambda; H_r, H_r^{(0)})\, g(\lambda)\, d\lambda = \int_{\mathbb{R}} \xi(\lambda; H, H^{(0)})\, g(\lambda)\, d\lambda, \quad g \in C_0(\mathbb{R}), \tag{1.3}$$

as well as the interesting assertion that the half-line spectral shift function may be recovered pointwise in terms of the following Cesáro limit:

$$\lim_{R\to\infty} \frac{1}{R} \int_0^R \xi(\lambda; H_r, H_r^{(0)})\, dr = \xi(\lambda; H, H^{(0)}), \quad \lambda \in \mathbb{R}\backslash(\sigma_p(H) \cup \{0\}), \tag{1.4}$$

where $\sigma_p(H)$ denotes the point spectrum of H. In [8], Gesztesy and one of the present authors extended the vague convergence result in (1.3) to all separated self-adjoint boundary conditions,

$$\cos(\alpha)u(0) + \sin(\alpha)u'(0) = 0, \quad \cos(\beta)u(r) + \sin(\beta)u'(r) = 0, \tag{1.5}$$

where $\alpha, \beta \in [0, \pi)$ are fixed, under the slightly weaker assumption that the potential V belongs to $L^1((0, \infty); dx)$. Actually, convergence is strengthened in [8] to the result that

$$\lim_{r\to\infty} \int_{\mathbb{R}} \frac{\xi(\lambda; H_r, H_r^{(0)})}{1 + \lambda^2}\, f(\lambda)\, d\lambda = \int_{\mathbb{R}} \frac{\xi(\lambda; H, H^{(0)})}{1 + \lambda^2}\, f(\lambda)\, d\lambda \tag{1.6}$$

for every bounded continuous function f on $\mathbb{R}$ and any set of separated self-adjoint boundary conditions at the endpoints of $(0, r)$. The statement in (1.6) means that $(1 + |\cdot|^2)^{-1}\xi(\,\cdot\,; H_r, H_r^{(0)})$ converges *weakly* to $(1 + |\cdot|^2)^{-1}\xi(\,\cdot\,; H, H^{(0)})$ as $r \to \infty$. Note that (1.6) immediately implies (1.3) by choosing $f(\lambda) = (1 + \lambda^2)g(\lambda)$ in (1.6).

Also, the inclusion of the weight $(1+\lambda^2)^{-1}$ in (1.6) is crucial; otherwise, the integrals may be divergent.

A closer look at [8] reveals that the arguments presented there extend to the case where the infinite volume is $\mathbb{R}$, and the finite intervals take the form $(-\ell,\ell)$, with separated self-adjoint boundary conditions at the endpoints (cf. [9, §4**(I)**]),

$$\cos(\alpha)u(-\ell)+\sin(\alpha)u'(-\ell)=0, \quad \cos(\beta)u(\ell)+\sin(\beta)u'(\ell)=0. \tag{1.7}$$

Infinite Fredholm determinants and convergence properties of resolvents of the finite interval Schrödinger operators as $r\to\infty$ (or $\ell\to\infty$) play a central role in [8]. This determinant approach led to the development of abstract criteria in [9] for vague convergence of spectral shift functions in terms of the convergence properties of associated sequences of Birman–Schwinger-type operators (i.e., resolvents conjugated from the left and/or right with suitable factors of the potential) in the Hilbert–Schmidt or trace classes.

Although the question of vague convergence of spectral shift functions is answered in [8] for all separated self-adjoint boundary conditions (1.7), which includes both Dirichlet (viz., $\alpha=\beta=0$) and Neumann (viz., $\alpha=\beta=\pi/2$) boundary conditions as special cases, the analogous problem for coupled self-adjoint boundary conditions, which include periodic boundary conditions as a special case, is not treated. Moreover, coupled boundary conditions are not discussed in the applications in [9].

In this paper, we extend (1.6) to all coupled boundary conditions of the form (1.1) and for $V\in L^1(\mathbb{R};dx)$ by employing Krein-type resolvent identities, in particular their precise form for regular Sturm–Liouville operators developed recently in [6], in order to verify and apply the abstract convergence criteria from [9]. A Krein-type resolvent identity relates the resolvent operators of two self-adjoint extensions of a symmetric operator with equal deficiency indices, and abstract identities of this type have been presented in various sources (cf., e.g., [1, §VII.84], [20, §14.6], and [22, Lemma 2.30]) and studied in both abstract and concrete applications by a number of authors (e.g., [2], [5], [6], [10], [15], [16], [18], and [19]).

In general, coupled self-adjoint boundary conditions for a regular Sturm–Liouville operator on the interval $(-\ell,\ell)$ are of the form (cf., e.g., [6, Theorem 2.5], [24, Theorem 13.15])

$$\begin{pmatrix} u(\ell) \\ u'(\ell)\end{pmatrix} = e^{i\phi}\begin{pmatrix} R_{1,1} & R_{1,2} \\ R_{2,1} & R_{2,2}\end{pmatrix}\begin{pmatrix} u(-\ell) \\ u'(-\ell)\end{pmatrix}, \tag{1.8}$$

where $\phi\in[0,2\pi)$ and the matrix $R=[R_{j,k}]_{1\leqslant j,k\leqslant 2}$ belongs to $SL_2(\mathbb{R})$, that is $R\in\mathbb{R}^{2\times 2}$ and $\det(R)=1$. Now, if $W\in L^1((-\ell,\ell);dx)$ and $H_{\ell,R,\phi}$ denotes the self-adjoint realization of the differential expression

$$\tau=-\frac{d^2}{dx^2}+W \tag{1.9}$$

in $L^2((-\ell,\ell);dx)$ with the boundary conditions in (1.8) and $H_{\ell,D}$ denotes the self-adjoint realization of τ in $L^2((-\ell,\ell);dx)$ with Dirichlet boundary conditions, then the difference of the resolvents of $H_{\ell,R,\phi}$ and $H_{\ell,D}$ is finite rank with rank at most equal to two. In fact, if $R_{1,2}=0$, then the difference is precisely rank one, owing to the fact that $H_{\ell,R,\phi}$ and $H_{\ell,D}$ are not *relatively prime* (see [1, §84]) with respect to the underlying minimal operator when $R_{1,2}=0$, and (cf. [6])

$$\big(H_{\ell,D}-zI_{L^2((-\ell,\ell);dx)}\big)^{-1}-\big(H_{\ell,R,\phi}-zI_{L^2((-\ell,\ell);dx)}\big)^{-1} \tag{1.10}$$

$$= q_{\ell,R,\phi}(z)^{-1}(u_{R,\phi}(\overline{z},\,\cdot\,),\,\cdot\,)_{L^2((-\ell,\ell);dx)}u_{R,\phi}(z,\,\cdot\,), \quad z\in\rho(H_{\ell,D})\cap\rho(H_{\ell,R,\phi}),$$

where $u_{R,\phi}(z,\,\cdot\,)$ is an spanning vector for $\ker(H_{\ell,\max}-zI_{L^2((-\ell,\ell);dx)})$ (the operator $H_{\ell,\max}$ is the maximal operator associated to τ in $L^2((-\ell,\ell);dx)$ and is defined precisely in Section 2) and $q_{\ell,R,\phi}(\,\cdot\,)$ is a nonvanishing complex-valued function on $\rho(H_{\ell,D})\cap\rho(H_{\ell,R,\phi})$.

In order to apply the abstract convergence criteria of [9], it is necessary to prove appropriate convergence results for Birman–Schwinger-type operators for the finite interval Schrödinger operator with coupled boundary conditions in the limit $\ell\to\infty$. To do this, we make use of the Krein identity in (1.10) to relate the corresponding Birman–Schwinger-type operators for the finite interval Schrödinger operators with coupled boundary conditions to the Birman–Schwinger-type operators for the finite-interval Schrödinger operators with Dirichlet boundary conditions, plus a rank one term. The required convergence properties of the Dirichlet Birman–Schwinger-type operators as $\ell\to\infty$ are known from [8], so we are left to analyze the limiting behavior as $\ell\to\infty$ of the remaining rank one term. A precise knowledge, in particular the explicit ℓ-dependence, of the factor $q_{\ell,R,\phi}(z)$ and the function $u_{\ell,R,\phi}(z,\,\cdot\,)$ is essential to this approach.

We should note that if $R_{1,2}\neq 0$, then the resolvent difference on the left-hand side in (1.10) is precisely rank two, and

$$\begin{aligned}&\big(H_{\ell,D}-zI_{L^2((-\ell,\ell);dx)}\big)^{-1}-\big(H_{\ell,R,\phi}-zI_{L^2((-\ell,\ell);dx)}\big)^{-1}\\&\quad=\sum_{j,k=1}^{2}\big[Q_{\ell,R,\phi}(z)^{-1}\big]_{j,k}(u_k(\overline{z},\,\cdot\,),\,\cdot\,)_{L^2((-\ell,\ell);dx)}u_j(z,\,\cdot\,), \qquad (1.11)\\&\qquad\qquad z\in\rho(H_{\ell,D})\cap\rho(H_{\ell,R,\phi}),\end{aligned}$$

where $\{u_{j,R,\phi}(z,\,\cdot\,)\}_{j=1,2}$ is an appropriate basis for $\ker(H_{\ell,\max}-zI_{L^2((-\ell,\ell);dx)})$ and $Q_{\ell,R,\phi}(\,\cdot\,)$ is a nonsingular $\mathbb{C}^{2\times 2}$-valued function on $\rho(H_{\ell,D})\cap\rho(H_{\ell,R,\phi})$. In this case, the problem of vague convergence of spectral shift functions is slightly more delicate due to the complicated nature of the coefficients in the rank two term of the Krein formula (1.11) and we will return to this elsewhere.

Next, we briefly summarize the organization and contents of each section of this paper: In Section 2, we rigorously define the self-adjoint Schrödinger operators H, $H^{(0)}$ in $L^2(\mathbb{R};dx)$ acting formally as $-d^2/dx^2+V$ and $-d^2/dx^2$, respectively, and their restrictions H_ℓ, $H_\ell^{(0)}$ to $(-\ell,\ell)$ with coupled self-adjoint boundary conditions of the form (1.1). We also introduce the restrictions $H_{\ell,D}$, $H_{\ell,D}^{(0)}$ to $(-\ell,\ell)$ with Dirichlet boundary conditions. We discuss their basic properties and recall Krein's resolvent identity which relates the resolvents of H_ℓ and $H_{\ell,D}$ via a rank one term. In Section 3, which contains the bulk of our major analysis, we use the Krein resolvent identity to study convergence properties in the limit $\ell\to\infty$ of the Birman–Schwinger-type operators associated to H_ℓ and $H_\ell^{(0)}$. The results in Lemmata 3.3, 3.4, and 3.5 are fundamental to our approach, and they are precisely the results that ultimately yield vague convergence of spectral shift functions and the analogue of (1.6). In Section 4, we combine the convergence results from Section 3 with the abstract convergence criteria from [9] to obtain weak and vague convergence of spectral shift functions in the limit $\ell\to\infty$ for the class of coupled boundary conditions in (1.1). To our knowledge, these are the first results of their type for classes of coupled boundary conditions. Appendix A recalls some basic convergence results for trace ideals that are used throughout this paper. For completeness, Appendix B contains a summary of the convergence criteria from [9], suitably tailored for the applications to Schrödinger operators in $L^2(\mathbb{R}; dx)$ and $L^2((-\ell, \ell); dx)$ to be studied.

Finally, we comment on some of the basic notation used throughout this paper. Let $\mathcal{H}$ be a separable complex Hilbert space, $(\cdot, \cdot)_{\mathcal{H}}$ the scalar product in $\mathcal{H}$ (linear in the second argument), and $I_{\mathcal{H}}$ the identity operator in $\mathcal{H}$.

If T is a linear operator mapping (a subspace of) a Hilbert space into another, then $\mathrm{dom}(T)$ and $\ker(T)$ denote the domain and kernel (i.e., null space) of T. The closure of a closable operator S is denoted by $\overline{S}$. The spectrum and resolvent set of a closed linear operator in a Hilbert space will be denoted by $\sigma(\cdot)$ and $\rho(\cdot)$, respectively. The point spectrum (i.e., the set of eigenvalues) of a linear operator T will be denoted by $\sigma_p(T)$. The quadratic form sum of two self-adjoint operators A and W will be denoted by $A +_q W$.

The convergence of bounded operators in the strong operator topology (i.e., pointwise limits) will be denoted throughout by s-lim. The Banach spaces of bounded and compact linear operators on a separable complex Hilbert space $\mathcal{H}$ are denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_\infty(\mathcal{H})$, respectively; the corresponding ℓ^p-based trace ideals will be denoted by $\mathcal{B}_p(\mathcal{H})$, their norms are abbreviated by $\|\cdot\|_{\mathcal{B}_p(\mathcal{H})}$, $p \in [1, \infty)$. Moreover, $\mathrm{tr}_{\mathcal{H}}(A)$ denotes the corresponding trace of a trace class operator $A \in \mathcal{B}_1(\mathcal{H})$.

For any closed finite interval $[a, b] \subset \mathbb{R}$, $AC([a, b])$ denotes the set of absolutely continuous functions defined on $[a, b]$. The symbol $\mathrm{sgn}(\cdot)$ denotes the signum function on $\mathbb{R}$,

$$\mathrm{sgn}(x) = \begin{cases} \dfrac{x}{|x|}, & x \in \mathbb{R}\backslash\{0\}, \\ 0, & x = 0. \end{cases} \tag{1.12}$$

We denote by $C(\mathbb{R})$ the space of continuous functions on $\mathbb{R}$, by $C_0(\mathbb{R})$ the continuous functions on $\mathbb{R}$ with compact support, and by $C_b(\mathbb{R})$ the bounded continuous functions on $\mathbb{R}$. $L^1_{\mathrm{loc}}(\mathbb{R}; dx)$ denotes the set of (equivalence classes of) locally integrable (with respect to Lebesgue measure) functions on $\mathbb{R}$, and $H^1(\mathbb{R})$ (resp., $H^1(a, b)$) is the Sobolev space of order one on $\mathbb{R}$ (resp., $(a, b) \subset \mathbb{R}$) (cf., e.g., [20, Appendix E]). If u is a function on a set Σ, then the restriction of u to a subset $\Omega \subset \Sigma$ will be denoted by $u|_\Omega$. Finally, "resp." is used as an abbreviation for "respectively," and "a.e" is used as an abbreviation for "almost everywhere" and "almost every."

2. One-Dimensional Schrödinger Operators, Their Properties, and Krein's Formula

In this preparatory section, we rigorously define the one-dimensional Schrödinger operators and their restrictions to finite intervals of the form $(-\ell, \ell)$, $\ell \in \mathbb{N}$, to be studied in the sequel and recall some of their basic properties. In particular, we recall a special case of Krein's resolvent identity which relates the resolvent of Schrödinger-type operators on $(-\ell, \ell)$ with the coupled boundary conditions in (1.1) to the resolvent of the Schrödinger-type operator with Dirichlet boundary conditions at the endpoints. We begin by introducing the following set of hypotheses which also introduce much of the notation to be employed in the sequel.

Hypothesis 2.1. *(i) Suppose*

$$V \in L^1(\mathbb{R}; dx) \ \text{ is real-valued a.e.,} \tag{2.1}$$

and

$$M := \int_{-\infty}^{\infty} |V(x)|\, dx. \tag{2.2}$$

For each $\ell \in \mathbb{N}$, let V_ℓ denote the restriction of V to $(-\ell, \ell)$ so that

$$V_\ell(x) = V|_{(-\ell,\ell)}(x) \ \text{ for a.e. } x \in (-\ell, \ell), \tag{2.3}$$

and define

$$\mathfrak{D}_\ell = \{f \in L^2((-\ell,\ell); dx) \,|\, f, f' \in AC([-\ell,\ell]),\ -f'' + V_\ell f \in L^2((-\ell,\ell); dx)\}. \tag{2.4}$$

Introduce the differential expression τ by

$$\tau = -\frac{d^2}{dx^2} + V(x), \tag{2.5}$$

and let $H_{\ell,\max}$ denote the maximal operator associated to τ in $L^2((-\ell,\ell); dx)$ so that

$$\begin{aligned} (H_{\ell,\max} f)(x) = -f''(x) + V(x) f(x) \ \text{ for a.e. } x \in (-\ell,\ell),\\ f \in \operatorname{dom}(H_{\ell,\max}) = \mathfrak{D}_\ell,\ \ell \in \mathbb{N}. \end{aligned} \tag{2.6}$$

(ii) Let $V(x)$ and $V_\ell(x)$, $\ell \in \mathbb{N}$, be factored according to

$$\begin{aligned} V(x) = u(x)v(x), \quad v(x) = |V(x)|^{1/2}, \quad u(x) = v(x) \operatorname{sgn}(V(x)),\\ \text{for a.e. } x \in \mathbb{R}, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} V_\ell(x) = V|_{(-\ell,\ell)}(x), \quad v_\ell(x) = v|_{(-\ell,\ell)}(x), \quad u_\ell(x) = u|_{(-\ell,\ell)}(x),\\ \text{for a.e. } x \in (-\ell,\ell),\ \ell \in \mathbb{N}. \end{aligned} \tag{2.8}$$

(iii) For each $\ell \in \mathbb{N}$, let $H_{\ell,D}$ denote the self-adjoint Dirichlet operator defined in $L^2((-\ell,\ell); dx)$ by

$$\begin{aligned} &(H_{\ell,D} f)(x) = -f''(x) + V(x) f(x) \ \text{ for a.e. } x \in (-\ell,\ell), \\ &f \in \operatorname{dom}(H_{\ell,D}) = \{g \in \mathfrak{D}_\ell \,|\, g(-\ell) = g(\ell) = 0\}. \end{aligned} \tag{2.9}$$

(iv) Fix $a \in \mathbb{R}\backslash\{0\}$, $b \in \mathbb{R}$, and $\phi \in [0, 2\pi)$. For each $\ell \in \mathbb{N}$, let H_ℓ denote the self-adjoint operator defined in $L^2((-\ell,\ell); dx)$ by

$$\begin{aligned} &(H_\ell f)(x) = -f''(x) + V(x) f(x) \ \text{ for a.e. } x \in (-\ell,\ell), \\ &f \in \operatorname{dom}(H_\ell) = \left\{ g \in \mathfrak{D}_\ell \,\middle|\, \begin{pmatrix} g(\ell) \\ g'(\ell) \end{pmatrix} = e^{i\phi} \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \begin{pmatrix} g(-\ell) \\ g'(-\ell) \end{pmatrix} \right\}, \quad \ell \in \mathbb{N}. \end{aligned} \tag{2.10}$$

Moreover, let

$$\lambda_\infty^{(0)} := -|b(a^{-1} + a)| \big[1 + 2|b(a^{-1} + a)|\big], \tag{2.11}$$

(v) Let H denote the self-adjoint operator defined in $L^2(\mathbb{R}; dx)$ by

$$\begin{aligned} &(Hf)(x) = -f''(x) + V(x) f(x) \ \text{ for a.e. } x \in \mathbb{R}, \\ &f \in \operatorname{dom}(H) = \big\{ g \in L^2(\mathbb{R}; dx) \,\big|\, g, g' \in AC([-R,R]) \text{ for all } R > 0, \\ &\qquad\qquad -g'' + Vg \in L^2(\mathbb{R}; dx) \big\}. \end{aligned} \tag{2.12}$$

Assuming Hypothesis 2.1, we introduce the following notation for the resolvent operators of $H_{\ell,D}$ and H_ℓ:

$$R_{\ell,D}(z) = \big(H_{\ell,D} - zI_{L^2((-\ell,\ell);dx)}\big)^{-1}, \quad z \in \rho(H_{\ell,D}), \tag{2.13}$$

$$R_\ell(z) = \big(H_\ell - zI_{L^2((-\ell,\ell);dx)}\big)^{-1}, \quad z \in \rho(H_\ell),\ \ell \in \mathbb{N}, \tag{2.14}$$

and for the full-line Schrödinger operator H:

$$R(z) = \big(H - zI_{L^2(\mathbb{R};dx)}\big)^{-1}, \quad z \in \rho(H). \tag{2.15}$$

In the special case when $V(x) = 0$ for a.e. $x \in \mathbb{R}$, we append the superscript (0), and write $H_{\ell,D}^{(0)}$, $H_\ell^{(0)}$, $H^{(0)}$, $R_{\ell,D}^{(0)}(\cdot)$, $R_\ell^{(0)}(\cdot)$, $R^{(0)}(\cdot)$ for the corresponding "free" Schrödinger operators and their resolvents.

In light of the assumption in (2.1), one immediately infers that

$$u, v \in L^2(\mathbb{R}; dx), \quad u_\ell, v_\ell \in L^2((-\ell,\ell); dx), \quad \ell \in \mathbb{N}. \tag{2.16}$$

Note that H_ℓ (resp., $H_{\ell,D}$) is the self-adjoint restriction of $H_{\ell,\max}$ with coupled (resp., Dirichlet) boundary conditions at the endpoints of $(-\ell,\ell)$. Alternatively, H_ℓ (resp., $H_{\ell,D}$) is the unique semibounded (from below) self-adjoint operator associated via the *KLMN* Theorem (cf., e.g., [22, Theorem 6.24 & Corollary 9.36]) with the closed, semibounded symmetric sesquilinear form $\mathfrak{q}_\ell$ (resp., $\mathfrak{q}_{\ell,D}$) defined by (cf., e.g., [6, (5.3)] and [20, §10.2])

$$\mathfrak{q}_\ell[f,g] = \int_{-\ell}^{\ell} \Big[\overline{f'(x)}g'(x) + V(x)\overline{f(x)}g(x)\Big]\,dx - b(a^{-1}+a)\overline{f(-\ell)}g(-\ell), \tag{2.17}$$

$$f, g \in \operatorname{dom}(\mathfrak{q}_\ell) = \big\{h \in H^1(-\ell,\ell) \,\big|\, h(\ell) = ae^{i\phi}h(-\ell)\big\}, \quad \ell \in \mathbb{N},$$

$$\Bigg(\text{resp., } \mathfrak{q}_{\ell,D}[f,g] = \int_{-\ell}^{\ell} \Big[\overline{f'(x)}g'(x) + V(x)\overline{f(x)}g(x)\Big]\,dx, \tag{2.18}$$

$$f, g \in \operatorname{dom}(\mathfrak{q}_{\ell,D}) = \big\{h \in H^1(-\ell,\ell) \,|\, h(-\ell) = h(\ell) = 0\big\}, \quad \ell \in \mathbb{N}\Bigg).$$

In particular, H_ℓ is the quadratic form sum of $H_\ell^{(0)}$ and the operator of multiplication by V_ℓ,

$$H_\ell = H_\ell^{(0)} +_q V_\ell, \quad \ell \in \mathbb{N}. \tag{2.19}$$

Similarly, H is the unique semibounded self-adjoint operator associated via the *KLMN* Theorem with the closed, semibounded symmetric sesquilinear form $\mathfrak{q}$ given by

$$\mathfrak{q}[f,g] = \int_{-\infty}^{\infty} \Big[\overline{f'(x)}g'(x) + \overline{f(x)}V(x)g(x)\Big]\,dx, \quad f, g \in \operatorname{dom}(\mathfrak{q}) = H^1(\mathbb{R}). \tag{2.20}$$

In fact, H is actually the quadratic form sum of $H^{(0)}$ and the operator of multiplication by V,

$$H = H^{(0)} +_q V. \tag{2.21}$$

A close look at the quadratic forms $\mathfrak{q}$, $\mathfrak{q}_\ell$, and $\mathfrak{q}_{\ell,D}$, combined with well-known estimates, reveals that H, H_ℓ, and $H_{\ell,D}$ are bounded from below *uniformly* in $\ell \in \mathbb{N}$.

Theorem 2.2. *Assume Hypothesis* 2.1. *If $\mathfrak{q}_\ell$, $\mathfrak{q}_{\ell,D}$, and $\mathfrak{q}$ are defined by* (2.17), (2.18)*, and* (2.20)*, respectively, then*

$$\mathfrak{q}_{\ell,D}[f,f] \geqslant -M(M+1)\|f\|^2_{L^2((-\ell,\ell);dx)}, \quad f \in \operatorname{dom}(\mathfrak{q}_{\ell,D}),\ \ell \in \mathbb{N}, \tag{2.22}$$

$$\mathfrak{q}[g,g] \geqslant -M(M+1)\|g\|^2_{L^2(\mathbb{R};dx)}, \quad g \in \operatorname{dom}(\mathfrak{q}), \tag{2.23}$$

and

$$\mathfrak{q}_\ell[h,h] \geqslant \big(-M(1+2M) + \lambda_\infty^{(0)}\big)\|h\|^2_{L^2((-\ell,\ell);dx)}, \quad h \in \operatorname{dom}(\mathfrak{q}_\ell),\ \ell \in \mathbb{N}. \tag{2.24}$$

In particular, H_ℓ and $H_{\ell,D}$ are uniformly bounded from below in $\ell \in \mathbb{N}$ by the constant

$$\lambda_\infty := -M(1+2M) + \lambda_\infty^{(0)}. \tag{2.25}$$

Proof. Fix $\ell \in \mathbb{N}$, and let us consider $\mathfrak{q}_{\ell,D}$ first. The result is obvious if $M = 0$, so we may assume without loss that $M > 0$. The explicit form of $\mathfrak{q}_{\ell,D}$ implies

$$\mathfrak{q}_{\ell,D}[f,f] \geqslant \|f'\|^2_{L^2((-\ell,\ell);dx)} - \int_{-\ell}^{\ell} |V(x)||f(x)|^2\,dx, \quad f \in \operatorname{dom}(\mathfrak{q}_{\ell,D}). \tag{2.26}$$

By [22, Lemma 9.32], one has

$$\begin{gathered}\sup_{x\in[n,n+1]} |f(x)|^2 \leqslant \varepsilon \int_n^{n+1} |f'(x')|^2\,dx' + \left(1 + \frac{1}{\varepsilon}\right)\int_n^{n+1} |f(x')|^2\,dx', \\ n \in \mathbb{Z},\ -\ell \leqslant n \leqslant \ell - 1,\ f \in H^1((-\ell,\ell)),\ \varepsilon > 0.\end{gathered} \tag{2.27}$$

Choosing $\varepsilon = M^{-1}$ and splitting the interval $[-\ell,\ell]$ into intervals of unit length, one obtains

$$\begin{aligned}
&\int_{-\ell}^{\ell} |V(x)||f(x)|^2\,dx \\
&\quad = \sum_{n=-\ell}^{\ell-1} \int_n^{n+1} |V(x)||f(x)|^2\,dx \\
&\quad \leqslant \sum_{n=-\ell}^{\ell-1} \left\{ M^{-1}\int_n^{n+1} |f'(x')|^2\,dx' + (1+M)\int_n^{n+1} |f(x')|^2\,dx' \right\} \int_n^{n+1} |V(x)|\,dx \\
&\leqslant M \sum_{n=-\ell}^{\ell-1} \left\{ M^{-1}\int_n^{n+1} |f'(x')|^2\,dx' + (1+M)\int_n^{n+1} |f(x')|^2\,dx' \right\} \\
&= \|f'\|^2_{L^2((-\ell,\ell);dx)} + M(M+1)\|f\|^2_{L^2((-\ell,\ell);dx)}, \quad f \in \operatorname{dom}(\mathfrak{q}_{\ell,D}).
\end{aligned} \tag{2.28}$$

Thus, upon combining (2.26) and (2.28),

$$\mathfrak{q}_{\ell,D}[f,f] \geqslant -M(M+1)\|f\|^2_{L^2((-\ell,\ell);dx)}, \quad f \in \operatorname{dom}(\mathfrak{q}_{\ell,D}). \tag{2.29}$$

Since $\ell \in \mathbb{N}$ was arbitrary, (2.22) follows. A similar argument yields (2.23). The operators $H_{\ell,D}$ are bounded from below by $-M(M+1)$ by [14, Theorem VI.2.6].

Next, we consider $\mathfrak{q}_\ell$ with $\ell \in \mathbb{N}$ fixed. If $b = 0$, then the same argument in (2.26)–(2.29) shows that $\mathfrak{q}_\ell[f,f] \geqslant -M(1+M)\|f\|^2_{L^2((-\ell,\ell);dx)}$, $f \in \operatorname{dom}(\mathfrak{q}_\ell)$, and (2.24) follows. Therefore, we may assume without loss that $b \neq 0$. We will also assume $M > 0$, the case $M = 0$ being slightly simpler. Then

$$\begin{aligned}\mathfrak{q}_\ell[f,f] \geqslant \|f'\|^2_{L^2((-\ell,\ell);dx)} - \int_{-\ell}^{\ell} |V(x)||f(x)|^2\,dx - |b(a^{-1}+a)||f(-\ell)|^2,& \\ f \in \operatorname{dom}(\mathfrak{q}_\ell).&\end{aligned} \tag{2.30}$$

Proceeding similar to (2.28), but choosing $\varepsilon = (2M)^{-1}$ in (2.27) instead, we obtain

$$\int_{-\ell}^{\ell} |V(x)||f(x)|^2\,dx \leqslant \frac{1}{2}\|f'\|^2_{L^2((-\ell,\ell);dx)} + M(1+2M)\|f\|^2_{L^2((-\ell,\ell);dx)}, \quad f \in \operatorname{dom}(\mathfrak{q}_\ell). \tag{2.31}$$

On the other hand, choosing $\varepsilon = (2|b(a^{-1}+a)|)^{-1}$ in (2.27) yields

$$-|b(a^{-1}+a)||f(-\ell)|^2 \geqslant -\frac{1}{2}\|f'\|^2_{L^2((-\ell,\ell);dx)} + \lambda_\infty^{(0)}\|f\|^2_{L^2((-\ell,\ell);dx)}, \quad f \in \operatorname{dom}(\mathfrak{q}_\ell). \tag{2.32}$$

Finally, combining (2.30), (2.31), (2.32), one obtains

$$\mathfrak{q}_\ell[f,f] \geqslant \lambda_\infty \|f\|^2_{L^2((-\ell,\ell);dx)}, \quad f \in \operatorname{dom}(\mathfrak{q}_\ell). \tag{2.33}$$

□

Remark 2.3. Of course, for each $\ell \in \mathbb{N}$, the operators $H_{\ell,D}^{(0)}$ and $H_\ell^{(0)}$ are bounded below (uniformly in $\ell \in \mathbb{N}$) by $\lambda_\infty^{(0)}$.

Using a Krein-type resolvent identity, for each fixed $\ell \in \mathbb{N}$, one may relate $R_\ell(z)$ to $R_{\ell,D}(z)$ for $z \in \rho(H_\ell) \cap \rho(H_{\ell,D})$. The precise form of these resolvent identities was worked out in detail in [6] for all self-adjoint restrictions of the maximal operator associated to τ, that is, for all separated and non-separated (i.e., coupled) self-adjoint boundary conditions. In the case of a general self-adjoint restriction of $H_{\ell,\max}$ with the coupled boundary conditions in (1.8), the resolvent difference $R_\ell(z) - R_{\ell,D}(z)$ is at most rank two. However, when $R_{1,2} = 0$, which is precisely the case for H_ℓ, the difference is actually rank one, owing to the fact that H_ℓ and $H_{\ell,D}$ are not relatively prime (see [1, §84]) with respect to the underlying minimal symmetric operator. For completeness, we recall the result for the special case of the coupled boundary conditions considered in this paper.

In order to state Krein's identity, one introduces for each $z \in \rho(H_{\ell,D})$ a distinguished basis for $\ker(H_{\ell,\max} - zI_{L^2((-\ell,\ell);dx)})$, denoted by $\{\psi_{j,\ell}(z,\cdot\,)\}_{j=1,2}$, by specifying the boundary values

$$\begin{aligned} &\psi_{1,\ell}(z,-\ell) = 0, \quad \psi_{1,\ell}(z,\ell) = 1, \\ &\psi_{2,\ell}(z,-\ell) = 1, \quad \psi_{2,\ell}(z,\ell) = 0, \end{aligned} \tag{2.34}$$

in addition to the requirement

$$H_{\ell,\max}\psi_{j,\ell}(z,\cdot\,) = z\psi_{j,\ell}(z,\cdot\,), \quad j \in \{1,2\}. \tag{2.35}$$

Of course, (2.35) implies that $\psi_{j,\ell}(z,\cdot\,) \in \operatorname{dom}(H_{\ell,\max})$ satisfies the ordinary differential equation

$$-\psi''_{j,\ell}(z,x) + V(x)\psi_{j,\ell}(z,x) = z\psi_{j,\ell}(z,x), \quad x \in (-\ell,\ell),\ j \in \{1,2\}, \quad z \in \rho(H_{\ell,D}),\ \ell \in \mathbb{N}. \tag{2.36}$$

In the special case when $V(x) = 0$ for a.e. $x \in \mathbb{R}$, we will follow our previously adopted convention (of appending the superscript "(0)") and denote the maximal operator by $H_{\ell,\max}^{(0)}$ and the distinguished basis for $\ker\big(H_{\ell,\max}^{(0)} - zI_{L^2((-\ell,\ell);dx)}\big)$ by

$\{\psi_{j,\ell}^{(0)}(z,\,\cdot\,)\}_{j=1,2}$. Actually, $\{\psi_{j,\ell}^{(0)}(z,\,\cdot\,)\}_{j=1,2}$ may be computed explicitly, and one finds for each $\ell\in\mathbb{N}$,

$$\begin{aligned}\psi_{1,\ell}^{(0)}(z,x) &= \frac{1}{2}\bigg[\frac{\cos(z^{1/2}x)}{\cos(z^{1/2}\ell)}+\frac{\sin(z^{1/2}x)}{\sin(z^{1/2}\ell)}\bigg], \\ \psi_{2,\ell}^{(0)}(z,x) &= \frac{1}{2}\bigg[\frac{\cos(z^{1/2}x)}{\cos(z^{1/2}\ell)}-\frac{\sin(z^{1/2}x)}{\sin(z^{1/2}\ell)}\bigg], \quad x\in[-\ell,\ell],\ \operatorname{Im}(z^{1/2})\geqslant 0,\ z\in\rho(H_{\ell,D}).\end{aligned} \tag{2.37}$$

Returning to the case of general V, and with the basis $\{\psi_{j,\ell}(z,\,\cdot\,)\}_{j=1,2}$ in hand, we now recall Krein's resolvent identity for $R_\ell(z)$ to $R_{\ell,D}(z)$.

Lemma 2.4 (Krein's Resolvent Formula, [6])**.** *Assume items (i), (iii), and (iv) in Hypothesis 2.1. If $\{\psi_{j,\ell}(z,\,\cdot\,)\}_{j=1,2}$ denotes the basis of $\ker(H_{\ell,\max}-zI_{L^2((-\ell,\ell);dx)})$ which satisfies (2.34) for $z\in\rho(H_{\ell,D})$, then*

$$\begin{aligned}q_\ell(z) &:= a^{-1}b + a^{-2}\psi_{2,\ell}'(z,-\ell) + e^{i\phi}a^{-1}\psi_{1,\ell}'(z,-\ell) \\ &\quad - e^{-i\phi}a^{-1}\psi_{2,\ell}'(z,\ell) - \psi_{1,\ell}'(z,\ell), \quad z\in\rho(H_\ell)\cap\rho(H_{\ell,D}),\ \ell\in\mathbb{N},\end{aligned} \tag{2.38}$$

is nonzero and

$$R_\ell(z) = R_{\ell,D}(z) + P_\ell(z), \quad z\in\rho(H_\ell)\cap\rho(H_{\ell,D}),\ \ell\in\mathbb{N}, \tag{2.39}$$

where $P_\ell(z)\in\mathcal{B}(L^2((-\ell,\ell);dx))$ is the rank one operator defined by

$$P_\ell(z) := -q_\ell(z)^{-1}(\psi_\ell(\overline{z},\,\cdot\,),\,\cdot\,)_{L^2((-\ell,\ell);dx)}\psi_\ell(z,\,\cdot\,), \quad z\in\rho(H_\ell)\cap\rho(H_{\ell,D}),\ \ell\in\mathbb{N}, \tag{2.40}$$

with

$$\psi_\ell(z,x) := e^{-i\phi}a^{-1}\psi_{2,\ell}(z,x) + \psi_{1,\ell}(z,x), \quad x\in[-\ell,\ell],\ z\in\rho(H_{\ell,D}),\ \ell\in\mathbb{N}. \tag{2.41}$$

Of course, in the free case, $V_\ell(x)\equiv 0$, the terms in the Krein formula (2.38)–(2.41) may be computed explicitly.

Example 2.5. *In the special case when $V(x)=0$ for a.e. $x\in(-\ell,\ell)$, the terms in* (2.40), (2.38), *and* (2.41) *may be computed explicitly, and one obtains*

$$R_\ell^{(0)}(z) = R_{\ell,D}^{(0)}(z) + P_\ell^{(0)}(z), \quad z\in\rho(H_\ell^{(0)})\cap\rho(H_{\ell,D}^{(0)}),\ \ell\in\mathbb{N}, \tag{2.42}$$

where

$$\begin{aligned}P_\ell^{(0)}(z) &:= -q_\ell^{(0)}(z)^{-1}\big(\psi_\ell^{(0)}(\overline{z},\,\cdot\,),\,\cdot\,\big)_{L^2((-\ell,\ell);dx)}\psi_\ell^{(0)}(z,\,\cdot\,), \\ &\qquad z\in\rho(H_\ell^{(0)})\cap\rho(H_{\ell,D}^{(0)}),\ \operatorname{Im}(z^{1/2})\geqslant 0,\ \ell\in\mathbb{N},\end{aligned} \tag{2.43}$$

with

$$\begin{aligned}q_\ell^{(0)}(z) &:= \frac{b}{a} + \frac{1+a^{-2}}{2}z^{1/2}[\tan(z^{1/2}\ell)-\cot(z^{1/2}\ell)] \\ &\quad + \frac{\cos(\phi)}{a}z^{1/2}[\cot(z^{1/2}\ell)+\tan(z^{1/2}\ell)], \quad z\in\rho(H_\ell^{(0)})\cap\rho(H_{\ell,D}^{(0)}), \\ &\qquad \operatorname{Im}(z^{1/2})\geqslant 0,\ \ell\in\mathbb{N},\end{aligned} \tag{2.44}$$

and

$$\begin{aligned}\psi_\ell^{(0)}(z,x) &:= c_+\frac{\cos(z^{1/2}x)}{\cos(z^{1/2}\ell)} + c_-\frac{\sin(z^{1/2}x)}{\sin(z^{1/2}\ell)}, \quad x\in[-\ell,\ell], \\ &\qquad z\in\rho(H_\ell^{(0)})\cap\rho(H_{\ell,D}^{(0)}),\ \operatorname{Im}(z^{1/2})\geqslant 0,\ \ell\in\mathbb{N},\end{aligned} \tag{2.45}$$

with

$$c_\pm := \frac{1 \pm e^{-i\phi} a^{-1}}{2}. \tag{2.46}$$

The assumptions on V in Hypothesis 2.1(i) (in particular, the fact that the conditions in (2.16) hold) imply that $R^{(0)}_{\ell,D}(z)v_\ell$ and $u_\ell R^{(0)}_{\ell,D} v_\ell$, defined initially only on the dense subspace $\operatorname{dom}(v_\ell)$, extend by continuity to bounded operators on all of $L^2((-\ell,\ell);dx)$ (i.e., their closures belong to $\mathcal{B}(L^2((-\ell,\ell);dx))$). In fact, one has the following Hilbert–Schmidt and trace class containments (cf., e.g. [8, (2.69), (3.12), and (3.14)]):

$$\begin{aligned} &u_\ell R^{(0)}_{\ell,D}(z),\ \overline{R^{(0)}_{\ell,D}(z)v_\ell} \in \mathcal{B}_2(L^2((-\ell,\ell);dx)), \\ &\qquad \overline{u_\ell R^{(0)}_{\ell,D}(z)v_\ell} \in \mathcal{B}_1(L^2((-\ell,\ell);dx)), \quad z \in \mathbb{C}\backslash\sigma(H^{(0)}_{\ell,D}),\ \ell \in \mathbb{N}, \end{aligned} \tag{2.47}$$

and there exist ℓ-independent constants $E_D < 0$ and $C_D > 0$ such that

$$\Big\| \overline{u_\ell R^{(0)}_{\ell,D}(z)v_\ell} \Big\|_{\mathcal{B}_1(L^2((-\ell,\ell);dx))} \leqslant C_D |z|^{-1/2}, \quad z \in (-\infty, E_D),\ \ell \in \mathbb{N}. \tag{2.48}$$

Analogous statements hold true for $R^{(0)}(z)v$ and $uR^{(0)}(z)v$ with

$$\begin{aligned} &uR^{(0)}(z),\ \overline{R^{(0)}(z)v} \in \mathcal{B}_2(L^2(\mathbb{R};dx)), \\ &\qquad \overline{uR^{(0)}(z)v} \in \mathcal{B}_1(L^2(\mathbb{R};dx)), \quad z \in \mathbb{C}\backslash[0,\infty), \end{aligned} \tag{2.49}$$

and

$$\Big\| \overline{uR^{(0)}(z)v} \Big\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} \leqslant C_\infty |z|^{-1/2}, \quad z \in (-\infty, E_\infty), \tag{2.50}$$

for suitable constants $E_\infty < 0$ and $C_\infty > 0$. An application of (2.42), combined with (2.47), yields immediate analogues of (2.47) and (2.48) for $H^{(0)}_\ell$.

Lemma 2.6. *Assume items* (i), (iii) *and* (iv) *in Hypothesis* 2.1 *hold. If* $z \in \rho(H^{(0)}_\ell)$, *then* $R^{(0)}_\ell(z)v_\ell$ *and* $u_\ell R^{(0)}_\ell(z)v_\ell$ *defined on* $\operatorname{dom}(v_\ell)$ *extend by continuity to bounded linear operators on* $L^2((-\ell,\ell);dx)$. *Moreover,*

$$u_\ell R^{(0)}_\ell(z),\ \overline{R^{(0)}_\ell(z)v_\ell} \in \mathcal{B}_2(L^2((-\ell,\ell);dx)), \tag{2.51}$$

$$\overline{u_\ell R^{(0)}_\ell(z)v_\ell} \in \mathcal{B}_1(L^2((-\ell,\ell);dx)), \quad z \in \rho(H^{(0)}_\ell),\ \ell \in \mathbb{N}, \tag{2.52}$$

and

$$\Big\| \overline{u_\ell R^{(0)}_\ell(z)v_\ell} \Big\|_{\mathcal{B}_1(L^2((-\ell,\ell);dx))} \leqslant C|z|^{-1/2}, \quad z \in (-\infty, E),\ \ell \in \mathbb{N}, \tag{2.53}$$

for suitable ℓ*-independent constants* $E < 0$ *and* $C > 0$.

Proof. Let $\ell \in \mathbb{N}$ be fixed for the remainder of this proof. Suppose $z \in \rho(H^{(0)}_\ell)$. Since $u_\ell \in L^2((-\ell,\ell);dx)$ and $R^{(0)}_\ell(z)$ is an integral operator with an integral kernel which is continuous (hence, bounded) on $[-\ell,\ell]\times[-\ell,\ell]$, the inclusion $u_\ell R^{(0)}_\ell(z) \in \mathcal{B}_2(L^2((-\ell,\ell);dx))$ holds by [23, Theorem 6.11]. Then

$$\overline{R^{(0)}_\ell(z)v_\ell} = \big[R^{(0)}_\ell(z)v_\ell\big]^{**} = \big[v_\ell R^{(0)}_\ell(\overline{z})\big]^* \in \mathcal{B}_2(L^2((-\ell,\ell);dx)), \tag{2.54}$$

since $v_\ell \in L^2((-\ell,\ell);dx)$ and $R^{(0)}_\ell(\overline{z})$ has a continuous (hence, bounded) integral kernel on $[-\ell,\ell]\times[-\ell,\ell]$.

It suffices to prove (2.52) for one $z \in \rho(H_\ell^{(0)})$ which we take to be $z = -k^2$ for some fixed $k > |\lambda_\infty|^{1/2}$ which guarantees $-k^2 < \lambda_\infty$. The desired containment then extends to all $z \in \rho(H_\ell^{(0)})$ by the first resolvent identity combined with (2.51). By Krein's resolvent formula (2.42),

$$\begin{aligned} \overline{u_\ell R_\ell^{(0)}(-k^2)v_\ell} &= \overline{u_\ell\big[R_{\ell,D}^{(0)}(-k^2) + P_\ell^{(0)}(-k^2)\big]v_\ell} \\ &= \overline{u_\ell R_{\ell,D}^{(0)}(-k^2)v_\ell} + \overline{u_\ell P_\ell^{(0)}(-k^2)v_\ell}. \end{aligned} \tag{2.55}$$

The splitting of the closure in (2.55) is justified by the fact that $u_\ell R_{\ell,D}^{(0)}(-k^2)v_\ell$ and $u_\ell P_\ell^{(0)}(-k^2)v_\ell$ are both bounded on the dense subspace $\operatorname{dom}(v_\ell)$, so the closures appearing on the right-hand sides in (2.55) are simply the continuous extensions of the underlying densely defined operators to all of $L^2((-\ell,\ell);dx)$. Moreover, for $f \in \operatorname{dom}(v_\ell)$, one computes

$$\begin{aligned} &u_\ell P_\ell^{(0)}(-k^2)v_\ell f \\ &\quad = -q_\ell^{(0)}(-k^2)^{-1}\big(v_\ell\psi_\ell^{(0)}(-k^2,\,\cdot\,),f\big)_{L^2((-\ell,\ell);dx)}\big[u_\ell\psi_\ell^{(0)}(-k^2,\,\cdot\,)\big] \\ &\quad = -q_\ell^{(0)}(-k^2)^{-1}\big(\Psi(-k^2,\,\cdot\,),f\big)_{L^2((-\ell,\ell);dx)}\Phi(-k^2,\,\cdot\,), \end{aligned} \tag{2.56}$$

where

$$\Psi(-k^2,x) = v_\ell(x)\psi_\ell^{(0)}(-k^2,x) \text{ and } \Phi(-k^2,x) = u_\ell(x)\psi_\ell^{(0)}(-k^2,x) \quad \text{for a.e. } x \in (-\ell,\ell). \tag{2.57}$$

Thus, $u_\ell P_\ell^{(0)}(-k^2)v_\ell$ is the restriction of the bounded rank one operator

$$-\,q_\ell^{(0)}(-k^2)^{-1}\big(\Psi_\ell(-k^2,\,\cdot\,),\,\cdot\,\big)_{L^2((-\ell,\ell);dx)}\Phi_\ell(-k^2,\,\cdot\,) \in \mathcal{B}_1(L^2((-\ell,\ell);dx)) \tag{2.58}$$

to the dense subspace $\operatorname{dom}(v_\ell)$. Therefore, by continuity,

$$\overline{u_\ell P_\ell^{(0)}(-k^2)v_\ell} = -q_\ell^{(0)}(-k^2)^{-1}\big(\Psi_\ell(-k^2,\,\cdot\,),\,\cdot\,\big)_{L^2((-\ell,\ell);dx)}\Phi_\ell(-k^2,\,\cdot\,). \tag{2.59}$$

Applying (2.55) with (2.47), (2.58), and (2.59), one arrives at

$$\overline{u_\ell R_\ell^{(0)}(-k^2)v_\ell} \in \mathcal{B}_1(L^2((-\ell,\ell);dx)), \tag{2.60}$$

and (2.52) follows.

In order to prove (2.53), let $z = -k^2$ with $k > |\lambda_\infty|^{1/2}$. Applying the simple estimate

$$|\Psi(-k^2,x)|^2 = |\Phi(-k^2,x)|^2 \leqslant (|c_+| + |c_-|)^2|V_\ell(x)| \quad \text{for a.e. } x \in (-\ell,\ell) \text{ and all } k > |\lambda_\infty|^{1/2}, \tag{2.61}$$

in conjunction with (2.59) and Proposition A.1, one infers

$$\begin{aligned} &\Big\|\overline{u_\ell P_\ell^{(0)}(-k^2)v_\ell}\Big\|_{\mathcal{B}_1(L^2((-\ell,\ell);dx))} \\ &\quad = |q_\ell^{(0)}(-k^2)|^{-1}\|\Psi(-k^2,\,\cdot\,)\|_{L^2((-\ell,\ell);dx)}\|\Phi(-k^2,\,\cdot\,)\|_{L^2((-\ell,\ell);dx)} \\ &\quad \leqslant (|c_+| + |c_-|)^2 M|q_\ell^{(0)}(-k^2)|^{-1}, \quad k > |\lambda_\infty|^{1/2}. \end{aligned} \tag{2.62}$$

On the other hand,

$$q_\ell^{(0)}(-k^2) = k\bigg\{\frac{a^{-1}b}{k} - \frac{1+a^{-2}}{2}[\tanh(k\ell) + \coth(k\ell)]$$

$$+\frac{\cos(\phi)}{2}[\coth(k\ell)-\tanh(k\ell)]\bigg\}, \quad k>|\lambda_\infty|^{1/2}, \tag{2.63}$$

and since the expression in braces converges to $-(1+a^{-2})$ and $k\to\infty$, and the convergence is uniform with respect to $\ell\in\mathbb{N}$, there exist ℓ-independent constants $C_0>0$ and $k_0>|\lambda_\infty|^{1/2}$ such that

$$|q_\ell^{(0)}(-k^2)|^{-1}\leqslant C_0k^{-1}, \quad k>k_0,\, \ell\in\mathbb{N}. \tag{2.64}$$

Thus, (2.55), (2.62), and (2.64) imply

$$\begin{aligned}\Big\|\overline{u_\ell R_\ell^{(0)}(-k^2)v_\ell}\Big\|_{\mathcal{B}_1(L^2((-\ell,\ell);dx))}\leqslant C_Dk^{-1}+(|c_+|+|c_-|)^2C_0Mk^{-1},&\\ k>\max\big\{|E_D|^{1/2},k_0\big\}.&\end{aligned} \tag{2.65}$$

Therefore, (2.53) follows with $E=\min\{E_D,-k_0^2\}$ and $C=C_D+(|c_+|+|c_-|)^2C_0M$. □

3. Convergence Properties of Resolvents

In this section, we study various convergence properties of the resolvents of the periodic restrictions $H_\ell^{(0)}$ and H_ℓ introduced in the previous section. Ultimately, the results of this section will combine to yield vague convergence for spectral shift functions in the limit, $\ell\to\infty$. In order to recall the corresponding known results for the Dirichlet restrictions $H_{\ell,D}^{(0)}$ and $H_{\ell,D}$, and state our new results for $H_\ell^{(0)}$ and H_ℓ, we first introduce the following ℓ-dependent direct sum decomposition of $L^2(\mathbb{R};dx)$.

Let $\ell\in\mathbb{N}$. If $f\in L^2((-\ell,\ell);dx)$ and $g\in L^2(\mathbb{R}\backslash(-\ell,\ell);dx)$, then we define $(f\oplus_\ell g)\in L^2(\mathbb{R};dx)$ by

$$(f\oplus_\ell g)(x)=\begin{cases}f(x), & \text{for a.e. } x\in(-\ell,\ell),\\ g(x), & \text{for a.e. } x\in\mathbb{R}\backslash(-\ell,\ell).\end{cases} \tag{3.1}$$

From the properties of the Lebesgue integral, it is clear that $(f\oplus_\ell g)\in L^2(\mathbb{R};dx)$ and that

$$\|f\oplus_\ell g\|^2_{L^2(\mathbb{R};dx)}=\|f\|^2_{L^2((-\ell,\ell);dx)}+\|g\|^2_{L^2(\mathbb{R}\backslash(-\ell,\ell);dx)}, \tag{3.2}$$

for all $f\in L^2((-\ell,\ell);dx)$ and all $g\in L^2(\mathbb{R}\backslash(-\ell,\ell);dx)$. In addition, every function $u\in L^2(\mathbb{R};dx)$ may be expressed in the form (3.1):

$$u=f\oplus_\ell g \quad \text{with} \quad f=u|_{(-\ell,\ell)} \quad \text{and} \quad g=u|_{\mathbb{R}\backslash(-\ell,\ell)}. \tag{3.3}$$

Thus, for each $\ell\in\mathbb{N}$, $L^2(\mathbb{R};dx)$ may be expressed as a *direct sum* of $L^2((-\ell,\ell);dx)$ and $L^2(\mathbb{R}\backslash(-\ell,\ell);dx)$:

$$L^2(\mathbb{R};dx)=L^2((-\ell,\ell);dx)\oplus_\ell L^2(\mathbb{R}\backslash(-\ell,\ell);dx). \tag{3.4}$$

By the additivity property of the Lebesgue integral, the inner product of two functions in $L^2(\mathbb{R};dx)$ may be expressed as a sum of individual inner products of their corresponding components: if $u,v\in L^2(\mathbb{R};dx)$ are given by

$$u=f_1\oplus_\ell g_1 \quad \text{and} \quad v=f_2\oplus_\ell g_2, \tag{3.5}$$

for some $f_j\in L^2((-\ell,\ell);dx)$ and $g_j\in L^2(\mathbb{R}\backslash(-\ell,\ell);dx)$, $j\in\{1,2\}$, then evidently

$$(u,v)_{L^2(\mathbb{R};dx)}=(f_1,f_2)_{L^2((-\ell,\ell);dx)}+(g_1,g_2)_{L^2(\mathbb{R}\backslash(-\ell,\ell);dx)}. \tag{3.6}$$

In the sequel, we shall make use of the fact that a function $u \in L^2(\mathbb{R}; dx)$ may be decomposed according to (3.3). Since the decomposition obviously depends on the value of $\ell \in \mathbb{N}$, and we intend to study various limiting phenomena as $\ell \to \infty$, we insist on the notation "$\oplus_\ell$" to bring out the explicit ℓ-dependence of the decomposition in (3.3).

If

$$A : \operatorname{dom}(A) \subseteq L^2((-\ell,\ell); dx) \to L^2((-\ell,\ell); dx) \tag{3.7}$$

and

$$B : \operatorname{dom}(B) \subseteq L^2(\mathbb{R}\backslash(-\ell,\ell); dx) \to L^2(\mathbb{R}\backslash(-\ell,\ell); dx) \tag{3.8}$$

are linear operators, then their direct sum $A \oplus_\ell B$ is defined in $L^2(\mathbb{R}; dx)$ according to the direct sum decomposition in (3.4) in the standard way by setting

$$(A \oplus_\ell B)f = (Af_1) \oplus_\ell (Bf_2), \quad f = f_1 \oplus_\ell f_2 \in \operatorname{dom}(A \oplus_\ell B) = \operatorname{dom}(A) \oplus_\ell \operatorname{dom}(B). \tag{3.9}$$

Next, we recall two important convergence results for the free Dirichlet restrictions $H_{\ell,D}^{(0)}$ that are proved in [8]. Actually, these results are proved in [8] for intervals of the form $(0,\ell)$, but as noted in [8], the situation for intervals of the form $(-\ell,\ell)$ is completely analogous. The first result is that $H_{\ell,D}^{(0)} \oplus_\ell 0$ converges to $H^{(0)}$ in the strong resolvent sense as $\ell \to \infty$.

Lemma 3.1 (Lemma 3.1 in [8]). *Assume items (i)–(iii) in Hypothesis* 2.1 *hold. Then the sequence* $\{H_{\ell,D}^{(0)} \oplus_\ell 0\}_{\ell=1}^\infty$ *converges to* $H^{(0)}$ *in the strong resolvent sense. That is, for each fixed* $z \in \mathbb{C}\backslash[0,\infty)$,

$$\operatorname*{s-lim}_{\ell\to\infty} \Big(\Big[H_{\ell,D}^{(0)} \oplus_\ell 0\Big] - zI_{L^2(\mathbb{R};dx)}\Big)^{-1} = R^{(0)}(z). \tag{3.10}$$

Lemma 3.2 (Lemmata 3.1 and 3.2 in [8]). *Assume items (i)–(iii) in Hypothesis* 2.1 *hold. For each fixed* $z \in \mathbb{C}\backslash[0,\infty)$, *the following convergence results hold in* $\mathcal{B}_2(L^2(\mathbb{R}; dx))$:

$$\lim_{\ell\to\infty} \Big\| \Big[u_\ell R_{\ell,D}^{(0)}(z) \oplus_\ell 0\Big] - uR^{(0)}(z) \Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} = 0, \tag{3.11}$$

$$\lim_{\ell\to\infty} \Big\| \Big[\overline{R_{\ell,D}^{(0)}(z)v_\ell} \oplus_\ell 0\Big] - \overline{R^{(0)}(z)v} \Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} = 0, \tag{3.12}$$

and the following convergence result holds in $\mathcal{B}_1(L^2(\mathbb{R}; dx))$:

$$\lim_{\ell\to\infty} \Big\| \Big[\overline{u_\ell R_{\ell,D}^{(0)}(z)v_\ell} \oplus_\ell 0\Big] - \overline{uR^{(0)}(z)v} \Big\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} = 0. \tag{3.13}$$

By applying Krein's resolvent identity (2.42), we obtain the following extension of Lemma 3.1 to the case of the coupled boundary conditions at the endpoints of $[-\ell,\ell]$ in (2.10).

Lemma 3.3. *Assume Hypothesis* 2.1. *The sequence* $\{H_\ell^{(0)} \oplus_\ell 0\}_{\ell=1}^\infty$ *converges to* $H^{(0)}$ *in the strong resolvent sense. That is, for each fixed* $z \in \mathbb{C}\backslash[\lambda_\infty^{(0)},\infty)$,

$$\operatorname*{s-lim}_{\ell\to\infty} \Big(\Big[H_\ell^{(0)} \oplus_\ell 0\Big] - zI_{L^2(\mathbb{R};dx)}\Big)^{-1} = R^{(0)}(z). \tag{3.14}$$

Proof. We begin by introducing some notation. For each $f \in L^2(\mathbb{R}; dx)$ and $\ell \in \mathbb{N}$, we define the function $f_{<\ell} \in L^2((-\ell,\ell); dx)$ by the requirement that

$$f_{<\ell}(x) = f(x) \ \text{ for a.e. } x \in (-\ell,\ell). \tag{3.15}$$

It suffices to prove (3.14) for just one $z \in \mathbb{C}\backslash[\lambda_\infty^{(0)},\infty)$, which we take to be $z=-k^2$ with $k>k_0$ fixed, where k_0 is the constant from (2.64). The result then follows for arbitrary $z \in \mathbb{C}\backslash[\lambda_\infty^{(0)},\infty)$ by the application of a standard resolvent identity (cf., e.g., [23, Exercise 7.8]). Therefore, we will show

$$\lim_{\ell\to 0}\Big\|\Big(\Big[H_\ell^{(0)}\oplus_\ell 0\Big]+k^2 I_{L^2(\mathbb{R};dx)}\Big)^{-1}f-R^{(0)}(-k^2)f\Big\|_{L^2(\mathbb{R};dx)}=0, \quad f\in L^2(\mathbb{R};dx). \tag{3.16}$$

For each $f\in L^2(\mathbb{R};dx)$, we use various properties of the direct sum and (2.42) to compute

$$\begin{aligned}
&\Big\|\Big(\Big[H_\ell^{(0)}\oplus_\ell 0\Big]+k^2 I_{L^2(\mathbb{R};dx)}\Big)^{-1}f-R^{(0)}(-k^2)f\Big\|_{L^2(\mathbb{R};dx)} \quad (3.17)\\
&\quad=\Big\|\Big[\Big(H_\ell^{(0)}+k^2 I_{L^2((-\ell,\ell);dx)}\Big)^{-1}\oplus_\ell -k^{-2}I_{L^2(\mathbb{R}\backslash(-\ell,\ell);dx)}\Big]f\\
&\qquad -R^{(0)}(-k^2)f\Big\|_{L^2(\mathbb{R};dx)}\\
&\quad=\Big\|\Big\{\Big[\Big(H_{\ell,D}^{(0)}+k^2 I_{L^2((-\ell,\ell);dx)}\Big)^{-1}+P_\ell^{(0)}(-k^2)\Big]\oplus_\ell -k^{-2}I_{L^2(\mathbb{R}\backslash(-\ell,\ell);dx)}\Big\}f\\
&\qquad -R^{(0)}(-k^2)f\Big\|_{L^2(\mathbb{R};dx)}\\
&\quad\leqslant\Big\|\Big(\Big[H_{\ell,D}^{(0)}\oplus_\ell 0\Big]+k^2 I_{L^2(\mathbb{R};dx)}\Big)^{-1}f-R^{(0)}(-k^2)f\Big\|_{L^2(\mathbb{R};dx)}\\
&\qquad+\Big\|\Big[P_\ell^{(0)}(-k^2)\oplus_\ell 0\Big]f\Big\|_{L^2(\mathbb{R};dx)},\quad \ell\in\mathbb{N}.
\end{aligned}$$

By Lemma 3.1,

$$\lim_{\ell\to 0}\Big\|\Big(\Big[H_{\ell,D}^{(0)}\oplus_\ell 0\Big]+k^2 I_{L^2(\mathbb{R};dx)}\Big)^{-1}f-R^{(0)}(-k^2)f\Big\|_{L^2(\mathbb{R};dx)}=0, \tag{3.18}$$

so in view of the inequality in (3.17), the claim in (3.16) reduces to showing that

$$\lim_{\ell\to\infty}\Big\|\Big[P_\ell^{(0)}(-k^2)\oplus_\ell 0\Big]f\Big\|_{L^2(\mathbb{R};dx)}=0,\quad f\in L^2(\mathbb{R};dx), \tag{3.19}$$

that is, that the sequence of operators $\big\{P_\ell^{(0)}(-k^2)\oplus_\ell 0\big\}_{\ell=1}^\infty$ converges strongly to the zero operator in $L^2(\mathbb{R};dx)$. To this end, we claim that the sequence of operators $\big\{P_\ell^{(0)}(-k^2)\oplus_\ell 0\big\}_{\ell=1}^\infty$ is uniformly bounded in $\mathcal{B}(L^2(\mathbb{R};dx))$. Indeed, by (2.43), and (2.45), with C_0 the constant from (2.64),

$$\begin{aligned}
&\Big\|P_\ell^{(0)}(-k^2)\oplus_\ell 0\Big\|_{\mathcal{B}(L^2(\mathbb{R};dx))} \quad (3.20)\\
&\quad=\Big\|P_\ell^{(0)}(-k^2)\Big\|_{\mathcal{B}(L^2((-\ell,\ell);dx))}\\
&\quad=\big|q_\ell^{(0)}(-k^2)^{-1}\big|\big\|\psi_\ell^{(0)}(-k^2,\,\cdot\,)\big\|^2_{L^2((-\ell,\ell);dx)}\\
&\quad\leqslant 2C_0k^{-1}\bigg[|c_+|^2\int_{-\ell}^{\ell}\frac{\cosh^2(kx)}{\cosh^2(k\ell)}\,dx+|c_-|^2\int_{-\ell}^{\ell}\frac{\sinh^2(kx)}{\sinh^2(k\ell)}\,dx\bigg]\\
&\quad=2C_0k^{-1}\big\{|c_+|^2\big[k^{-1}\tanh(k\ell)+\ell\,\mathrm{sech}^2(k\ell)\big]\\
&\qquad+|c_-|^2\big[k^{-1}\coth(k\ell)-\ell\,\mathrm{csch}^2(k\ell)\big]\big\},\quad \ell\in\mathbb{N},
\end{aligned}$$

which is clearly bounded by a constant, say $C_1 > 0$, uniformly in $\ell \in \mathbb{N}$. Therefore, by [23, Exercise 4.28], it suffices to prove the convergence in (3.19) for all f from a dense subspace of $L^2(\mathbb{R}; dx)$, which we take to be $L^2(\mathbb{R}; dx) \cap L^1(\mathbb{R}; dx)$. Let $f \in L^2(\mathbb{R}; dx) \cap L^1(\mathbb{R}; dx)$ and note that

$$\begin{aligned}
&\Big\| \Big[P_\ell^{(0)}(-k^2) \oplus_\ell 0\Big] f \Big\|^2_{L^2(\mathbb{R};dx)} \qquad (3.21)\\
&\quad = \Big\| P_\ell^{(0)}(-k^2) f_{<\ell} \Big\|^2_{L^2((-\ell,\ell);dx)}\\
&\quad \leqslant C_0^2 k^{-2} \Big| \big(\psi_\ell^{(0)}(-k^2,\,\cdot\,), f_{<\ell}\big)_{L^2((-\ell,\ell);dx)} \Big|^2 \big\| \psi_\ell^{(0)}(-k^2,\,\cdot\,) \big\|^2_{L^2((-\ell,\ell);dx)}\\
&\quad \leqslant 2C_0^2 k^{-2} \big\{ |c_+|^2 \big[k^{-1}\tanh(k\ell) + \ell \operatorname{sech}^2(k\ell)\big]\\
&\qquad + |c_-|^2 \big[k^{-1}\coth(k\ell) - \ell \operatorname{csch}^2(k\ell)\big] \big\} \Big| \big(\psi_\ell^{(0)}(-k^2,\,\cdot\,), f_{<\ell}\big)_{L^2((-\ell,\ell);dx)} \Big|^2\\
&\quad \leqslant C_0 C_1 k^{-1} \Big| \big(\psi_\ell^{(0)}(-k^2,\,\cdot\,), f_{<\ell}\big)_{L^2((-\ell,\ell);dx)} \Big|^2, \quad \ell \in \mathbb{N}.
\end{aligned}$$

Clearly,

$$\begin{aligned}
&\bigg| \chi_{(-\ell,\ell)}(x) \bigg[c_+ \frac{\cosh(kx)}{\cosh(k\ell)} + c_- \frac{\sinh(kx)}{\sinh(k\ell)} \bigg] f(x) \bigg|\\
&\qquad \leqslant (|c_+| + |c_-|)|f(x)| \text{ for a.e. } x \in \mathbb{R},\ \ell \in \mathbb{N}, \qquad (3.22)
\end{aligned}$$

and

$$\lim_{\ell\to\infty} \chi_{(-\ell,\ell)}(x) \bigg[c_+ \frac{\cosh(kx)}{\cosh(k\ell)} + c_- \frac{\sinh(kx)}{\sinh(k\ell)} \bigg] f(x) = 0 \text{ for a.e. } x \in \mathbb{R}, \qquad (3.23)$$

so an application of dominated convergence yields

$$\lim_{\ell\to\infty} \big(\psi_\ell^{(0)}(-k^2,\,\cdot\,), f_{<\ell}\big)_{L^2((-\ell,\ell);dx)} = \lim_{\ell\to\infty} \int_{-\ell}^{\ell} \psi_\ell^{(0)}(-k^2,x) f_{<\ell}(x)\, dx = 0. \quad (3.24)$$

Finally, (3.24) and the last inequality in (3.21) imply

$$\lim_{\ell\to\infty} \Big\| \Big[P_\ell^{(0)}(-k^2) \oplus_\ell 0\Big] f \Big\|^2_{L^2(\mathbb{R};dx)} = 0, \quad f \in L^2(\mathbb{R}; dx) \cap L^1(\mathbb{R}; dx), \qquad (3.25)$$

which yields (3.19) and completes the proof. □

Another application of Krein's resolvent formula (2.42) yields the extensions of (3.11) and (3.12) to the case of the coupled boundary conditions at the endpoints of $[-\ell, \ell]$ in (2.10). The proofs of these results rely on two classic convergence results for the trace ideals $\mathcal{B}_p(\mathcal{H})$ which we recall in Appendix A for completeness.

Lemma 3.4. *Assume Hypothesis* 2.1. *For each fixed $z \in \mathbb{C}\backslash[\lambda_\infty^{(0)}, \infty)$, the following convergence results hold in $\mathcal{B}_2(L^2(\mathbb{R}; dx))$:*

$$\lim_{\ell\to\infty} \Big\| \Big[u_\ell R_\ell^{(0)}(z) \oplus_\ell 0\Big] - u R^{(0)}(z) \Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} = 0, \qquad (3.26)$$

$$\lim_{\ell\to\infty} \Big\| \Big[\overline{R_\ell^{(0)}(z) v_\ell} \oplus_\ell 0\Big] - \overline{R^{(0)}(z) v} \Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} = 0. \qquad (3.27)$$

Proof. It suffices to prove (3.26) for one $z \in \mathbb{C}\backslash[\lambda_\infty^{(0)}, \infty)$. In fact, suppose that

$$\lim_{\ell\to\infty} \Big\| \Big[u_\ell R_\ell^{(0)}(z_0) \oplus_\ell 0\Big] - u R^{(0)}(z_0) \Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} = 0 \qquad (3.28)$$

for some fixed $z_0 \in \mathbb{C}\backslash[\lambda_\infty^{(0)}, \infty)$. We claim that (3.28) actually implies (3.26) for all $z \in \mathbb{C}\backslash[\lambda_\infty^{(0)}, \infty)$. Indeed, for any $z \in \mathbb{C}\backslash[\lambda_\infty^{(0)}, \infty)$, the first resolvent identity implies

$$\begin{aligned}
&\Big[u_\ell R_\ell^{(0)}(z) \oplus_\ell 0\Big] - uR^{(0)}(z) \\
&\quad = \Big[u_\ell R_\ell^{(0)}(z_0) \oplus_\ell 0\Big] - uR^{(0)}(z_0) \\
&\qquad + (z - z_0)\Big\{ \Big[u_\ell R_\ell^{(0)}(z_0) \oplus_\ell 0\Big]\Big(\big[H_\ell^{(0)} \oplus_\ell 0\big] - zI_{L^2(\mathbb{R};dx)}\Big)^{-1} \\
&\qquad\quad - uR^{(0)}(z_0)R^{(0)}(z)\Big\}, \quad \ell \in \mathbb{N},
\end{aligned} \tag{3.29}$$

so that

$$\begin{aligned}
&\Big\|\Big[u_\ell R_\ell^{(0)}(z) \oplus_\ell 0\Big] - uR^{(0)}(z)\Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} \\
&\quad \leqslant \Big\|\Big[u_\ell R_\ell^{(0)}(z_0) \oplus_\ell 0\Big] - uR^{(0)}(z_0)\Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} \\
&\qquad + |z - z_0|\Big\|\Big[u_\ell R_\ell^{(0)}(z_0) \oplus_\ell 0\Big]\Big(\big[H_\ell^{(0)} \oplus_\ell 0\big] - zI_{L^2(\mathbb{R};dx)}\Big)^{-1} \\
&\qquad\quad - uR^{(0)}(z_0)R^{(0)}(z)\Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))}, \quad \ell \in \mathbb{N}.
\end{aligned} \tag{3.30}$$

The first term on the right-hand side of the inequality in (3.30) goes to zero as $\ell \to \infty$ by (3.28). To show the norm in the second term on the right-hand side of the inequality in (3.30) goes to zero as $\ell \to \infty$, we apply Grümm's Theorem (i.e., Theorem A.2) with the choices $p = 2$ and

$$\begin{aligned}
&A = uR^{(0)}(z_0), \quad A_\ell = \Big[u_\ell R_\ell^{(0)}(z_0) \oplus_\ell 0\Big], \quad \ell \in \mathbb{N}, \\
&B = R^{(0)}(z), \qquad B_\ell = \Big(\Big[H_\ell^{(0)} \oplus_\ell 0\Big] - zI_{L^2(\mathbb{R};dx)}\Big)^{-1}, \quad \ell \in \mathbb{N}.
\end{aligned} \tag{3.31}$$

If $\operatorname{dist}(\zeta, \Omega)$ denotes the distance from a point $\zeta \in \mathbb{C}$ to a subset $\Omega \subset \mathbb{C}$, then

$$\begin{aligned}
\|B_\ell\|_{\mathcal{B}(L^2(\mathbb{R};dx))} &= \Big\|R_\ell^{(0)}(z) \oplus_\ell (-z^{-1})I_{L^2(\mathbb{R}\backslash(-\ell,\ell);dx)}\Big\|_{\mathcal{B}(L^2(\mathbb{R};dx))} \\
&\leqslant \Big\|R_\ell^{(0)}(z)\Big\|_{\mathcal{B}(L^2((-\ell,\ell);dx))} \\
&\quad + \Big\|(-z^{-1})I_{L^2(\mathbb{R}\backslash(-\ell,\ell);dx)}\Big\|_{\mathcal{B}(L^2(\mathbb{R}\backslash(-\ell,\ell);dx))} \\
&= \operatorname{dist}\big(z, \sigma\big(H_\ell^{(0)}\big)\big)^{-1} + |z|^{-1} \\
&\leqslant \operatorname{dist}(z, [\lambda_\infty^{(0)}, \infty))^{-1} + |z|^{-1}, \quad \ell \in \mathbb{N},
\end{aligned} \tag{3.32}$$

shows that

$$\sup_{\ell \in \mathbb{N}} \|B_\ell\|_{\mathcal{B}(L^2(\mathbb{R};dx))} < \infty. \tag{3.33}$$

In (3.32), we have used the standard norm estimate for the resolvent of a self-adjoint operator (cf., e.g., [22, Problem 3.5 on p. 111]). Moreover, Lemma 3.3 implies $\operatorname{s-lim}_{\ell\to\infty} B_\ell = B$. Therefore, the choices in (3.31) satisfy the hypotheses of Grümm's Theorem. Consequently, one infers that

$$\begin{aligned}
\lim_{\ell\to\infty} &\Big\|\Big[u_\ell R_\ell^{(0)}(z_0) \oplus_\ell 0\Big]\Big(\big[H_\ell^{(0)} \oplus_\ell 0\big] - zI_{L^2(\mathbb{R};dx)}\Big)^{-1} \\
&\qquad - uR^{(0)}(z_0)R^{(0)}(z)\Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} = 0,
\end{aligned} \tag{3.34}$$

and (3.30) implies

$$\lim_{\ell\to\infty}\Big\|\Big[u_\ell R_\ell^{(0)}(z)\oplus_\ell 0\Big]-uR^{(0)}(z)\Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))}=0. \tag{3.35}$$

We will now prove that (3.28) holds for $z_0=-k^2$ with $k>k_0$ fixed, where k_0 is again the constant from (2.64). By Krein's resolvent formula (2.42),

$$\begin{aligned}&\Big[u_\ell R_\ell^{(0)}(-k^2)\oplus_\ell 0\Big]-uR^{(0)}(-k^2)\\ &\quad=\Big[u_\ell\Big(R_{\ell,D}^{(0)}(-k^2)+P_\ell^{(0)}(-k^2)\Big)\oplus_\ell 0\Big]-uR^{(0)}(-k^2)\\ &\quad=\Big[u_\ell R_{\ell,D}^{(0)}(-k^2)\oplus_\ell 0\Big]-uR^{(0)}(-k^2)+\Big[u_\ell P_\ell^{(0)}(-k^2)\oplus_\ell 0\Big],\quad \ell\in\mathbb{N}.\end{aligned} \tag{3.36}$$

Applying the triangle inequality for norms,

$$\begin{aligned}&\Big\|\Big[u_\ell R_\ell^{(0)}(-k^2)\oplus_\ell 0\Big]-uR^{(0)}(-k^2)\Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))}\\ &\quad\leqslant\Big\|\Big[u_\ell R_{\ell,D}^{(0)}(-k^2)\oplus_\ell 0\Big]-uR^{(0)}(-k^2)\Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))}\\ &\qquad+\Big\|u_\ell P_\ell^{(0)}(-k^2)\oplus_\ell 0\Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))},\quad \ell\in\mathbb{N}.\end{aligned} \tag{3.37}$$

By (3.11), the first term on the right-hand side of the inequality in (3.37) converges to zero as $\ell\to\infty$. Thus, it suffices to show

$$\lim_{\ell\to\infty}\Big\|u_\ell P_\ell^{(0)}(-k^2)\oplus_\ell 0\Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))}=0. \tag{3.38}$$

To this end, note that

$$\begin{aligned}&\Big\|u_\ell P_\ell^{(0)}(-k^2)\oplus_\ell 0\Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))}\\ &\quad=\Big\|u_\ell P_\ell^{(0)}(-k^2)\Big\|_{\mathcal{B}_2(L^2((-\ell,\ell);dx))}\\ &\quad=\big|q_\ell^{(0)}(-k^2)^{-1}\big|\Big\|u_\ell\big(\psi_\ell^{(0)}(-k^2,\,\cdot\,),\,\cdot\,\big)_{L^2((-\ell,\ell);dx)}\psi_\ell^{(0)}(-k^2,\,\cdot\,)\Big\|_{\mathcal{B}_2(L^2((-\ell,\ell);dx))}\\ &\quad\leqslant C_0k^{-1}\Big\|\big(\psi_\ell^{(0)}(-k^2,\,\cdot\,),\,\cdot\,\big)_{L^2((-\ell,\ell);dx)}u_\ell\psi_\ell^{(0)}(-k^2,\,\cdot\,)\Big\|_{\mathcal{B}_2(L^2((-\ell,\ell);dx))}\\ &\quad=C_0k^{-1}\big\|\psi_\ell^{(0)}(-k^2,\,\cdot\,)\big\|_{L^2((-\ell,\ell);dx)}\big\|u_\ell\psi_\ell^{(0)}(-k^2,\,\cdot\,)\big\|_{L^2((-\ell,\ell);dx)},\quad \ell\in\mathbb{N},\end{aligned} \tag{3.39}$$

where we have used the simple fact that

$$\|A\oplus_\ell 0\|_{\mathcal{B}_p(L^2(\mathbb{R};dx))}=\|A\|_{\mathcal{B}_p(L^2((-\ell,\ell);dx))},\quad A\in\mathcal{B}_p(L^2((-\ell,\ell);dx)),\ p\in[1,\infty),\ \ell\in\mathbb{N}. \tag{3.40}$$

On the other hand,

$$\begin{aligned}\big\|\psi_\ell^{(0)}(-k^2,\,\cdot\,)\big\|_{L^2((-\ell,\ell);dx)}&\leqslant\big\{2|c_+|^2\big[k^{-1}\tanh(k\ell)+\ell\,\mathrm{sech}^2(k\ell)\big]\\ &\qquad+2|c_-|^2\big[k^{-1}\coth(k\ell)-\ell\,\mathrm{csch}^2(k\ell)\big]\big\}^{1/2}\\ &\leqslant C_2,\quad \ell\in\mathbb{N},\end{aligned} \tag{3.41}$$

for some constant $C_2>0$. In addition,

$$\big\|u_\ell\psi_\ell^{(0)}(-k^2,\,\cdot\,)\big\|^2_{L^2((-\ell,\ell);dx)}$$

$$
\begin{aligned}
&= \int_{\ell}^{\ell} |u_\ell(x)|^2 \bigg| c_+ \frac{\cosh(kx)}{\cosh(k\ell)} + c_- \frac{\sinh(kx)}{\sinh(k\ell)} \bigg|^2 \, dx \\
&= \int_{-\infty}^{\infty} \chi_{(-\ell,\ell)}(x) |V(x)| \bigg| \bigg[c_+ \frac{\cosh(kx)}{\cosh(k\ell)} + c_- \frac{\sinh(kx)}{\sinh(k\ell)} \bigg] \bigg|^2 \, dx, \quad \ell \in \mathbb{N}. \qquad (3.42)
\end{aligned}
$$

An application of dominated convergence, using $V \in L^1(\mathbb{R}; dx)$, then yields

$$
\lim_{\ell \to \infty} \big\| u_\ell \psi_\ell^{(0)}(-k^2, \cdot\,) \big\|^2_{L^2((-\ell,\ell); dx)} = 0. \qquad (3.43)
$$

Therefore, (3.38) follows from (3.39), (3.41), and (3.43).

The claim in (3.27) actually follows for all $z \in \mathbb{C} \backslash [\lambda_\infty^{(0)}, \infty)$ by a simple adjoint argument. Indeed, one notes that

$$
\begin{aligned}
\overline{R_\ell^{(0)}(z) v_\ell} &= \big(v_\ell R_\ell^{(0)}(\overline{z}) \big)^*, \\
\overline{R^{(0)}(z) v} &= \big(v R^{(0)}(\overline{z}) \big)^*, \quad z \in \mathbb{C} \backslash [\lambda_\infty^{(0)}, \infty), \qquad (3.44)
\end{aligned}
$$

and since $\|A^*\|_{\mathcal{B}_2(\mathcal{H})} = \|A\|_{\mathcal{B}_2(\mathcal{H})}$, $A \in \mathcal{B}_2(\mathcal{H})$, one obtains

$$
\begin{aligned}
&\Big\| \Big[\overline{R_\ell^{(0)}(z) v_\ell} \oplus_\ell 0 \Big] - \overline{R^{(0)}(z) v} \Big\|_{\mathcal{B}_2(L^2(\mathbb{R}; dx))} \\
&\quad = \Big\| \Big[v_\ell R_\ell^{(0)}(\overline{z}) \oplus_\ell 0 \Big] - v R^{(0)}(\overline{z}) \Big\|_{\mathcal{B}_2(L^2(\mathbb{R}; dx))}, \quad z \in \mathbb{C} \backslash [\lambda_\infty^{(0)}, \infty). \qquad (3.45)
\end{aligned}
$$

By repeating the proof of (3.26) with u_ℓ and u replaced by v_ℓ and v, respectively, one infers that

$$
\lim_{\ell \to \infty} \Big\| \Big[v_\ell R_\ell^{(0)}(\overline{z}) \oplus_\ell 0 \Big] - v R^{(0)}(\overline{z}) \Big\|_{\mathcal{B}_2(L^2(\mathbb{R}; dx))} = 0, \quad z \in \mathbb{C} \backslash [\lambda_\infty^{(0)}, \infty), \qquad (3.46)
$$

and (3.27) follows. □

Next, we extend (3.13) to the coupled boundary conditions in (2.10).

Lemma 3.5. *Assume Hypothesis* 2.1. *For each fixed $z \in \mathbb{C} \backslash [\lambda_\infty^{(0)}, \infty)$, the following convergence result holds in $\mathcal{B}_1(L^2(\mathbb{R}; dx))$:*

$$
\lim_{\ell \to \infty} \Big\| \Big[\overline{u_\ell R_\ell^{(0)}(z) v_\ell} \oplus_\ell 0 \Big] - \overline{u R^{(0)}(z) v} \Big\|_{\mathcal{B}_1(L^2(\mathbb{R}; dx))} = 0. \qquad (3.47)
$$

Proof. It suffices to prove (3.47) for one $z \in \mathbb{C} \backslash [\lambda_\infty^{(0)}, \infty)$. To see this, suppose that

$$
\lim_{\ell \to \infty} \Big\| \Big[\overline{u_\ell R_\ell^{(0)}(z_0) v_\ell} \oplus_\ell 0 \Big] - \overline{u R^{(0)}(z_0) v} \Big\|_{\mathcal{B}_1(L^2(\mathbb{R}; dx))} = 0 \qquad (3.48)
$$

for some fixed $z_0 \in \mathbb{C} \backslash [\lambda_\infty^{(0)}, \infty)$. If $z \in \mathbb{C} \backslash [\lambda_\infty^{(0)}, \infty)$, then (3.48) actually implies (3.47). Indeed, by the first resolvent identity,

$$
\begin{aligned}
&\Big[\overline{u_\ell R_\ell^{(0)}(z) v_\ell} \oplus_\ell 0 \Big] - \overline{u R^{(0)}(z) v} \qquad (3.49) \\
&\quad = \Big[\overline{u_\ell R_\ell^{(0)}(z_0) v_\ell} \oplus_\ell 0 \Big] - \overline{u R^{(0)}(z_0) v} \\
&\qquad + (z - z_0) \Big\{ \Big[u_\ell R_\ell^{(0)}(z_0) \oplus_\ell 0 \Big] \Big[\overline{R_\ell^{(0)}(z) v_\ell} \oplus_\ell 0 \Big] - u R^{(0)}(z_0) \overline{R^{(0)}(z) v} \Big\}, \quad \ell \in \mathbb{N}.
\end{aligned}
$$

As a result, one obtains the following estimate:

$$
\Big\| \Big[\overline{u_\ell R_\ell^{(0)}(z) v_\ell} \oplus_\ell 0 \Big] - \overline{u R^{(0)}(z) v} \Big\|_{\mathcal{B}_1(L^2(\mathbb{R}; dx))}
$$

$$\begin{aligned}
&\leqslant \Big\| \Big[\overline{u_\ell R_\ell^{(0)}(z_0) v_\ell} \oplus_\ell 0\Big] - \overline{u R^{(0)}(z_0) v} \Big\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} \\
&\quad + |z - z_0| \Big\| \Big[u_\ell R_\ell^{(0)}(z_0) \oplus_\ell 0\Big] \Big[\overline{R_\ell^{(0)}(z) v_\ell} \oplus_\ell 0\Big] \\
&\qquad - u R^{(0)}(z_0) \overline{R^{(0)}(z) v} \Big\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))}, \quad \ell \in \mathbb{N}.
\end{aligned} \tag{3.50}$$

In light of (3.48), the first term on the right-hand side of the inequality in (3.50) goes to zero as $\ell \to \infty$. By (3.26) and (3.27), one has

$$\begin{aligned}
\lim_{\ell\to\infty} \Big\| \Big[u_\ell R_\ell^{(0)}(z_0) \oplus_\ell 0\Big] - u R^{(0)}(z_0) \Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} &= 0, \\
\lim_{\ell\to\infty} \Big\| \Big[\overline{R_\ell^{(0)}(z) v_\ell} \oplus_\ell 0\Big] - \overline{R^{(0)}(z) v} \Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} &= 0,
\end{aligned} \tag{3.51}$$

so the second term on the right-hand side of the inequality in (3.50) goes to zero as $\ell \to \infty$ by a direct application of Lemma A.3. Finally, (3.47) follows from (3.50) by an application of the squeeze theorem.

To show (3.47) holds for $z = -k^2$ with $k > k_0$, where k_0 is the constant in (2.64), we compute

$$\begin{aligned}
&\Big[\overline{u_\ell R_\ell^{(0)}(-k^2) v_\ell} \oplus_\ell 0\Big] - \overline{u R^{(0)}(-k^2) v} \\
&\quad = \Big[\overline{u_\ell R_{\ell,D}^{(0)}(-k^2) v_\ell} \oplus_\ell 0\Big] - \overline{u R^{(0)}(-k^2) v} + \Big[\overline{u_\ell P_\ell^{(0)}(-k^2) v_\ell} \oplus_\ell 0\Big], \quad \ell \in \mathbb{N},
\end{aligned} \tag{3.52}$$

where the splitting of the closure is justified by the fact that for each $\ell \in \mathbb{N}$, $u_\ell R_{\ell,D}^{(0)}(-k^2) v_\ell$ and $u_\ell P_\ell^{(0)}(-k^2) v_\ell$ are bounded operators with domains equal to the dense subspace $\operatorname{dom}(v_\ell) \subset L^2((-\ell,\ell);dx)$, so all closures involved are simply the continuous extensions of the underlying operators to all of $L^2((-\ell,\ell);dx)$. By (3.52),

$$\begin{aligned}
&\Big\| \Big[\overline{u_\ell R_\ell^{(0)}(-k^2) v_\ell} \oplus_\ell 0\Big] - \overline{u R^{(0)}(-k^2) v} \Big\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} \\
&\quad \leqslant \Big\| \Big[\overline{u_\ell R_{\ell,D}^{(0)}(-k^2) v_\ell} \oplus_\ell 0\Big] - \overline{u R^{(0)}(-k^2) v} \Big\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} \\
&\qquad + \Big\| \overline{u_\ell P_\ell^{(0)}(-k^2) v_\ell} \oplus_\ell 0 \Big\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))}, \quad \ell \in \mathbb{N}.
\end{aligned} \tag{3.53}$$

In light of (3.13) and (3.53), to prove (3.47) for $z = -k^2$, it suffices to show

$$\lim_{\ell\to\infty} \Big\| \overline{u_\ell P_\ell^{(0)}(-k^2) v_\ell} \oplus_\ell 0 \Big\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} = 0. \tag{3.54}$$

The closure $\overline{u_\ell P_\ell^{(0)}(-k^2) v_\ell}$ was computed explicitly in (2.59). By the equality in (2.62) and the estimate in (2.64), one obtains

$$\begin{aligned}
&\Big\| \overline{u_\ell P_\ell^{(0)}(-k^2) v_\ell} \Big\|_{\mathcal{B}_1(L^2((-\ell,\ell);dx))} \\
&\quad \leqslant C_0 k^{-1} \|\Psi(-k^2,\,\cdot\,)\|_{L^2((-\ell,\ell);dx)} \|\Phi(-k^2,\,\cdot\,)\|_{L^2((-\ell,\ell);dx)} \\
&\quad = C_0 k^{-1} \int_{-\infty}^{\infty} \chi_{(-\ell,\ell)}(x) |V(x)| \bigg| \bigg[c_+ \frac{\cosh(kx)}{\cosh(k\ell)} + c_- \frac{\sinh(kx)}{\sinh(k\ell)}\bigg] \bigg|^2 \, dx, \quad \ell \in \mathbb{N},
\end{aligned} \tag{3.55}$$

so (3.54) follows from an application of dominated convergence, using $V \in L^1(\mathbb{R};dx)$ once more.

□

4. Convergence Properties of Spectral Shift Functions

In this section, assuming Hypothesis 2.1, we introduce the spectral shift functions for the pairs $(H, H^{(0)})$ and $(H_\ell, H_\ell^{(0)})$, $\ell \in \mathbb{N}$, and apply the convergence properties of resolvents developed in Section 3 and the abstract convergence criteria from [9] (and summarized in Appendix B) to obtain weak and vague convergence results for the spectral shift functions in the limit $\ell \to \infty$.

As a consequence of Lemma 2.6, one notes that for each fixed $\ell \in \mathbb{N}$, the conditions in [9, Hypothesis 2.1] hold upon making the identifications

$$B = H_\ell, \quad A = H_\ell^{(0)}, \quad W = V_\ell, \quad W_1 = u_\ell, \quad W_2 = v_\ell. \tag{4.1}$$

As a result, the resolvents of H_ℓ and $H_\ell^{(0)}$ satisfy the following Kato-type resolvent identity:

$$\begin{aligned} R_\ell(z) = R_\ell^{(0)}(z) - \overline{R_\ell^{(0)}(z)v_\ell}\Big[I_{L^2((-\ell,\ell);dx)} + \overline{u_\ell R_\ell^{(0)}(z)v_\ell}\Big]^{-1} u_\ell R_\ell^{(0)}(z),& \\ z \in \rho(H_\ell) \cap \rho(H_\ell^{(0)}),\ \ell \in \mathbb{N}.& \end{aligned} \tag{4.2}$$

By Lemma 2.6 (viz., (2.51)), and the fact that the product of two Hilbert–Schmidt operators is trace class, H_ℓ and $H_\ell^{(0)}$ are resolvent comparable,

$$\big[R_\ell(z) - R_\ell^{(0)}(z)\big] \in \mathcal{B}_1(L^2((-\ell,\ell);dx)), \quad z \in \rho(H_\ell) \cap \rho(H_\ell^{(0)}),\ \ell \in \mathbb{N}. \tag{4.3}$$

In a similar vein, the resolvents of H and $H^{(0)}$ satisfy

$$R(z) = R^{(0)}(z) - \overline{R^{(0)}(z)v}\Big[I_{L^2(\mathbb{R};dx)} + \overline{uR^{(0)}(z)v}\Big]^{-1} uR^{(0)}(z), \quad z \in \rho(H), \tag{4.4}$$

and

$$\big[R(z) - R^{(0)}(z)\big] \in \mathcal{B}_1(L^2(\mathbb{R};dx)), \quad z \in \rho(H). \tag{4.5}$$

The resolvent comparability condition in (4.3) guarantees the existence of a real-valued spectral shift function $\xi(\,\cdot\,; H_\ell, H_\ell^{(0)})$ for each $\ell \in \mathbb{N}$ which satisfies

$$\int_{-\infty}^{\infty} \frac{|\xi(\lambda; H_\ell, H_\ell^{(0)})|}{1+\lambda^2}\, d\lambda < \infty \quad \text{and} \quad \xi(\lambda; H_\ell, H_\ell^{(0)}) = 0, \quad \lambda \in (-\infty, \lambda_\infty), \tag{4.6}$$

and

$$\begin{aligned} \operatorname{tr}_{L^2((-\ell,\ell);dx)}\big(R_\ell(z) - R_\ell^{(0)}(z)\big) = -\int_{\mathbb{R}} \frac{\xi(\lambda; H_\ell, H_\ell^{(0)})}{(\lambda - z)^2}\, d\lambda,& \\ z \in \rho(H_\ell) \cap \rho(H_\ell^{(0)}),\ \ell \in \mathbb{N}.& \end{aligned} \tag{4.7}$$

Similarly, by (4.5) there exists a real-valued spectral shift function $\xi(\,\cdot\,; H, H^{(0)})$ which satisfies

$$\int_{-\infty}^{\infty} \frac{|\xi(\lambda; H, H^{(0)})|}{1+\lambda^2}\, d\lambda < \infty \quad \text{and} \quad \xi(\lambda; H, H^{(0)}) = 0, \quad \lambda \in (-\infty, \lambda_\infty), \tag{4.8}$$

and

$$\operatorname{tr}_{L^2(\mathbb{R};dx)}\big(R(z) - R^{(0)}(z)\big) = -\int_{\mathbb{R}} \frac{\xi(\lambda; H, H^{(0)})}{(\lambda - z)^2}\, d\lambda, \quad z \in \rho(H). \tag{4.9}$$

The conditions in (4.6)–(4.9) guarantee that $\xi(\,\cdot\,; H_\ell, H_\ell^{(0)})$ and $\xi(\,\cdot\,; H, H^{(0)})$ are uniquely determined almost everywhere. Moreover, for a large class of functions f (e.g., any f such that the Fourier transform $\hat{f} \in L^1(\mathbb{R}; (1+|p|)\,dp)$) one has

$$\begin{aligned} &\big[f(H_\ell) - f(H_\ell^{(0)})\big] \in \mathcal{B}_1(L^2((-\ell,\ell); dx)), \quad \ell \in \mathbb{N}, \\ &\big[f(H) - f(H^{(0)})\big] \in \mathcal{B}_1(L^2(\mathbb{R}; dx)), \end{aligned} \tag{4.10}$$

and the following trace formulas hold:

$$\begin{aligned} \operatorname{tr}_{L^2((-\ell,\ell);dx)}\big(f(H_\ell) - f(H_\ell^{(0)})\big) &= \int_{\mathbb{R}} f'(\lambda)\,\xi(\lambda; H_\ell, H_\ell^{(0)})\,d\lambda, \quad \ell \in \mathbb{N}, \\ \operatorname{tr}_{L^2(\mathbb{R};dx)}\big(f(H) - f(H^{(0)})\big) &= \int_{\mathbb{R}} f'(\lambda)\,\xi(\lambda; H, H^{(0)})\,d\lambda, \end{aligned} \tag{4.11}$$

For these, and many other pertinent facts related to spectral shift functions, we refer to [25, Chapter 8].

At this point, assuming Hypothesis 2.1, we may verify the conditions of Hypothesis B.1 and apply Theorem B.2 and Corollary B.5. In the notation employed in Hypothesis B.1,

$$\begin{aligned} &v \text{ corresponds to } V_1^*, && u \text{ corresponds to } V_2, \\ &v_\ell \text{ corresponds to } V_{1,\ell}^*, && u_\ell \text{ corresponds to } V_{2,\ell}, \quad \ell \in \mathbb{N}, \\ &H^{(0)} \text{ corresponds to } A^{(0)}, && H_\ell^{(0)} \text{ corresponds to } A_\ell^{(0)},\ \ell \in \mathbb{N}. \end{aligned} \tag{4.12}$$

Under the correspondences in (4.12), it is clear that items (i)–(iii) hold in Hypothesis B.1. In particular, (B.2) and (B.3) hold in light of the fact that functions in $\operatorname{dom}(|H^{(0)}|^{1/2})$ and $\operatorname{dom}(|H_\ell^{(0)}|^{1/2})$ are bounded on $\mathbb{R}$ and $(-\ell,\ell)$, respectively, and $u, v \in L^2(\mathbb{R}; dx)$, while $u_\ell, v_\ell \in L^2((-\ell,\ell); dx)$. Item (iv) in Hypothesis B.1 follows from (2.49), (2.50), and Lemma 2.6. The strong resolvent convergence condition in item (v) of Hypothesis B.1 follows from Lemma 3.3, and conditions (B.11)–(B.13) follow from Lemmata 3.4 and 3.5. Plainly, item (vii) in Hypothesis B.1 is satisfied since V is a.e. real-valued. Finally, item $(viii)$ in Hypothesis B.1 follows from the fact that the operator of multiplication by a function from $L^1(\mathbb{R}; dx)$ (resp., $L^1((-\ell,\ell); dx)$) is infinitesimally form bounded with respect to $H^{(0)}$ (resp., $H_\ell^{(0)}$).

Having verified the criteria of Hypothesis B.1, we are now in a position to invoke Theorem B.2 and Corollary B.5 to obtain weak and vague convergence results for the spectral shift functions $\{\xi(\,\cdot\,; H_\ell, H_\ell^{(0)})\}_{\ell=1}^\infty$ as $\ell \to \infty$.

Theorem 4.1. *Assume Hypothesis* 2.1. *If* $f \in C_b(\mathbb{R})$, *then*

$$\lim_{\ell\to\infty} \int_{\mathbb{R}} \frac{\xi(\lambda; H_\ell, H_\ell^{(0)})}{\lambda^2+1} f(\lambda)\,d\lambda = \int_{\mathbb{R}} \frac{\xi(\lambda; H, H^{(0)})}{\lambda^2+1} f(\lambda)\,d\lambda. \tag{4.13}$$

The result of Theorem 4.1 is that

$$\frac{\xi(\,\cdot\,; H_\ell, H_\ell^{(0)})}{|\,\cdot\,|^2+1} \text{ converges } \textit{weakly} \text{ to } \frac{\xi(\,\cdot\,; H, H^{(0)})}{|\,\cdot\,|^2+1} \text{ as } \ell \to \infty. \tag{4.14}$$

By applying Corollary B.4, the continuity assumption on f in Theorem 4.1 may be relaxed.

Corollary 4.2. *Assume Hypothesis* 2.1. *Then* (4.13) *holds for any bounded Borel measurable function g that is continuous almost everywhere with respect to Lebesgue* *measure on* $\mathbb{R}$*. In particular,*

$$\lim_{\ell\to\infty}\int_S \frac{\xi(\lambda; H_\ell, H_\ell^{(0)})}{1+\lambda^2}\,d\lambda = \int_S \frac{\xi(\lambda; H, H^{(0)})}{1+\lambda^2}\,d\lambda \tag{4.15}$$

holds for any set $S \subseteq \mathbb{R}$ *that is boundaryless with respect to Lebesgue measure* (*i.e., any set* $S \subseteq \mathbb{R}$ *for which the boundary of* S *has Lebesgue measure equal to zero*)*.*

Finally, an application of Corollary B.5 yields the following result.

Corollary 4.3. *Assume Hypothesis* 2.1*. If* g *is a bounded Borel measurable function that is compactly supported and Lebesgue almost everywhere continuous on* $\mathbb{R}$*, then*

$$\lim_{\ell\to\infty}\int_{\mathbb{R}} \xi(\lambda; H_\ell, H_\ell^{(0)})\, g(\lambda)\,d\lambda = \int_{\mathbb{R}} \xi(\lambda; H, H^{(0)})\, g(\lambda)\,d\lambda. \tag{4.16}$$

Of course, (4.16) holds in particular when $g \in C_0(\mathbb{R})$, so:

$$\xi(\,\cdot\,; H_\ell, H_\ell^{(0)}) \text{ converges } \textit{vaguely} \text{ to } \xi(\,\cdot\,; H, H^{(0)}) \text{ as } \ell\to\infty. \tag{4.17}$$

Appendix A. Background Results for $\mathcal{B}_p(\mathcal{H})$

In this appendix, we collect some well-known facts pertaining to the trace ideals $\mathcal{B}_p(\mathcal{H})$, $p \in [1,\infty)$, which are used extensively throughout this paper. In the first result, we compute the $\mathcal{B}_p$-norm of a rank-one operator. Though elementary, this result plays a fundamental role in handling the rank-one terms which appear as a result of Krein's resolvent identity in the proofs of Lemmata 3.3–3.5.

Proposition A.1. *Let* $\mathcal{H}$ *denote a Hilbert space with inner product* $(\cdot,\cdot)_{\mathcal{H}}$*. If* $\phi,\psi \in \mathcal{H}\backslash\{0\}$*, then the lone nonzero singular value of* $A = (\psi,\cdot)_{\mathcal{H}}\phi$*,* $\operatorname{dom}(A) = \mathcal{H}$*, is* $s_1(A) = \|\psi\|_{\mathcal{H}}\|\phi\|_{\mathcal{H}}$*. In particular, for each* $p \in [1,\infty)$*,*

$$\|A\|_{\mathcal{B}_p(\mathcal{H})} = \|\psi\|_{\mathcal{H}}\|\phi\|_{\mathcal{H}}. \tag{A.1}$$

Proof. In light of the identities

$$\begin{aligned}(f, Ag)_{\mathcal{H}} &= \big(f, (\psi, g)_{\mathcal{H}}\phi\big)_{\mathcal{H}} = (\psi, g)_{\mathcal{H}}(f,\phi)_{\mathcal{H}}\\ &= \big(\overline{(f,\phi)_{\mathcal{H}}}\psi, g\big)_{\mathcal{H}} = \big((\phi, f)_{\mathcal{H}}\psi, g\big)_{\mathcal{H}}, \quad f, g\in\mathcal{H},\end{aligned} \tag{A.2}$$

the adjoint of A is $A^* = (\phi,\cdot)_{\mathcal{H}}\psi$. Therefore,

$$A^*Af = A^*(\psi, f)_{\mathcal{H}}\phi = \|\phi\|_{\mathcal{H}}^2(\psi, f)_{\mathcal{H}}\psi, \quad f\in\mathcal{H}, \tag{A.3}$$

that is, $A^*A = \|\phi\|_{\mathcal{H}}^2(\psi,\cdot)\psi$. It follows that the lone nonzero eigenvalue of A^*A is $\|\phi\|_{\mathcal{H}}^2\|\psi\|_{\mathcal{H}}^2$, and A has one nonzero singular value, namely $s_1(A) = \|\phi\|_{\mathcal{H}}\|\psi\|_{\mathcal{H}}$. Since the $\mathcal{B}_p$-norm of A is the ℓ^p-norm of its sequence of singular values, we have $\|A\|_{\mathcal{B}_p(\mathcal{H})} = s_1(A)$, and the result follows. □

The following convergence results are classic and deal with sequences of products in the trace ideals.

Theorem A.2 (Grümm's Theorem, [11])**.** *Let* $p \in [1,\infty)$*,* $A \in \mathcal{B}_p(\mathcal{H})$*, and suppose that* $\{A_\ell\}_{\ell=1}^\infty \subset \mathcal{B}_p(\mathcal{H})$ *with* $\lim_{\ell\to\infty}\|A_\ell - A\|_{\mathcal{B}_p(\mathcal{H})} = 0$*. If* $B \in \mathcal{B}(\mathcal{H})$*,* $\{B_\ell\}_{\ell=1}^\infty \subset \mathcal{B}(\mathcal{H})$ *with* $\sup_{\ell\in\mathbb{N}}\|B_\ell\|_{\mathcal{B}(\mathcal{H})} < \infty$ *and* $\operatorname{s-lim}_{\ell\to\infty} B_\ell = B$*, then*

$$\lim_{\ell\to\infty}\|A_\ell B_\ell - AB\|_{\mathcal{B}_p(\mathcal{H})} = \lim_{\ell\to\infty}\|B_\ell A_\ell - BA\|_{\mathcal{B}_p(\mathcal{H})} = 0. \tag{A.4}$$

Lemma A.3. *Let $p,q,r \in [1,\infty)$ with $p^{-1}+q^{-1}=r^{-1}$. If $\{A_\ell\}_{\ell=1}^\infty \subset \mathcal{B}_p(\mathcal{H})$, $\{B_\ell\}_{\ell=1}^\infty \subset \mathcal{B}_q(\mathcal{H})$, $A \in \mathcal{B}_p(\mathcal{H})$, and $B \in \mathcal{B}_q(\mathcal{H})$ with*

$$\lim_{\ell\to\infty} \|A_\ell - A\|_{\mathcal{B}_p(\mathcal{H})} = 0 \quad \text{and} \quad \lim_{\ell\to\infty} \|B_\ell - B\|_{\mathcal{B}_q(\mathcal{H})} = 0, \tag{A.5}$$

then

$$\lim_{\ell\to\infty} \|A_\ell B_\ell - AB\|_{\mathcal{B}_r(\mathcal{H})} = 0. \tag{A.6}$$

The proof of Lemma A.3 is a simple exercise which makes use of Hölder's inequality for the trace ideals (cf., e.g., [21, Theorem 2.8]).

Appendix B. Criteria for Vague and Weak Convergence of Spectral Shift Functions

In this appendix, we recall the criteria for convergence of sequences of spectral shift functions introduced in [9]. For clarity, we state the criteria and the corresponding convergence results in the context in which they are applied in this paper, that is for pairs of self-adjoint operators acting in the Hilbert spaces $L^2((-\ell,\ell);dx)$ and $L^2(\mathbb{R};dx)$.

Hypothesis B.1 (Hypothesis 3.1 in [9])**.** *Let $\mathcal{H} := L^2(\mathbb{R};dx)$.*
(i) For each $\ell \in \mathbb{N}$, decompose $\mathcal{H}$ according to

$$L^2(\mathbb{R};dx) = L^2((-\ell,\ell);dx) \oplus_\ell L^2(\mathbb{R}\backslash(-\ell,\ell);dx), \tag{B.1}$$

and write $\mathcal{H}_\ell := L^2((-\ell,\ell);dx)$ and $\mathcal{H}_\ell^c = L^2(\mathbb{R}\backslash(-\ell,\ell);dx)$.
(ii) Let $A^{(0)}$ be a self-adjoint operator in $\mathcal{H}$, and for each $\ell \in \mathbb{N}$, let $A_\ell^{(0)}$ be self-adjoint operators in $\mathcal{H}_\ell$. In addition, suppose that $A^{(0)}$ is bounded from below in $\mathcal{H}$, and that for each $\ell \in \mathbb{N}$, $A_\ell^{(0)}$ is bounded from below in $\mathcal{H}_\ell$.
(iii) Suppose that V_1, and V_2 are closed operators in $\mathcal{H}$, and for each $\ell \in \mathbb{N}$, assume that $V_{1,\ell}$, and $V_{2,\ell}$ are closed operators in $\mathcal{H}_\ell$ such that

$$\operatorname{dom}(V_1) \cap \operatorname{dom}(V_2) \supseteq \operatorname{dom}(|A^{(0)}|^{1/2}), \tag{B.2}$$

$$\operatorname{dom}(V_{1,\ell}) \cap \operatorname{dom}(V_{2,\ell}) \supseteq \operatorname{dom}(|A_\ell^{(0)}|^{1/2}), \quad \ell \in \mathbb{N}, \tag{B.3}$$

where

$$V = V_1^* V_2 \ \text{ is a self-adjoint operator in } \mathcal{H}, \tag{B.4}$$

and for each $\ell \in \mathbb{N}$,

$$V_\ell = V_{1,\ell}^* V_{2,\ell} \ \text{ is a self-adjoint operator in } \mathcal{H}_\ell. \tag{B.5}$$

(iv) Suppose

$$\overline{V_2(A^{(0)} - zI_{\mathcal{H}})^{-1}V_1^*},\ \overline{V_{2,\ell}(A_\ell^{(0)} - zI_{\mathcal{H}_\ell})^{-1}V_{1,\ell}^*} \oplus_\ell 0 \in \mathcal{B}_1(\mathcal{H}), \quad \ell \in \mathbb{N}, \tag{B.6}$$

$$V_2(A^{(0)} - zI_{\mathcal{H}})^{-1},\ V_{2,\ell}(A_\ell^{(0)} - zI_{\mathcal{H}_\ell})^{-1} \oplus_\ell 0 \in \mathcal{B}_2(\mathcal{H}), \quad \ell \in \mathbb{N}, \tag{B.7}$$

$$\overline{(A^{(0)} - zI_{\mathcal{H}})^{-1}V_1^*},\ \overline{(A_\ell^{(0)} - zI_{\mathcal{H}_\ell})^{-1}V_{1,\ell}^*} \oplus_\ell 0 \in \mathcal{B}_2(\mathcal{H}), \quad \ell \in \mathbb{N}, \tag{B.8}$$

for some (and hence for all) $z \in \mathbb{C}\backslash\mathbb{R}$. In addition, assume that

$$\begin{aligned} &\lim_{z\downarrow-\infty} \Big\| \Big[\overline{V_2(A^{(0)} - zI_{\mathcal{H}})^{-1}V_1^*} \Big\|_{\mathcal{B}_1(\mathcal{H})} = 0, \\ &\lim_{z\downarrow-\infty} \Big\| \overline{V_{2,\ell}(A_\ell^{(0)} - zI_{\mathcal{H}_\ell})^{-1}V_{1,\ell}^*} \oplus_\ell 0 \Big\|_{\mathcal{B}_1(\mathcal{H})} = 0, \quad \ell \in \mathbb{N}. \end{aligned} \tag{B.9}$$

(v) Assume that for some (and hence for all) $z \in \mathbb{C}\backslash\mathbb{R}$*,*

$$\operatorname*{s-lim}_{\ell\to\infty}\bigg[(A_\ell^{(0)} - zI_{\mathcal{H}_\ell})^{-1} \oplus_\ell \frac{-1}{z} I_{\mathcal{H}_\ell^c}\bigg] = (A^{(0)} - zI_{\mathcal{H}})^{-1}. \tag{B.10}$$

(vi) Suppose that for some (and hence for all) $z \in \mathbb{C}\backslash\mathbb{R}$*,*

$$\lim_{\ell\to\infty}\Big\|\Big[\overline{V_{2,\ell}(A_\ell^{(0)} - zI_{\mathcal{H}_\ell})^{-1}V_{1,\ell}^*} \oplus_\ell 0\Big] - \overline{V_2(A^{(0)} - zI_{\mathcal{H}})^{-1}V_1^*}\Big\|_{\mathcal{B}_1(\mathcal{H})} = 0, \tag{B.11}$$

$$\lim_{\ell\to\infty}\Big\|\Big[V_{2,\ell}(A_\ell^{(0)} - zI_{\mathcal{H}_\ell})^{-1} \oplus_\ell 0\Big] - V_2(A^{(0)} - zI_{\mathcal{H}})^{-1}\Big\|_{\mathcal{B}_2(\mathcal{H})} = 0, \tag{B.12}$$

$$\lim_{\ell\to\infty}\Big\|\Big[\overline{(A_\ell^{(0)} - zI_{\mathcal{H}_\ell})^{-1}V_{1,\ell}^*} \oplus_\ell 0\Big] - \overline{(A^{(0)} - z)^{-1}V_1^*}\Big\|_{\mathcal{B}_2(\mathcal{H})} = 0. \tag{B.13}$$

(vii) Suppose that

$$\begin{aligned} &(V_2 f, V_1 g)_{\mathcal{H}} = (V_1 f, V_2 g)_{\mathcal{H}}, \quad f, g \in \operatorname{dom}(V_1) \cap \operatorname{dom}(V_2), \\ &(V_{2,\ell} f, V_{1,\ell} g)_{\mathcal{H}} = (V_{1,\ell} f, V_{2,\ell} g)_{\mathcal{H}}, \quad f, g \in \operatorname{dom}(V_{1,\ell}) \cap \operatorname{dom}(V_{2,\ell}), \quad \ell \in \mathbb{N}. \end{aligned} \tag{B.14}$$

(viii) Decomposing V, V_ℓ*,* $\ell \in \mathbb{N}$*, into their positive and negative parts,*

$$V_\pm = (1/2)[|V| \pm V], \quad V_{\ell,\pm} = (1/2)[|V_\ell| \pm V_\ell], \ \ell \in \mathbb{N}, \tag{B.15}$$

$V_\pm$ *are assumed to be infinitesimally form bounded with respect to* $A^{(0)}$*, and for each* $\ell \in \mathbb{N}$*,* $V_{\ell,\pm}$ *are assumed to be infinitesimally form bounded with respect to* $A_\ell^{(0)}$*.*

These hypotheses permit one to identify A and A_ℓ with the quadratic form sums of $A^{(0)}$ with $V_1^* V_2$ and $A_\ell^{(0)}$ with $V_{1,\ell}^* V_{2,\ell}$, respectively:

$$\begin{aligned} A &= A^{(0)} +_q V_1^* V_2, \\ A_\ell &= A_\ell^{(0)} +_q V_{1,\ell}^* V_{2,\ell}, \quad \ell \in \mathbb{N}. \end{aligned} \tag{B.16}$$

Assuming Hypothesis B.1, the pairs $(A, A^{(0)})$ and $(A_\ell, A_\ell^{(0)})$, $\ell \in \mathbb{N}$, are resolvent comparable in the sense that

$$\big[(A - zI_{\mathcal{H}})^{-1} - (A^{(0)} - zI_{\mathcal{H}})^{-1}\big] \in \mathcal{B}_1(\mathcal{H}), \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{B.17}$$

and

$$\big[(A_\ell - zI_{\mathcal{H}})^{-1} - (A_\ell^{(0)} - zI_{\mathcal{H}})^{-1}\big] \in \mathcal{B}_1(\mathcal{H}), \quad z \in \mathbb{C}\backslash\mathbb{R}, \ \ell \in \mathbb{N}. \tag{B.18}$$

Therefore, Hypothesis B.1 guarantees the existence of real-valued spectral shift functions $\xi(\,\cdot\,; A, A^{(0)})$ and $\xi(\,\cdot\,; A_\ell, A_\ell^{(0)})$, $\ell \in \mathbb{N}$, which satisfy

$$\begin{aligned} \operatorname{tr}_{\mathcal{H}}\big((A - zI_{\mathcal{H}})^{-1} - (A^{(0)} - zI_{\mathcal{H}})^{-1}\big) = -\int_{\mathbb{R}} \frac{\xi(\lambda; A, A^{(0)})}{(\lambda - z)^2}\, d\lambda,& \\ z \in \rho(A) \cap \rho(A^{(0)}),& \end{aligned} \tag{B.19}$$

and

$$\begin{aligned} \operatorname{tr}_{\mathcal{H}_\ell}\big((A_\ell - zI_{\mathcal{H}})^{-1} - (A_\ell^{(0)} - zI_{\mathcal{H}})^{-1}\big) = -\int_{\mathbb{R}} \frac{\xi(\lambda; A_\ell, A_\ell^{(0)})}{(\lambda - z)^2}\, d\lambda,& \\ z \in \rho(A_\ell) \cap \rho(A_\ell^{(0)}), \ \ell \in \mathbb{N},& \end{aligned} \tag{B.20}$$

and are determined uniquely (a.e.) by the conditions

$$\begin{aligned} &\xi(\lambda; A, A^{(0)}) = 0, \quad \lambda < \inf[\sigma(A) \cup \sigma(A^{(0)})], \\ &\xi(\,\cdot\,; A, A^{(0)}) \in L^1(\mathbb{R}; (1+\lambda^2)^{-1}), \end{aligned} \tag{B.21}$$

and

$$\begin{aligned} &\xi(\lambda; A_\ell, A_\ell^{(0)}) = 0, \quad \lambda < \inf[\sigma(A_\ell) \cup \sigma(A_\ell^{(0)})], \\ &\xi(\,\cdot\,; A_\ell, A_\ell^{(0)}) \in L^1(\mathbb{R}; (1+\lambda^2)^{-1}), \quad \ell \in \mathbb{N}. \end{aligned} \tag{B.22}$$

Moreover, for a large class of functions f, for example, any f such that the Fourier transform $\hat{f} \in L^1(\mathbb{R}; (1+|p|)\,dp)$, Krein's trace formula holds:

$$\operatorname{tr}_{\mathcal{H}}(f(A) - f(A^{(0)})) = \int_{\mathbb{R}} f'(\lambda)\,\xi(\lambda; A, A^{(0)})\,d\lambda, \tag{B.23}$$

$$\operatorname{tr}_{\mathcal{H}}(f(A_\ell) - f(A_\ell^{(0)})) = \int_{\mathbb{R}} f'(\lambda)\,\xi(\lambda; A_\ell, A_\ell^{(0)})\,d\lambda, \quad \ell \in \mathbb{N}, \tag{B.24}$$

We refer to [25, Chapter 8] for these, as well as many other, properties of spectral shift functions.

Under the assumptions in Hypothesis B.1, the following convergence results hold for the sequence of spectral shift functions $\{\xi(\,\cdot\,; A_\ell, A_\ell^{(0)})\}_{\ell=1}^{\infty}$.

Theorem B.2 (Theorem 3.13 in [9])**.** *Assume Hypothesis* B.1*. Then*

$$\lim_{\ell\to\infty} \int_{\mathbb{R}} \frac{\xi(\lambda; A_\ell, A_\ell^{(0)})}{\lambda^2+1} f(\lambda)\,d\lambda = \int_{\mathbb{R}} \frac{\xi(\lambda; A, A^{(0)})}{\lambda^2+1} f(\lambda)\,d\lambda, \quad f \in C_b(\mathbb{R}). \tag{B.25}$$

The factor $(1+\lambda^2)^{-1}$ is essential in (B.25). Without it, the integrals need not exist. As a consequence of Theorem B.2, $\{\xi(\,\cdot\,; A_\ell, A_\ell^{(0)})\}_{\ell=1}^{\infty}$ converges vaguely to $\xi(\,\cdot\,; A, A^{(0)})$ as $\ell \to \infty$, which is the content of the following corollary.

Corollary B.3 (Corollary 3.11 in [9])**.** *Assume Hypothesis* B.1 *and let* $g \in C_0(\mathbb{R})$*. Then*

$$\lim_{\ell\to\infty} \int_{\mathbb{R}} \xi(\lambda; A_\ell, A_\ell^{(0)})\,g(\lambda)\,d\lambda = \int_{\mathbb{R}} \xi(\lambda; A, A^{(0)})\,g(\lambda)\,d\lambda. \tag{B.26}$$

Actually, the continuity assumption in Theorem B.2 may be relaxed, as in the following result.

Corollary B.4 (Corollary 3.14 in [9])**.** *Assume Hypothesis* B.1*. Then convergence in* (B.25) *holds for any bounded Borel measurable function that is continuous almost everywhere with respect to Lebesgue measure on* $\mathbb{R}$*. In particular,*

$$\lim_{\ell\to\infty} \int_S \frac{\xi(\lambda; A_\ell, A_\ell^{(0)})}{1+\lambda^2}\,d\lambda = \int_S \frac{\xi(\lambda; A, A^{(0)})}{1+\lambda^2}\,d\lambda \tag{B.27}$$

holds for any set $S \subseteq \mathbb{R}$ *that is boundaryless with respect to Lebesgue measure* (*i.e., any set* $S \subseteq \mathbb{R}$ *for which the boundary of* S *has Lebesgue measure equal to zero*)*.*

Corollary B.5 (Corollary 3.15 in [9])**.** *Assume Hypothesis* B.1*. If* g *is a bounded Borel measurable function that is compactly supported and Lebesgue almost everywhere continuous on* $\mathbb{R}$*, then*

$$\lim_{\ell\to\infty} \int_{\mathbb{R}} \xi(\lambda; A_\ell, A_\ell^{(0)})\,g(\lambda)\,d\lambda = \int_{\mathbb{R}} \xi(\lambda; A, A^{(0)})\,g(\lambda)\,d\lambda. \tag{B.28}$$

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