A JOST–PAIS-TYPE REDUCTION OF (MODIFIED) FREDHOLM DETERMINANTS FOR SEMI-SEPARABLE OPERATORS IN INFINITE DIMENSIONS

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Dedicated with great pleasure to Lev Aronovich Sakhnovich on the occasion of his 80th birthday.

Abstract. We study the analog of semi-separable integral kernels in $\mathcal{H}$ of the type

$$K(x,x') = \begin{cases} F_1(x)G_1(x'), & a < x' < x < b, \\ F_2(x)G_2(x'), & a < x < x' < b, \end{cases}$$

where $-\infty \leqslant a < b \leqslant \infty$, and for a.e. $x \in (a,b)$, $F_j(x) \in \mathcal{B}_2(\mathcal{H}_j, \mathcal{H})$ and $G_j(x) \in \mathcal{B}_2(\mathcal{H}, \mathcal{H}_j)$ such that $F_j(\cdot)$ and $G_j(\cdot)$ are uniformly measurable, and

$$\|F_j(\cdot)\|_{\mathcal{B}_2(\mathcal{H}_j,\mathcal{H})} \in L^2((a,b)), \ \|G_j(\cdot)\|_{\mathcal{B}_2(\mathcal{H},\mathcal{H}_j)} \in L^2((a,b)), \quad j = 1,2,$$

with $\mathcal{H}$ and $\mathcal{H}_j$, $j = 1,2$, complex, separable Hilbert spaces. Assuming that $K(\cdot,\cdot)$ generates a Hilbert–Schmidt operator $\boldsymbol{K}$ in $L^2((a,b);\mathcal{H})$, we derive the analog of the Jost–Pais reduction theory that succeeds in proving that the modified Fredholm determinant $\det_{2,L^2((a,b);\mathcal{H})}(\boldsymbol{I} - \alpha\boldsymbol{K})$, $\alpha \in \mathbb{C}$, naturally reduces to appropriate Fredholm determinants in the Hilbert spaces $\mathcal{H}$ (and $\mathcal{H} \oplus \mathcal{H}$).

Some applications to Schrödinger operators with operator-valued potentials are provided.

1. Introduction

Lev A. Sakhnovich's contributions to analysis in general are legendary, including, in particular, fundamental results in interpolation theory, spectral and inverse spectral theory, canonical systems, integrable systems and nonlinear evolution equations, integral equations, stochastic processes, applications to statistical physics, and the list goes on and on (see, e.g., [30]–[34], and the literature cited therein). Since integral operators frequently play a role in his research interests, we hope our modest contribution to semi-separable operators in infinite dimensions will create some joy for him.

The principal topic in this paper concerns semi-separable integral operators and their associated Fredholm determinants. In a nutshell, suppose that $\mathcal{H}$ and $\mathcal{H}_j$, $j = 1,2$, are complex, separable Hilbert spaces, that $-\infty \leqslant a < b \leqslant \infty$, and

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introduce the semi-separable integral kernel in $\mathcal{H}$,

$$K(x,x') = \begin{cases} F_1(x)G_1(x'), & a < x' < x < b, \\ F_2(x)G_2(x'), & a < x < x' < b, \end{cases}$$

where for a.e. $x \in (a,b)$, $F_j(x) \in \mathcal{B}_2(\mathcal{H}_j, \mathcal{H})$ and $G_j(x) \in \mathcal{B}_2(\mathcal{H}, \mathcal{H}_j)$ such that $F_j(\cdot)$ and $G_j(\cdot)$ are uniformly measurable (i.e., measurable with respect to the uniform operator topology), and

$$\|F_j(\cdot)\|_{\mathcal{B}_2(\mathcal{H}_j,\mathcal{H})} \in L^2((a,b)), \ \|G_j(\cdot)\|_{\mathcal{B}_2(\mathcal{H},\mathcal{H}_j)} \in L^2((a,b)), \quad j=1,2.$$

Assuming that $K(\cdot,\cdot)$ generates a Hilbert–Schmidt operator $\boldsymbol{K}$ in $L^2((a,b);\mathcal{H})$, we derive the analog of the Jost–Pais reduction theory that naturally reduces the modified Fredholm determinant $\det_{2,L^2((a,b);\mathcal{H})}(\boldsymbol{I} - \alpha\boldsymbol{K})$, $\alpha \in \mathbb{C}$, to appropriate Fredholm determinants in the Hilbert spaces $\mathcal{H}$ (and $\mathcal{H} \oplus \mathcal{H}$) as described in detail in Theorem 2.12 and Corollary 2.13. For instance, we will prove the following remarkable abstract version of the Jost–Pais-type reduction of modified Fredholm determinants [22] (see also [7], [11], [26], [36]),

$$\begin{aligned} &\det_{2,L^2((a,b);\mathcal{H})}(\boldsymbol{I} - \alpha\boldsymbol{K}) \\ &\quad = \det_{\mathcal{H}_1}\bigg(I_{\mathcal{H}_1} - \alpha\int_a^b dx\, G_1(x)\widehat{F}_1(x,\alpha)\bigg)\exp\bigg(\alpha\int_a^b dx\ \operatorname{tr}_{\mathcal{H}}(F_1(x)G_1(x))\bigg) \\ &\quad = \det_{\mathcal{H}_2}\bigg(I_{\mathcal{H}_2} - \alpha\int_a^b dx\, G_2(x)\widehat{F}_2(x,\alpha)\bigg)\exp\bigg(\alpha\int_a^b dx\ \operatorname{tr}_{\mathcal{H}}(F_2(x)G_2(x))\bigg), \end{aligned} \tag{1.1}$$

in Theorem 2.12, where $\widehat{F}_1(\cdot;\alpha)$ and $\widehat{F}_2(\cdot;\alpha)$ are defined via the Volterra integral equations

$$\widehat{F}_1(x;\alpha) = F_1(x) - \alpha\int_x^b dx'\, H(x,x')\widehat{F}_1(x';\alpha), \tag{1.2}$$

$$\widehat{F}_2(x;\alpha) = F_2(x) + \alpha\int_a^x dx'\, H(x,x')\widehat{F}_2(x';\alpha). \tag{1.3}$$

The analog of (1.1) in the case where $\boldsymbol{K}$ is a trace class operator in $L^2((a,b);\mathcal{H})$ was recently derived in [5] (cf. Corollary 2.13).

Section 2 focuses on our abstract results on semi-separable operators in infinite dimensions and represents the bulk of this paper. In particular, we will derive (1.1) and additional variants of it in Theorem 2.12, the principal new result of this paper. Section 3 then presents some applications to Schrödinger operators with operator-valued potentials on $\mathbb{R}$ and $(0,\infty)$.

2. Semiseparable Operators and Reduction Theory for Fredholm Determinants

In this section we describe one of the basic tools in this paper: a reduction theory for (modified) Fredholm determinants that permits one to reduce (modified) Fredholm determinants in the Hilbert space $L^2((a,b);\mathcal{H})$ to those in the Hilbert space $\mathcal{H}$, as described in detail in Theorem 2.12 and in Corollary 2.13. More precisely, we focus on a particular set of Hilbert–Schmidt operators $\boldsymbol{K}$ in $L^2((a,b);\mathcal{H})$ with $\mathcal{B}(\mathcal{H})$-valued semi-separable integral kernels (with $\mathcal{H}$ a complex, separable Hilbert space, generally of infinite dimension) and show how to naturally reduce the Fredholm determinant $\det_{2,L^2((a,b);\mathcal{H})}(\boldsymbol{I}-\alpha\boldsymbol{K})$, $\alpha\in\mathbb{C}$, to appropriate Fredholm determinants in Hilbert spaces $\mathcal{H}$ and $\mathcal{H}\oplus\mathcal{H}$ (in fact, we will describe a slightly more general framework below).

In our treatment we closely follow the approaches presented in Gohberg, Goldberg, and Kaashoek [14, Ch. IX] and Gohberg, Goldberg, and Krupnik [17, Ch. XIII] (see also [18]), and especially, in [11], where the particular case $\dim(\mathcal{H})<\infty$ was treated in detail. Our treatment of the case $\dim(\mathcal{H})=\infty$ in this section closely follows the one in [5] in the case where $\boldsymbol{K}$ is a trace class operator in $L^2((a,b);\mathcal{H})$.

Next, we briefly summarize some of the notation used in this section: $\mathcal{H}$ and $\mathcal{K}$ denote separable, complex Hilbert spaces, $(\cdot,\cdot)_{\mathcal{H}}$ represents the scalar product in $\mathcal{H}$ (linear in the second argument), and $I_{\mathcal{H}}$ is the identity operator in $\mathcal{H}$.

If T is a linear operator mapping (a subspace of) a Hilbert space into another, then $\operatorname{dom}(T)$ and $\ker(T)$ denote the domain and kernel (i.e., null space) of T. The closure of a closable operator S is denoted by $\overline{S}$. The spectrum, essential spectrum, and resolvent set of a closed linear operator in a Hilbert space will be denoted by $\sigma(\cdot)$, $\sigma_{ess}(\cdot)$, and $\rho(\cdot)$, respectively.

The Banach spaces of bounded and compact linear operators between complex, separable Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ are denoted by $\mathcal{B}(\mathcal{H},\mathcal{K})$ and $\mathcal{B}_\infty(\mathcal{H},\mathcal{K})$, respectively, and the corresponding ℓ^p-based trace ideals will be denoted by $\mathcal{B}_p(\mathcal{H},\mathcal{K})$, $p>0$. When $\mathcal{H}=\mathcal{K}$, we simply write $\mathcal{B}(\mathcal{H})$, $\mathcal{B}_\infty(\mathcal{H})$ and $\mathcal{B}_p(\mathcal{H})$, $p>0$, respectively. The spectral radius of $T\in\mathcal{B}(\mathcal{H},\mathcal{K})$ is denoted by $\operatorname{spr}(T)$. Moreover, $\det_{\mathcal{H}}(I_{\mathcal{H}}-A)$, and $\operatorname{tr}_{\mathcal{H}}(A)$ denote the standard Fredholm determinant and the corresponding trace of a trace class operator $A\in\mathcal{B}_1(\mathcal{H})$. Modified Fredholm determinants are denoted by $\det_{k,\mathcal{H}}(I_{\mathcal{H}}-A)$, $A\in\mathcal{B}_k(\mathcal{H})$, $k\in\mathbb{N}$, $k\geqslant 2$.

For reasons of brevity, for operator-valued functions that are measurable with respect to the uniform operator topology, we typically use the short cut uniformly measurable.

Before setting up the basic formalism for this section, we state the following elementary result:

Lemma 2.1. *Let $\mathcal{H}$ and $\mathcal{H}'$ be complex, separable Hilbert spaces and and $-\infty\leqslant a<b\leqslant\infty$. Suppose that for a.e. $x\in(a,b)$, $F(x)\in\mathcal{B}(\mathcal{H}',\mathcal{H})$ and $G(x)\in\mathcal{B}(\mathcal{H},\mathcal{H}')$ with $F(\cdot)$ and $G(\cdot)$ uniformly measurable, and*

$$\|F(\cdot)\|_{\mathcal{B}(\mathcal{H}',\mathcal{H})}\in L^2((a,b)),\ \|G(\cdot)\|_{\mathcal{B}(\mathcal{H},\mathcal{H}')}\in L^2((a,b)). \tag{2.1}$$

Consider the integral operator $\boldsymbol{S}$ in $L^2((a,b);\mathcal{H})$ with $\mathcal{B}(\mathcal{H})$-valued separable integral kernel of the type

$$S(x,x')=F(x)G(x')\ \text{ for a.e. } x,x'\in(a,b). \tag{2.2}$$

Then

$$\boldsymbol{S}\in\mathcal{B}\big(L^2((a,b);\mathcal{H})\big). \tag{2.3}$$

Proof. Let $f\in L^2((a,b);\mathcal{H})$, then for a.e. $x\in(a,b)$, and any integral operator $\boldsymbol{T}$ in $L^2((a,b);\mathcal{H})$ with $\mathcal{B}(\mathcal{H})$-valued integral kernel $T(\cdot\,,\cdot)$, one obtains

$$\begin{aligned}\|(\boldsymbol{T}f)(x)\|_{\mathcal{H}}&\leqslant\int_a^b dx'\,\|T(x,x')\|_{\mathcal{B}(\mathcal{H})}\|f(x')\|_{\mathcal{H}}\\&\leqslant\bigg(\int_a^b dx'\,\|T(x,x')\|^2_{\mathcal{B}(\mathcal{H})}\bigg)^{1/2}\bigg(\int_a^b dx''\,\|f(x'')\|^2_{\mathcal{H}}\bigg)^{1/2},\end{aligned} \tag{2.4}$$

and hence,

$$\int_a^b dx\, \|(\boldsymbol{T}f)(x)\|_{\mathcal{H}}^2 \leqslant \bigg[\int_a^b dx \int_a^b dx'\, \|T(x,x')\|_{\mathcal{B}(\mathcal{H})}^2\bigg] \int_a^b dx''\, \|f(x'')\|_{\mathcal{H}}^2, \tag{2.5}$$

yields $\boldsymbol{T} \in \mathcal{B}(L^2((a,b);\mathcal{H}))$ whenever $\Big[\int_a^b dx \int_a^b dx'\, \|T(x,x')\|_{\mathcal{B}(\mathcal{H})}^2\Big] < \infty$, implying

$$\|\boldsymbol{T}\|_{\mathcal{B}(L^2((a,b);\mathcal{H}))} \leqslant \bigg(\int_a^b dx \int_a^b dx'\, \|T(x,x')\|_{\mathcal{B}(\mathcal{H})}^2\bigg)^{1/2}. \tag{2.6}$$

Thus, using the special form (2.2) of $\boldsymbol{S}$ implies

$$\begin{aligned} \|\boldsymbol{S}\|_{\mathcal{B}(L^2((a,b);\mathcal{H}))}^2 &\leqslant \int_a^b dx \int_a^b dx'\, \|S(x,x')\|_{\mathcal{B}(\mathcal{H})}^2 \\ &= \int_a^b dx \int_a^b dx'\, \|F(x)G(x')\|_{\mathcal{B}(\mathcal{H})}^2 \\ &\leqslant \int_a^b dx\, \|F(x)\|_{\mathcal{B}(\mathcal{H}',\mathcal{H})}^2 \int_a^b dx'\, \|G(x')\|_{\mathcal{B}(\mathcal{H},\mathcal{H}')}^2 < \infty. \end{aligned} \tag{2.7}$$

□

At this point we now make the following initial set of assumptions:

Hypothesis 2.2. *Let $\mathcal{H}$ and $\mathcal{H}_j$, $j=1,2$, be complex, separable Hilbert spaces and $-\infty \leqslant a < b \leqslant \infty$. Suppose that for a.e. $x \in (a,b)$, $F_j(x) \in \mathcal{B}(\mathcal{H}_j,\mathcal{H})$ and $G_j(x) \in \mathcal{B}(\mathcal{H},\mathcal{H}_j)$ such that $F_j(\cdot)$ and $G_j(\cdot)$ are uniformly measurable, and*

$$\|F_j(\cdot)\|_{\mathcal{B}(\mathcal{H}_j,\mathcal{H})} \in L^2((a,b)),\ \|G_j(\cdot)\|_{\mathcal{B}(\mathcal{H},\mathcal{H}_j)} \in L^2((a,b)), \quad j=1,2. \tag{2.8}$$

Given Hypothesis 2.2, we introduce in $L^2((a,b);\mathcal{H})$ the operator

$$(\boldsymbol{K}f)(x) = \int_a^b dx'\, K(x,x')f(x') \text{ for a.e. } x \in (a,b),\ f \in L^2((a,b);\mathcal{H}), \tag{2.9}$$

with $\mathcal{B}(\mathcal{H})$-valued semi-separable integral kernel $K(\cdot,\cdot)$ defined by

$$K(x,x') = \begin{cases} F_1(x)G_1(x'), & a < x' < x < b, \\ F_2(x)G_2(x'), & a < x < x' < b. \end{cases} \tag{2.10}$$

The operator $\boldsymbol{K}$ is bounded,

$$\boldsymbol{K} \in \mathcal{B}\big(L^2((a,b);\mathcal{H})\big). \tag{2.11}$$

In fact, using (2.6) and (2.10), one readily verifies

$$\begin{aligned} \int_a^b dx \int_a^b dx' \|K(x,x')\|_{\mathcal{B}(\mathcal{H})}^2 &= \int_a^b dx \bigg(\int_a^x + \int_x^b\bigg) dx' \|K(x,x')\|_{\mathcal{B}(\mathcal{H})}^2 \\ \leqslant \sum_{j=1}^2 \int_a^b dx \|F_j(x)\|_{\mathcal{B}(\mathcal{H}_j,\mathcal{H})}^2 &\int_a^b dx' \|G_j(x')\|_{\mathcal{B}(\mathcal{H},\mathcal{H}_j)}^2 < \infty. \end{aligned} \tag{2.12}$$

Associated with $\boldsymbol{K}$ we also introduce the bounded Volterra operators $\boldsymbol{H}_a$ and $\boldsymbol{H}_b$ in $L^2((a,b);\mathcal{H})$ defined by

$$(\boldsymbol{H}_a f)(x) = \int_a^x dx'\, H(x,x')f(x'), \tag{2.13}$$

$$(\boldsymbol{H}_b f)(x) = -\int_x^b dx'\, H(x,x')f(x'); \quad f \in L^2((a,b);\mathcal{H}), \tag{2.14}$$

with $\mathcal{B}(\mathcal{H})$-valued (triangular) integral kernel

$$H(x,x') = F_1(x)G_1(x') - F_2(x)G_2(x'). \tag{2.15}$$

Moreover, introducing the bounded operator block matrices[1]

$$C(x) = (F_1(x) \;\; F_2(x)), \tag{2.16}$$

$$B(x) = (G_1(x) \;\; -G_2(x))^\top, \tag{2.17}$$

one verifies

$$H(x,x') = C(x)B(x'), \;\text{ where } \begin{cases} a < x' < x < b & \text{for } \boldsymbol{H}_a, \\ a < x < x' < b & \text{for } \boldsymbol{H}_b, \end{cases} \tag{2.18}$$

and

$$K(x,x') = \begin{cases} C(x)(I_{\mathcal{H}_1\oplus\mathcal{H}_2} - P_0)B(x'), & a < x' < x < b, \\ -C(x)P_0B(x'), & a < x < x' < b, \end{cases} \tag{2.19}$$

with

$$P_0 = \begin{pmatrix} 0 & 0 \\ 0 & I_{\mathcal{H}_2} \end{pmatrix}. \tag{2.20}$$

The next result proves that, as expected, $\boldsymbol{H}_a$ and $\boldsymbol{H}_b$ are quasi-nilpotent (i.e., have vanishing spectral radius) in $L^2((a,b);\mathcal{H})$:

Lemma 2.3. *Assume Hypothesis* 2.2. *Then $\boldsymbol{H}_a$ and $\boldsymbol{H}_b$ are quasi-nilpotent in $L^2((a,b);\mathcal{H})$, equivalently,*

$$\sigma(\boldsymbol{H}_a) = \sigma(\boldsymbol{H}_b) = \{0\}. \tag{2.21}$$

Proof. It suffices to discuss $\boldsymbol{H}_a$. Then estimating the norm of $H_a^n(x,x')$, $n \in \mathbb{N}$, (i.e., the integral kernel for $\boldsymbol{H}_a^n$) in a straightforward manner (cf. (2.13), (2.15)) yields for a.e. $x, x' \in (a,b)$,

$$\begin{aligned} \big\|H_a^n(x,x')\big\|_{\mathcal{B}(\mathcal{H})} &\leqslant 2^n \max_{j=1,2}\big(\|F_j(x)\|_{\mathcal{B}(\mathcal{H}_j,\mathcal{H})}\big) \max_{k=1,2}\big(\|G_k(x')\|_{\mathcal{B}(\mathcal{H},\mathcal{H}_k)}\big) \\ &\quad\times \frac{1}{(n-1)!}\bigg[\int_a^x dx'' \max_{1\leqslant \ell,m\leqslant 2}\big(\|G_\ell(x'')\|_{\mathcal{B}(\mathcal{H},\mathcal{H}_\ell)}\|F_m(x'')\|_{\mathcal{B}(\mathcal{H}_m,\mathcal{H})}\big)\bigg]^{(n-1)}, \\ &\hspace{20em} n \in \mathbb{N}. \end{aligned} \tag{2.22}$$

Thus, applying (2.6), one verifies

$$\begin{aligned} \big\|\boldsymbol{H}_a^n\big\|_{\mathcal{B}(L^2((a,b);\mathcal{H}))} &\leqslant \bigg(\int_a^b dx \int_a^b dx'\, \|H_a^n(x,x')\|^2_{\mathcal{B}(\mathcal{H})}\bigg)^{1/2} \\ &\leqslant \max_{j=1,2}\bigg(\int_a^b dx\, \|F_j(x)\|_{\mathcal{B}(\mathcal{H}_j,\mathcal{H})}\bigg)^{1/2} \max_{k=1,2}\bigg(\int_a^b dx'\, \|G_k(x')\|_{\mathcal{B}(\mathcal{H},\mathcal{H}_k)}\bigg)^{1/2} \\ &\quad\times \frac{2^n}{(n-1)!} \max_{1\leqslant\ell,m\leqslant 2}\bigg(\int_a^b dx''\|G_\ell(x'')\|_{\mathcal{B}(\mathcal{H},\mathcal{H}_\ell)}\|F_m(x'')\|_{\mathcal{B}(\mathcal{H}_m,\mathcal{H})}\bigg)^{(n-1)}, \\ &\hspace{20em} n \in \mathbb{N}, \end{aligned} \tag{2.23}$$

and hence

$$\operatorname{spr}(\boldsymbol{H}_a) = \lim_{n\to\infty}\big\|\boldsymbol{H}_a^n\big\|^{1/n}_{\mathcal{B}(L^2((a,b);\mathcal{H}))} = 0 \tag{2.24}$$

[1] $M^\top$ denotes the transpose of the operator matrix M.

(where $\operatorname{spr}(\,\cdot\,)$ abbreviates the spectral radius). Thus, $\boldsymbol{H}_a$ and $\boldsymbol{H}_b$ are quasi-nilpotent in $L^2((a,b);\mathcal{H})$ which in turn is equivalent to (2.21). $\square$

Next, introducing the linear maps

$$Q\colon \mathcal{H}_2 \mapsto L^2((a,b);\mathcal{H}), \quad (Qw)(x) = F_2(x)w, \quad w \in \mathcal{H}_2, \tag{2.25}$$

$$R\colon L^2((a,b);\mathcal{H}) \mapsto \mathcal{H}_2, \quad (Rf) = \int_a^b dx'\, G_2(x')f(x'), \quad f \in L^2((a,b);\mathcal{H}), \tag{2.26}$$

$$S\colon \mathcal{H}_1 \mapsto L^2((a,b);\mathcal{H}), \quad (Sv)(x) = F_1(x)v, \quad v \in \mathcal{H}_1, \tag{2.27}$$

$$T\colon L^2((a,b);\mathcal{H}) \mapsto \mathcal{H}_1, \quad (Tf) = \int_a^b dx'\, G_1(x')f(x'), \quad f \in L^2((a,b);\mathcal{H}), \tag{2.28}$$

one easily verifies the following elementary result (cf. [14, Sect. IX.2], [17, Sect. XIII.6] in the case $\dim(\mathcal{H}) < \infty$):

Lemma 2.4. *Assume Hypothesis* 2.2. *Then*

$$\boldsymbol{K} = \boldsymbol{H}_a + QR \tag{2.29}$$

$$= \boldsymbol{H}_b + ST. \tag{2.30}$$

To describe the inverse of $\boldsymbol{I} - \alpha\boldsymbol{H}_a$ and $\boldsymbol{I} - \alpha\boldsymbol{H}_b$, $\alpha \in \mathbb{C}$, one introduces the block operator matrix $A(\cdot)$ in $\mathcal{H}_1 \oplus \mathcal{H}_2$

$$A(x) = \begin{pmatrix} G_1(x)F_1(x) & G_1(x)F_2(x) \\ -G_2(x)F_1(x) & -G_2(x)F_2(x) \end{pmatrix} \tag{2.31}$$

$$= B(x)C(x) \text{ for a.e. } x \in (a,b) \tag{2.32}$$

and considers the linear evolution equation in $\mathcal{H}_1 \oplus \mathcal{H}_2$,

$$\begin{cases} u'(x) = \alpha A(x)u(x), \quad \alpha \in \mathbb{C}, \text{ for a.e. } x \in (a,b), \\ u(x_0) = u_0 \in \mathcal{H}_1 \oplus \mathcal{H}_2 \end{cases} \tag{2.33}$$

for some $x_0 \in (a,b)$. Since $A(x) \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ for a.e. $x \in (a,b)$, $A(\cdot)$ is uniformly measurable, and $\|A(\cdot)\|_{\mathcal{B}(\mathcal{H}_1\oplus\mathcal{H}_2)} \in L^1((a,b))$, Theorems 1.1 and 1.4 in [27] (see also [21], which includes a discussion of a nonlinear extension of (2.33)) apply and yield the existence of a unique propagator $U(\,\cdot\,,\,\cdot\,;\alpha)$ on $(a,b)\times(a,b)$ satisfying the following conditions:

$$U(\cdot\,,\cdot\,;\alpha) : (a,b)\times(a,b) \to \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2) \text{ is uniformly (i.e., norm) continuous.} \tag{2.34}$$

$$\text{There exists } C_\alpha > 0 \text{ such that for all } x,x' \in (a,b),\ \|U(x,x';\alpha)\|_{\mathcal{B}(\mathcal{H})} \leqslant C_\alpha. \tag{2.35}$$

$$\text{For all } x,x',x'' \in (a,b),\ U(x,x';\alpha)U(x',x'';\alpha) = U(x,x'';\alpha), \tag{2.36}$$
$$U(x,x;\alpha) = I_{\mathcal{H}_1\oplus\mathcal{H}_2}.$$

For all $u \in \mathcal{H}_1 \oplus \mathcal{H}_2$, $\alpha \in \mathbb{C}$,

$$U(x,\cdot\,;\alpha)u, U(\cdot\,,x;\alpha)u \in W^{1,1}((a,b);\mathcal{H}_1 \oplus \mathcal{H}_2), \quad x \in (a,b), \tag{2.37}$$

and

$$\text{for a.e. } x \in (a,b),\ (\partial/\partial x)U(x,x';\alpha)u = \alpha A(x)U(x,x';\alpha)u, \quad x' \in (a,b), \tag{2.38}$$

$$\text{for a.e. } x' \in (a,b),\ (\partial/\partial x')U(x,x';\alpha)u = -\alpha U(x,x';\alpha)A(x')u, \quad x \in (a,b). \tag{2.39}$$

Hence, $u(\,\cdot\,;\alpha)$ defined by

$$u(x;\alpha) = U(x,x_0;\alpha)u_0, \quad x \in (a,b), \tag{2.40}$$

is the unique solution of (2.33), satisfying

$$u(\,\cdot\,;\alpha) \in W^{1,1}((a,b);\mathcal{H}_1 \oplus \mathcal{H}_2). \tag{2.41}$$

In fact, an explicit construction (including the proof of uniqueness and that of the properties of (2.34)–(2.39)) of $U(\cdot\,,\cdot\,;\alpha)$ can simply be obtained by a norm-convergent iteration of

$$U(x,x';\alpha) = I_{\mathcal{H}_1\oplus\mathcal{H}_2} + \alpha \int_{x'}^{x} dx''\, A(x'')U(x'',x';\alpha), \quad x,x' \in (a,b). \tag{2.42}$$

Moreover, because of the integrability assumptions made in Hypothesis 2.2, (2.33)-(2.42) extend to $x,x' \in [a,b)$ (resp., $x,x' \in (a,b]$) if $a > -\infty$ (resp., $b < \infty$) and permit taking norm limits of $U(x,x';\alpha)$ as x,x' to $-\infty$ if $a = -\infty$ (resp., $+\infty$ if $b = +\infty$), see also Remark 2.6.

The next result appeared in [14, Sect. IX.2], [17, Sects. XIII.5, XIII.6] in the special case $\dim(\mathcal{H}) < \infty$:

Theorem 2.5. *Assume Hypothesis* 2.2. *Then,*
(i) $\boldsymbol{I} - \alpha\boldsymbol{H}_a$ *and* $\boldsymbol{I} - \alpha\boldsymbol{H}_b$ *are boundedly invertible for all* $\alpha \in \mathbb{C}$ *and*

$$(\boldsymbol{I} - \alpha\boldsymbol{H}_a)^{-1} = \boldsymbol{I} + \alpha\boldsymbol{J}_a(\alpha), \tag{2.43}$$

$$(\boldsymbol{I} - \alpha\boldsymbol{H}_b)^{-1} = \boldsymbol{I} + \alpha\boldsymbol{J}_b(\alpha), \tag{2.44}$$

$$(\boldsymbol{J}_a(\alpha)f)(x) = \int_a^x dx'\, J(x,x';\alpha)f(x'), \tag{2.45}$$

$$(\boldsymbol{J}_b(\alpha)f)(x) = -\int_x^b dx'\, J(x,x';\alpha)f(x'); \quad f \in L^2((a,b);\mathcal{H}), \tag{2.46}$$

$$J(x,x';\alpha) = C(x)U(x,x';\alpha)B(x'), \ \ \text{where} \ \begin{cases} a < x' < x < b & \text{for } \boldsymbol{J}_a(\alpha), \\ a < x < x' < b & \text{for } \boldsymbol{J}_b(\alpha). \end{cases} \tag{2.47}$$

(ii) Let $\alpha \in \mathbb{C}$. *Then* $\boldsymbol{I} - \alpha\boldsymbol{K}$ *is boundedly invertible if and only if* $I_{\mathcal{H}_2} - \alpha R(\boldsymbol{I} - \alpha\boldsymbol{H}_a)^{-1}Q$ *is. Similarly,* $\boldsymbol{I} - \alpha\boldsymbol{K}$ *is boundedly invertible if and only if* $I_{\mathcal{H}_1} - \alpha T(\boldsymbol{I} - \alpha\boldsymbol{H}_b)^{-1}S$ *is. In particular,*

$$(\boldsymbol{I} - \alpha\boldsymbol{K})^{-1} = (\boldsymbol{I} - \alpha\boldsymbol{H}_a)^{-1} + \alpha(\boldsymbol{I} - \alpha\boldsymbol{H}_a)^{-1}QR(\boldsymbol{I} - \alpha\boldsymbol{K})^{-1} \tag{2.48}$$

$$= (\boldsymbol{I} - \alpha\boldsymbol{H}_a)^{-1} + \alpha(\boldsymbol{I} - \alpha\boldsymbol{H}_a)^{-1}Q\big[I_{\mathcal{H}_2} - \alpha R(\boldsymbol{I} - \alpha\boldsymbol{H}_a)^{-1}Q\big]^{-1}R(\boldsymbol{I} - \alpha\boldsymbol{H}_a)^{-1} \tag{2.49}$$

$$= (\boldsymbol{I} - \alpha\boldsymbol{H}_b)^{-1} + \alpha(\boldsymbol{I} - \alpha\boldsymbol{H}_b)^{-1}ST(\boldsymbol{I} - \alpha\boldsymbol{K})^{-1} \tag{2.50}$$

$$= (\boldsymbol{I} - \alpha\boldsymbol{H}_b)^{-1} + \alpha(\boldsymbol{I} - \alpha\boldsymbol{H}_b)^{-1}S\big[I_{\mathcal{H}_1} - \alpha T(\boldsymbol{I} - \alpha\boldsymbol{H}_b)^{-1}S\big]^{-1}T(\boldsymbol{I} - \alpha\boldsymbol{H}_b)^{-1}. \tag{2.51}$$

Proof. To prove the results (2.43)–(2.47) it suffices to focus on $\boldsymbol{H}_a$. Let $f \in L^2((a,b);\mathcal{H})$. Then using $H(x,x') = C(x)B(x')$ and $A(x) = B(x)C(x)$ (cf. (2.18) and (2.32)) one computes (for some $x_0 \in (a,b)$) with the help of (2.38),

$$\big((\boldsymbol{I} - \alpha\boldsymbol{H}_a)(\boldsymbol{I} + \alpha\boldsymbol{J}_a(\alpha))f\big)(x) = f(x) - \alpha\int_a^x dx'\, C(x)B(x')f(x')$$

$$
\begin{aligned}
&+ \alpha \int_a^x dx'\, C(x)U(x,x';\alpha)B(x')f(x') \\
&- \alpha^2 \int_a^x dx'\, C(x)B(x') \int_a^{x'} dx''\, C(x')U(x',x'';\alpha)B(x'')f(x'') \\
&= f(x) - \alpha \int_a^x dx'\, C(x)B(x')f(x') + \alpha \int_a^x dx'\, C(x)U(x,x';\alpha)B(x')f(x') \\
&- \alpha^2 \int_a^x dx'\, C(x)B(x')C(x')U(x',x_0;\alpha) \int_a^{x'} dx''\, U(x_0,x'';\alpha)B(x'')f(x'') \\
&= f(x) - \alpha \int_a^x dx'\, C(x)B(x')f(x') + \alpha \int_a^x dx'\, C(x)U(x,x';\alpha)B(x')f(x') \\
&- \alpha \int_a^x dx'\, C(x)[(\partial/\partial x')U(x',x_0;\alpha)] \int_a^{x'} dx''\, U(x_0,x'';\alpha)B(x'')f(x'') \\
&= f(x) - \alpha \int_a^x dx'\, C(x)B(x')f(x') + \alpha \int_a^x dx'\, C(x)U(x,x';\alpha)B(x')f(x') \\
&- \alpha C(x)\bigg[U(x',x_0;\alpha) \int_a^{x'} dx''\, U(x_0,x'';\alpha)B(x'')f(x'')\bigg|_{x'=a}^{x} \\
&\qquad - \int_a^x dx'\, U(x',x_0;\alpha)U(x_0,x';\alpha)B(x')f(x')\bigg] \\
&= f(x) \ \text{ for a.e. } x \in (a,b). \qquad (2.52)
\end{aligned}
$$

In the same manner one proves

$$
\big((\boldsymbol{I} + \alpha \boldsymbol{J}_a(\alpha))(\boldsymbol{I} - \alpha \boldsymbol{H}_a)f\big)(x) = f(x) \ \text{ for a.e. } x \in (a,b). \qquad (2.53)
$$

By (2.29) and (2.30), $\boldsymbol{K} - \boldsymbol{H}_a$ and $\boldsymbol{K} - \boldsymbol{H}_b$ factor into QR and ST, respectively. Consequently, (2.48) and (2.50) follow from the second resolvent identity, while (2.49) and (2.51) are direct applications of Kato's resolvent equation for factored perturbations (cf. [10, Sect. 2]). □

Remark 2.6. Even though this will not be used in this paper, we mention for completeness that if $(\boldsymbol{I} - \alpha \boldsymbol{K})^{-1} \in \mathcal{B}\big(L^2((a,b);\mathcal{H})\big)$, and if $U(\cdot\,,a;\alpha)$ is defined by

$$
U(x,a;\alpha) = I_{\mathcal{H}_1 \oplus \mathcal{H}_2} + \alpha \int_a^x dx'\, A(x')U(x',a;\alpha), \quad x \in (a,b), \qquad (2.54)
$$

and partitioned with respect to $\mathcal{H}_1 \oplus \mathcal{H}_2$ as

$$
U(x,a;\alpha) = \begin{pmatrix} U_{1,1}(x,a;\alpha) & U_{1,2}(x,a;\alpha) \\ U_{2,1}(x,a;\alpha) & U_{2,2}(x,a;\alpha) \end{pmatrix}, \quad x \in (a,b), \qquad (2.55)
$$

then

$$
(\boldsymbol{I} - \alpha \boldsymbol{K})^{-1} = \boldsymbol{I} + \alpha \boldsymbol{L}(\alpha), \qquad (2.56)
$$

$$
(\boldsymbol{L}(\alpha)f)(x) = \int_a^b dx'\, L(x,x';\alpha)f(x'), \qquad (2.57)
$$

$$
L(x,x';\alpha) = \begin{cases} C(x)U(x,a;\alpha)(I - P(\alpha))U(x',a;\alpha)^{-1}B(x'), & a < x' < x < b, \\ -C(x)U(x,a;\alpha)P(\alpha)U(x',a;\alpha)^{-1}B(x'), & a < x < x' < b, \end{cases} \qquad (2.58)
$$

where

$$P(\alpha) = \begin{pmatrix} 0 & 0 \\ U_{2,2}(b,a;\alpha)^{-1}U_{2,1}(b,a;\alpha) & I_{\mathcal{H}_2} \end{pmatrix}, \tag{2.59}$$

with $U(b,a;\alpha) = \text{n-lim}_{x\uparrow b} U(x,a;\alpha)$. (Here n-lim abbreviates the limit in the norm topology.) These results can be shown as in the finite-dimensional case treated in [14, Ch. IX].

Lemma 2.7. *Assume Hypothesis* 2.2 *and introduce, for* $\alpha \in \mathbb{C}$ *and a.e.* $x \in (a,b)$, *the Volterra integral equations*

$$\widehat{F}_1(x;\alpha) = F_1(x) - \alpha \int_x^b dx'\, H(x,x')\widehat{F}_1(x';\alpha), \tag{2.60}$$

$$\widehat{F}_2(x;\alpha) = F_2(x) + \alpha \int_a^x dx'\, H(x,x')\widehat{F}_2(x';\alpha). \tag{2.61}$$

Then there exist unique a.e. solutions on (a,b), $\widehat{F}_j(\cdot\,;\alpha) \in \mathcal{B}(\mathcal{H}_j,\mathcal{H})$, *of* (2.60), (2.61) *such that* $\widehat{F}_j(\cdot\,;\alpha)$ *are uniformly measurable, and*

$$\big\|\widehat{F}_j(\cdot\,;\alpha)\big\|_{\mathcal{B}(\mathcal{H}_j,\mathcal{H})} \in L^2((a,b)), \quad j=1,2. \tag{2.62}$$

Proof. Introducing,

$$\begin{aligned} &\widehat{F}_{1,0}(x;\alpha) = F_1(x), \\ &\widehat{F}_{1,n}(x;\alpha) = -\alpha \int_x^b dx'\, H(x,x')\widehat{F}_{1,n-1}(x';\alpha), \quad n\in\mathbb{N}, \end{aligned} \tag{2.63}$$

$$\begin{aligned} &\widehat{F}_{2,0}(x;\alpha) = F_2(x), \\ &\widehat{F}_{2,n}(x;\alpha) = \alpha \int_a^x dx'\, H(x,x')\widehat{F}_{2,n-1}(x';\alpha), \quad n\in\mathbb{N}, \end{aligned} \tag{2.64}$$

for a.e. $x \in (a,b)$, the familiar iteration procedure (in the scalar or matrix-valued context) yields for fixed $x \in (a,b)$ except for a set of Lebesgue measure zero,

$$\begin{aligned} \big\|\widehat{F}_{1,n}(x;\alpha)\big\|_{\mathcal{B}(\mathcal{H}_1,\mathcal{H})} &\leqslant (2|\alpha|)^n \max_{j=1,2}\big(\|F_j(x)\|_{\mathcal{B}(\mathcal{H}_j,\mathcal{H})}\big) \\ &\quad\times \frac{1}{n!}\bigg[\int_x^b dx' \max_{1\leqslant k,\ell\leqslant 2}\big(\|G_k(x')\|_{\mathcal{B}(\mathcal{H},\mathcal{H}_k)}\|F_\ell(x')\|_{\mathcal{B}(\mathcal{H}_\ell,\mathcal{H})}\big)\bigg]^n, \quad n\in\mathbb{N}, \end{aligned} \tag{2.65}$$

$$\begin{aligned} \big\|\widehat{F}_{2,n}(x;\alpha)\big\|_{\mathcal{B}(\mathcal{H}_2,\mathcal{H})} &\leqslant (2|\alpha|)^n \max_{j=1,2}\big(\|F_j(x)\|_{\mathcal{B}(\mathcal{H}_j,\mathcal{H})}\big) \\ &\quad\times \frac{1}{n!}\bigg[\int_a^x dx' \max_{1\leqslant k,\ell\leqslant 2}\big(\|G_k(x')\|_{\mathcal{B}(\mathcal{H},\mathcal{H}_k)}\|F_\ell(x')\|_{\mathcal{B}(\mathcal{H}_\ell,\mathcal{H})}\big)\bigg]^n, \quad n\in\mathbb{N}. \end{aligned} \tag{2.66}$$

Thus, the norm convergent expansions

$$\widehat{F}_j(x;\alpha) = \sum_{n=0}^\infty \widehat{F}_{j,n}(x;\alpha), \quad j=1,2, \text{ for a.e. } x\in(a,b), \tag{2.67}$$

yield the bounds

$$\begin{aligned} \big\|\widehat{F}_j(x;\alpha)\big\|_{\mathcal{B}(\mathcal{H}_j,\mathcal{H})} &\leqslant \max_{k=1,2}\big(\|F_k(x)\|_{\mathcal{B}(\mathcal{H}_k,\mathcal{H})}\big) \\ &\quad\times \max_{1\leqslant \ell,m\leqslant 2} \exp\bigg(2|\alpha|\int_a^b dx' \|G_\ell(x')\|_{\mathcal{B}(\mathcal{H},\mathcal{H}_\ell)}\|F_m(x')\|_{\mathcal{B}(\mathcal{H}_m,\mathcal{H})}\bigg) \end{aligned} \tag{2.68}$$

for a.e. $x \in (a,b)$. As in the scalar case (resp., as in the proof of Theorem 2.5) one shows that (2.67) uniquely satisfies (2.60), (2.61) $\square$

Lemma 2.8. *Assume Hypothesis* 2.2*, let* $\alpha \in \mathbb{C}$*, and introduce*

$$U(x;\alpha) = \begin{pmatrix} I_{\mathcal{H}_1} - \alpha \int_x^b dx'\, G_1(x')\widehat{F}_1(x';\alpha) & \alpha \int_a^x dx'\, G_1(x')\widehat{F}_2(x';\alpha) \\ \alpha \int_x^b dx'\, G_2(x')\widehat{F}_1(x';\alpha) & I_{\mathcal{H}_2} - \alpha \int_a^x dx'\, G_2(x')\widehat{F}_2(x';\alpha) \end{pmatrix}, \quad x \in (a,b). \tag{2.69}$$

If

$$\left[I_{\mathcal{H}_1} - \alpha \int_a^b dx\, G_1(x)\widehat{F}_1(x;\alpha)\right]^{-1} \in \mathcal{B}(\mathcal{H}_1), \tag{2.70}$$

or equivalently,

$$\left[I_{\mathcal{H}_2} - \alpha \int_a^b dx\, G_2(x)\widehat{F}_2(x;\alpha)\right]^{-1} \in \mathcal{B}(\mathcal{H}_2), \tag{2.71}$$

then

$$U(a;\alpha),\, U(b;\alpha),\, U(x;\alpha),\ x \in (a,b), \tag{2.72}$$

are boundedly invertible in $\mathcal{H}_1 \oplus \mathcal{H}_2$*. In particular,*

$$U(x,x';\alpha) = U(x;\alpha)U(x';\alpha)^{-1}, \quad x,x' \in (a,b), \tag{2.73}$$

is the propagator for the evolution equation (2.33) *satisfying* (2.34)*–*(2.42)*, and* (2.73) *extends by norm continuity to* $x,x' \in \{a,b\}$*.*

Proof. Since

$$U(a;\alpha) = \begin{pmatrix} I_{\mathcal{H}_1} - \alpha \int_a^b dx'\, G_1(x')\widehat{F}_1(x';\alpha) & 0 \\ \alpha \int_a^b dx'\, G_2(x')\widehat{F}_1(x';\alpha) & I_{\mathcal{H}_2} \end{pmatrix}, \tag{2.74}$$

the operator $U(a;\alpha)$ is boundedly invertible in $\mathcal{H}_1 \oplus \mathcal{H}_2$ if and only if $\big[I_{\mathcal{H}_1} - \alpha \int_a^b dx'\, G_1(x')\widehat{F}_1(x';\alpha)\big]$ is boundedly invertible in $\mathcal{H}_1$. (One recalls that a bounded 2×2 block operator $D = \begin{pmatrix} D_{1,1} & 0 \\ D_{2,1} & I_{\mathcal{H}_2} \end{pmatrix}$ in $\mathcal{H}_1 \oplus \mathcal{H}_2$ is boundedly invertible if and only if $D_{1,1}$ is boundedly invertible in $\mathcal{H}_1$, with $D^{-1} = \begin{pmatrix} D_{1,1}^{-1} & 0 \\ -D_{2,1}D_{1,1}^{-1} & I_{\mathcal{H}_2} \end{pmatrix}$ if D is boundedly invertible.) Similarly,

$$U(b;\alpha) = \begin{pmatrix} I_{\mathcal{H}_1} & \alpha \int_a^b dx'\, G_1(x')\widehat{F}_2(x';\alpha) \\ 0 & I_{\mathcal{H}_2} - \alpha \int_a^b dx'\, G_2(x')\widehat{F}_2(x';\alpha) \end{pmatrix} \tag{2.75}$$

is boundedly invertible in $\mathcal{H}_1 \oplus \mathcal{H}_2$ if and only if $\big[I_{\mathcal{H}_2} - \alpha \int_a^b dx'\, G_2(x')\widehat{F}_2(x';\alpha)\big]$ is in $\mathcal{H}_2$. (Again, one recalls that a bounded 2×2 block operator $E = \begin{pmatrix} I_{\mathcal{H}_1} & E_{1,2} \\ 0 & E_{2,2} \end{pmatrix}$ in $\mathcal{H}_1 \oplus \mathcal{H}_2$ is boundedly invertible if and only if $E_{2,2}$ is boundedly invertible in $\mathcal{H}_2$, with $E^{-1} = \begin{pmatrix} I_{\mathcal{H}_1} & -E_{1,2}E_{2,2}^{-1} \\ 0 & E_{2,2}^{-1} \end{pmatrix}$ if E is boundedly invertible.)

The equivalence of (2.70) and (2.71) has been settled in Theorem 2.5 (ii).

Next, differentiating the entries on the right-hand side of (2.69) with respect to x and using the Volterra integral equations (2.60), (2.61) yields

$$(d/dx)U(x;\alpha)u = \alpha A(x)U(x;\alpha)u \text{ for a.e. } x \in (a,b). \tag{2.76}$$

Thus, by uniqueness of the propagator $U(\cdot\,,\cdot\,;\alpha)$, extended by norm continuity to $x = a$ (cf. Remark 2.6), one obtains that

$$U(x,a;\alpha) = U(x;\alpha)U(a;\alpha)^{-1}, \quad x \in (a,b). \tag{2.77}$$

Thus, $U(x;\alpha) = U(x,a;\alpha)U(a;\alpha)$ is boundedly invertible for all $x \in (a,b)$ since $U(x,a;\alpha)$, $x \in (a,b)$ is by construction (using norm continuity and the transitivity property in (2.36)), and $U(a;\alpha)$ is boundedly invertible by hypothesis. Consequently, once more by uniqueness of the propagator $U(\cdot\,,\cdot\,;\alpha)$, one obtains that

$$U(x,x';\alpha) = U(x;\alpha)U(x';\alpha)^{-1}, \quad x,x' \in (a,b). \tag{2.78}$$

Again by norm continuity, (2.78) extends to $x,x' \in \{a,b\}$. $\square$

In the special case where $\mathcal{H}$ and $\mathcal{H}_j$, $j = 1,2$, are finite-dimensional, the Volterra integral equations (2.60), (2.61) and the operator U in (2.69) were introduced in [11].

Lemma 2.9. *Let $\mathcal{H}$ and $\mathcal{H}'$ be complex, separable Hilbert spaces and $-\infty \leqslant a < b \leqslant \infty$. Suppose that for a.e. $x \in (a,b)$, $F(x) \in \mathcal{B}_2(\mathcal{H}',\mathcal{H})$ and $G(x) \in \mathcal{B}_2(\mathcal{H},\mathcal{H}')$ with $F(\cdot)$ and $G(\cdot)$ weakly measurable, and*

$$\|F(\cdot)\|_{\mathcal{B}_2(\mathcal{H}',\mathcal{H})} \in L^2((a,b)),\ \|G(\cdot)\|_{\mathcal{B}_2(\mathcal{H},\mathcal{H}')} \in L^2((a,b)). \tag{2.79}$$

Consider the integral operator $\boldsymbol{S}$ in $L^2((a,b);\mathcal{H})$ with $\mathcal{B}_1(\mathcal{H})$-valued separable integral kernel of the type

$$S(x,x') = F(x)G(x') \text{ for a.e. } x,x' \in (a,b). \tag{2.80}$$

Then

$$\boldsymbol{S} \in \mathcal{B}_1\big(L^2((a,b);\mathcal{H})\big). \tag{2.81}$$

Proof. Since the Hilbert space of Hilbert–Schmidt operators, $\mathcal{B}_2(\mathcal{H}',\mathcal{H})$, is separable, weak measurability of $F(\cdot)$ implies $\mathcal{B}_2(\mathcal{H}',\mathcal{H})$-measurability by Pettis' theorem (cf., e.g., [2, Theorem 1.1.1], [6, Theorem II.1.2], [20, 3.5.3]), and analogously for $G(\cdot)$.

Next, one introduces (in analogy to (2.25)–(2.28)) the linear operators

$$Q_F\colon \mathcal{H}' \mapsto L^2((a,b);\mathcal{H}), \quad (Q_F w)(x) = F(x)w, \quad w \in \mathcal{H}', \tag{2.82}$$

$$R_G\colon L^2((a,b);\mathcal{H}) \mapsto \mathcal{H}', \quad (R_G f) = \int_a^b dx'\, G(x')f(x'), \quad f \in L^2((a,b);\mathcal{H}), \tag{2.83}$$

such that

$$\boldsymbol{S} = Q_F R_G. \tag{2.84}$$

Thus, with $\{v_n\}_{n\in\mathbb{N}}$ a complete orthonormal system in $\mathcal{H}'$, using the monotone convergence theorem, one concludes that

$$\begin{aligned} \|Q_F\|^2_{\mathcal{B}_2(\mathcal{H}',L^2((a,b);\mathcal{H}))} &= \sum_{n\in\mathbb{N}} \|Q_F v_n\|^2_{L^2((a,b);\mathcal{H})} \\ &= \sum_{n\in\mathbb{N}} \int_a^b dx\, \|F(x)v_n\|^2_{\mathcal{H}} = \int_a^b dx\, \sum_{n\in\mathbb{N}} (v_n, F(x)^*F(x)v_n)_{\mathcal{H}'} \end{aligned}$$

$$= \int_a^b dx\, \operatorname{tr}_{\mathcal{H}'}\big(F(x)^* F(x)\big) = \int_a^b dx\, \big\|F(x)^* F(x)\big\|_{\mathcal{B}_1(\mathcal{H}')}$$
$$= \int_a^b dx\, \|F(x)\|^2_{\mathcal{B}_2(\mathcal{H}',\mathcal{H})} < \infty. \tag{2.85}$$

The same argument applied to R_G^* (which is of the form Q_{G^*}, i.e., given by (2.82) with $F(\cdot)$ replaced by $G(\cdot)^*$) then proves $R_G^* \in \mathcal{B}_2(\mathcal{H}', L^2((a,b);\mathcal{H}))$. Hence,

$$Q_F \in \mathcal{B}_2(\mathcal{H}', L^2((a,b);\mathcal{H})), \quad R_G \in \mathcal{B}_2(L^2((a,b);\mathcal{H}), \mathcal{H}'), \tag{2.86}$$

together with the factorization (2.84), prove (2.81). □

Next, we strengthen our assumptions as follows:

Hypothesis 2.10. *Let $\mathcal{H}$ and $\mathcal{H}_j$, $j=1,2$, be complex, separable Hilbert spaces and $-\infty \leqslant a < b \leqslant \infty$. Suppose that for a.e. $x \in (a,b)$, $F_j(x) \in \mathcal{B}_2(\mathcal{H}_j,\mathcal{H})$ and $G_j(x) \in \mathcal{B}_2(\mathcal{H},\mathcal{H}_j)$ such that $F_j(\cdot)$ and $G_j(\cdot)$ are weakly measurable, and*

$$\|F_j(\cdot)\|_{\mathcal{B}_2(\mathcal{H}_j,\mathcal{H})} \in L^2((a,b)),\ \|G_j(\cdot)\|_{\mathcal{B}_2(\mathcal{H},\mathcal{H}_j)} \in L^2((a,b)), \quad j=1,2. \tag{2.87}$$

As an immediate consequence of Hypothesis 2.10 one infers the following facts.

Lemma 2.11. *Assume Hypothesis 2.10 and $\alpha \in \mathbb{C}$. Then, for a.e. $x \in (a,b)$, $\widehat{F}_j(x;\alpha) \in \mathcal{B}_2(\mathcal{H}_j,\mathcal{H})$, $\widehat{F}_j(\cdot\,;\alpha)$ are $\mathcal{B}_2(\mathcal{H}_j,\mathcal{H})$-measurable, and*

$$\big\|\widehat{F}_j(\cdot\,;\alpha)\big\|_{\mathcal{B}_2(\mathcal{H}_j,\mathcal{H})} \in L^2((a,b)), \quad j=1,2. \tag{2.88}$$

Moreover,

$$\begin{split}\int_c^d dx\, G_j(x)F_k(x),\ \int_c^d dx\, G_j(x)\widehat{F}_k(x;\alpha) \in \mathcal{B}_1(\mathcal{H}_k,\mathcal{H}_j),\\ 1 \leqslant j,k \leqslant 2,\ c,d \in (a,b)\cup\{a,b\},\end{split} \tag{2.89}$$

and

$$QR, ST \in \mathcal{B}_1\big(L^2((a,b);\mathcal{H})\big), \tag{2.90}$$
$$\boldsymbol{K}, \boldsymbol{H}_a, \boldsymbol{H}_b \in \mathcal{B}_2\big(L^2((a,b);\mathcal{H})\big). \tag{2.91}$$

Moreover,

$$\begin{split}\operatorname{tr}_{L^2((a,b);\mathcal{H})}(QR) = \operatorname{tr}_{\mathcal{H}_2}(RQ) &= \int_a^b dx\ \operatorname{tr}_{\mathcal{H}_2}(G_2(x)F_2(x))\\ &= \int_a^b dx\ \operatorname{tr}_{\mathcal{H}}(F_2(x)G_2(x)),\end{split} \tag{2.92}$$

and

$$\begin{split}\operatorname{tr}_{L^2((a,b);\mathcal{H})}(ST) = \operatorname{tr}_{\mathcal{H}_1}(TS) &= \int_a^b dx\ \operatorname{tr}_{\mathcal{H}_1}(G_1(x)F_1(x))\\ &= \int_a^b dx\ \operatorname{tr}_{\mathcal{H}_1}(G_1(x)F_1(x)).\end{split} \tag{2.93}$$

Proof. As in the proof of Lemma 2.9, one concludes that weak measurability of $\widehat{F}_j(\cdot\,;\alpha)$, $j=1,2$, implies their $\mathcal{B}_2(\mathcal{H}_j,\mathcal{H})$-measurability by Pettis' theorem. The properties concerning $\widehat{F}_j(\cdot\,;\alpha)$, $j=1,2$, then follow as in the proof of Lemma 2.7, systematically replacing $\|\cdot\|_{\mathcal{B}(\mathcal{H}_j,\mathcal{H})}$ by $\|\cdot\|_{\mathcal{B}_2(\mathcal{H}_j,\mathcal{H})}$, $j=1,2$.

Applying Lemma 2.9, relations (2.89) are now an immediate consequence of Hypothesis 2.10 and the fact that

$$\big\|G_j(\cdot)\widehat{F}_k(\cdot\,;\alpha)\big\|_{\mathcal{B}_1(\mathcal{H}_k,\mathcal{H}_j)} \in L^1((a,b)), \quad 1 \leqslant j,k \leqslant 2. \tag{2.94}$$

The proof of Lemma 2.9 (see (2.85)) yields

$$\begin{aligned} &S \in \mathcal{B}_2(\mathcal{H}_1, L^2((a,b);\mathcal{H})),\ Q \in \mathcal{B}_2(\mathcal{H}_2, L^2((a,b);\mathcal{H})),\\ &T \in \mathcal{B}_2(L^2((a,b);\mathcal{H}),\mathcal{H}_1),\ R \in \mathcal{B}_2(L^2((a,b);\mathcal{H}),\mathcal{H}_2), \end{aligned} \tag{2.95}$$

and (2.90) follows.

Next, for any integral operator $\boldsymbol{T}$ in $L^2((a,b);\mathcal{H})$, with integral kernel satisfying $\|T(\cdot\,,\cdot)\|_{\mathcal{B}_2(\mathcal{H})} \in L^2((a,b)\times(a,b);d^2x)$, one infers (cf. [4, Theorem 11.6]) that $\boldsymbol{T} \in \mathcal{B}_2\big(L^2((a,b);\mathcal{H})\big)$ and

$$\|\boldsymbol{T}\|_{\mathcal{B}_2(L^2((a,b);\mathcal{H}))} = \bigg(\int_a^b dx \int_a^b dx'\, \|T(x,x')\|^2_{\mathcal{B}_2(\mathcal{H})}\bigg)^{1/2}. \tag{2.96}$$

Given Lemma 2.9 and the fact (2.96), one readily concludes (2.91).

Finally, the first equality in both (2.92) (resp., (2.93)) follows from cyclicity of the trace. The other equalities throughout (2.92) and (2.93) follow from computing appropriate traces. For example, taking an orthonormal basis $\{v_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}_2$, one computes

$$\begin{aligned} \operatorname{tr}_{\mathcal{H}_2}(RQ) &= \sum_{n\in\mathbb{N}} (v_n, RQv_n)_{\mathcal{H}_2} = \sum_{n\in\mathbb{N}} \bigg(v_n, \int_a^b dx\, G_2(x)(Qv_n)(x)\bigg)_{\mathcal{H}_2}\\ &= \int_a^b dx \sum_{n\in\mathbb{N}} (v_n, G_2(x)(Qv_n)(x))_{\mathcal{H}_2} = \int_a^b dx \sum_{n\in\mathbb{N}} (v_n, G_2(x)F_2(x)v_n)_{\mathcal{H}_2}\\ &= \int_a^b dx\, \operatorname{tr}_{\mathcal{H}_2}(G_2(x)F_2(x)). \end{aligned} \tag{2.97}$$

□

In the following we use many of the standard properties of Fredholm determinants, 2-modified Fredholm determinants, and traces. For the Fredholm determinant and trace,

$$\det_{\mathcal{K}}(I_{\mathcal{K}} - A) = \prod_{n\in\mathcal{J}} (1 - \lambda_n(A)), \quad A \in \mathcal{B}_1(\mathcal{K}), \tag{2.98}$$

where $\{\lambda_n(A)\}_{n\in\mathcal{J}}$ is an enumeration of the non-zero eigenvalues of A, listed in non-increasing order according to their moduli, and $\mathcal{J} \subseteq \mathbb{N}$ is an appropriate indexing set.

$$\det_{\mathcal{K}}((I_{\mathcal{K}} - A)(I_{\mathcal{K}} - B)) = \det_{\mathcal{K}}(I_{\mathcal{K}} - A)\det_{\mathcal{K}}(I_{\mathcal{K}} - B), \quad A, B \in \mathcal{B}_1(\mathcal{K}), \tag{2.99}$$

$$\det_{\mathcal{K}}(I_{\mathcal{K}} - AB) = \det_{\mathcal{K}'}(I_{\mathcal{K}'} - BA), \quad \operatorname{tr}_{\mathcal{K}}(AB) = \operatorname{tr}_{\mathcal{K}'}(BA) \tag{2.100}$$

for all $A \in \mathcal{B}_1(\mathcal{K}',\mathcal{K})$, $B \in \mathcal{B}(\mathcal{K},\mathcal{K}')$ such that $AB \in \mathcal{B}_1(\mathcal{K})$, $BA \in \mathcal{B}_1(\mathcal{K}')$,

and

$$\det_{\mathcal{K}}(I_{\mathcal{K}} - A) = \det_{\mathcal{K}_2}(I_{\mathcal{K}_2} - D) \ \text{ for } \ A = \begin{pmatrix} 0 & C \\ 0 & D \end{pmatrix},\ D \in \mathcal{B}_1(\mathcal{K}_2),\ \mathcal{K} = \mathcal{K}_1 \dotplus \mathcal{K}_2, \tag{2.101}$$

since

$$I_{\mathcal{H}} - A = \begin{pmatrix} I_{\mathcal{K}_1} & -C \\ 0 & I_{\mathcal{K}_2} - D \end{pmatrix} = \begin{pmatrix} I_{\mathcal{K}_1} & 0 \\ 0 & I_{\mathcal{K}_2} - D \end{pmatrix} \begin{pmatrix} I_{\mathcal{K}_1} & -C \\ 0 & I_{\mathcal{K}_2} \end{pmatrix}. \tag{2.102}$$

For 2-modified Fredholm determinants,

$$\det_{2,\mathcal{K}}(I_{\mathcal{K}} - A) = \prod_{n\in\mathcal{J}} (1 - \lambda_n(A))e^{\lambda_n(A)}, \quad A \in \mathcal{B}_2(\mathcal{K}), \tag{2.103}$$

where $\{\lambda_n(A)\}_{n\in\mathcal{J}}$ is an enumeration of the non-zero eigenvalues of A, listed in non-increasing order according to their moduli, and $\mathcal{J} \subseteq \mathbb{N}$ is an appropriate indexing set,

$$\det_{2,\mathcal{K}}(I_{\mathcal{K}} - A) = \det_{\mathcal{K}}((I_{\mathcal{K}} - A)\exp(A)), \quad A \in \mathcal{B}_2(\mathcal{K}), \tag{2.104}$$

$$\det_{2,\mathcal{K}}((I_{\mathcal{K}} - A)(I_{\mathcal{K}} - B)) = \det_{2,\mathcal{K}}(I_{\mathcal{K}} - A)\det_{2,\mathcal{K}}(I_{\mathcal{K}} - B)e^{-\operatorname{tr}_{\mathcal{K}}(AB)}, \quad A, B \in \mathcal{B}_2(\mathcal{K}), \tag{2.105}$$

$$\det_{2,\mathcal{K}}(I_{\mathcal{K}} - A) = \det_{\mathcal{K}}(I_{\mathcal{K}} - A)e^{\operatorname{tr}_{\mathcal{K}}(A)}, \quad A \in \mathcal{B}_1(\mathcal{K}). \tag{2.106}$$

Here $\mathcal{K}$, $\mathcal{K}'$, and $\mathcal{K}_j$, $j = 1, 2$, are complex, separable Hilbert spaces, $\mathcal{B}(\mathcal{K})$ denotes the set of bounded linear operators on $\mathcal{K}$, $\mathcal{B}_p(\mathcal{K})$, $p \geqslant 1$, denote the usual trace ideals of $\mathcal{B}(\mathcal{K})$, and $I_{\mathcal{K}}$ denotes the identity operator in $\mathcal{K}$. Moreover, $\det_{\mathcal{K}}(I_{\mathcal{K}} - A)$, $A \in \mathcal{B}_1(\mathcal{K})$, denotes the standard Fredholm determinant, with $\operatorname{tr}_{\mathcal{K}}(A)$, $A \in \mathcal{B}_1(\mathcal{K})$, the corresponding trace, and $\det_{2,\mathcal{K}}(I_{\mathcal{K}} - A)$ the 2-modified Fredholm determinant of a Hilbert–Schmidt operator $A \in \mathcal{B}_2(\mathcal{K})$. Finally, $\dotplus$ in (2.101) denotes a direct, but not necessary orthogonal, sum decomposition of $\mathcal{K}$ into $\mathcal{K}_1$ and $\mathcal{K}_2$. (We refer, e.g., to [15], [16], [17, Ch. XIII], [19, Sects. IV.1 & IV.2], [29, Ch. 17], [35], [37, Ch. 3] for these facts).

Theorem 2.12. *Assume Hypothesis* 2.10 *and let* $\alpha \in \mathbb{C}$. *Then*

$$\det_{2,L^2((a,b);\mathcal{H})}(\boldsymbol{I} - \alpha\boldsymbol{H}_a) = \det_{2,L^2((a,b);\mathcal{H})}(\boldsymbol{I} - \alpha\boldsymbol{H}_b) = 1. \tag{2.107}$$

Assume, in addition, that U *is given by* (2.69). *Then*

$$\begin{aligned}
&\det_{2,L^2((a,b);\mathcal{H})}(\boldsymbol{I} - \alpha\boldsymbol{K}) \\
&\quad = \det_{\mathcal{H}_1}\big(I_{\mathcal{H}_1} - \alpha T(\boldsymbol{I} - \alpha\boldsymbol{H}_b)^{-1}S\big)\exp\big(\alpha\operatorname{tr}_{L^2((a,b);\mathcal{H})}(ST)\big) && (2.108)\\
&\quad = \det_{\mathcal{H}_1}\bigg(I_{\mathcal{H}_1} - \alpha\int_a^b dx\, G_1(x)\widehat{F}_1(x,\alpha)\bigg)\exp\bigg(\alpha\int_a^b dx\, \operatorname{tr}_{\mathcal{H}}(F_1(x)G_1(x))\bigg) && (2.109)\\
&\quad = \det_{\mathcal{H}_1\oplus\mathcal{H}_2}(U(a,\alpha))\exp\bigg(\alpha\int_a^b dx\, \operatorname{tr}_{\mathcal{H}}(F_1(x)G_1(x))\bigg) && (2.110)\\
&\quad = \det_{\mathcal{H}_2}\big(I_{\mathcal{H}_2} - \alpha R(\boldsymbol{I} - \alpha\boldsymbol{H}_a)^{-1}Q\big)\exp\big(\alpha\operatorname{tr}_{L^2((a,b);\mathcal{H})}(QR)\big) && (2.111)\\
&\quad = \det_{\mathcal{H}_2}\bigg(I_{\mathcal{H}_2} - \alpha\int_a^b dx\, G_2(x)\widehat{F}_2(x,\alpha)\bigg)\exp\bigg(\alpha\int_a^b dx\, \operatorname{tr}_{\mathcal{H}}(F_2(x)G_2(x))\bigg) && (2.112)\\
&\quad = \det_{\mathcal{H}_1\oplus\mathcal{H}_2}(U(b;\alpha))\exp\bigg(\alpha\int_a^b dx\, \operatorname{tr}_{\mathcal{H}}(F_2(x)G_2(x))\bigg). && (2.113)
\end{aligned}$$

Proof. Since $\boldsymbol{H}_a$ and $\boldsymbol{H}_b$ are quasi-nilpotent, they have no non-zero eigenvalues. Therefore, (2.107) follows from the representation of the 2-modified Fredholm determinant given in (2.103).

Next, one observes

$$\boldsymbol{I} - \alpha\boldsymbol{K} = (\boldsymbol{I} - \alpha\boldsymbol{H}_a)[\boldsymbol{I} - \alpha(\boldsymbol{I} - \alpha\boldsymbol{H}_a)^{-1}QR] \tag{2.114}$$

$$= (\boldsymbol{I} - \alpha\boldsymbol{H}_b)[\boldsymbol{I} - \alpha(\boldsymbol{I} - \alpha\boldsymbol{H}_b)^{-1}ST]. \tag{2.115}$$

Using the various properties of determinants given in (2.104)–(2.106) and (2.115), one computes

$$\begin{aligned}
&{\det}_{2,L^2((a,b);\mathcal{H})}(\boldsymbol{I} - \alpha\boldsymbol{K}) \\
&\quad = {\det}_{2,L^2((a,b);\mathcal{H})}\big((\boldsymbol{I} - \alpha\boldsymbol{H}_b)\big[\boldsymbol{I} - \alpha(\boldsymbol{I} - \boldsymbol{H}_b)^{-1}ST\big]\big) \\
&\quad = {\det}_{2,L^2((a,b);\mathcal{H})}(\boldsymbol{I} - \alpha\boldsymbol{H}_b)\,{\det}_{2,L^2((a,b);\mathcal{H})}\big(\boldsymbol{I} - \alpha(\boldsymbol{I} - \boldsymbol{H}_b)^{-1}ST\big) \\
&\qquad \times \exp\big(-\operatorname{tr}_{L^2((a,b);\mathcal{H})}\big(\alpha^2\boldsymbol{H}_b(\boldsymbol{I} - \boldsymbol{H}_b)^{-1}ST\big)\big) \\
&\quad = {\det}_{L^2((a,b);\mathcal{H})}\big(\boldsymbol{I} - \alpha(\boldsymbol{I} - \boldsymbol{H}_b)^{-1}ST\big)\exp\big(\operatorname{tr}_{L^2((a,b);\mathcal{H})}\big(\alpha(\boldsymbol{I} - \boldsymbol{H}_b)^{-1}ST\big)\big) \\
&\qquad \times \exp\big(-\operatorname{tr}_{L^2((a,b);\mathcal{H})}\big(\alpha^2\boldsymbol{H}_b(\boldsymbol{I} - \boldsymbol{H}_b)^{-1}ST\big)\big) \qquad (2.116) \\
&\quad = {\det}_{L^2((a,b);\mathcal{H})}\big(\boldsymbol{I} - \alpha(\boldsymbol{I} - \boldsymbol{H}_b)^{-1}ST\big)\exp\big(\alpha\operatorname{tr}_{L^2((a,b);\mathcal{H})}(ST)\big) \\
&\quad = {\det}_{\mathcal{H}_1}\big(I_{\mathcal{H}_1} - \alpha T(\boldsymbol{I} - \boldsymbol{H}_b)^{-1}S\big)\exp\big(\alpha\operatorname{tr}_{L^2((a,b);\mathcal{H})}(ST)\big) \qquad (2.117) \\
&\quad = {\det}_{\mathcal{H}_1}\left(I_{\mathcal{H}_1} - \alpha\int_a^b dx\, G_1(x)\widehat{F}_1(x;\alpha)\right)\exp\left(\alpha\int_a^b dx\, \operatorname{tr}_{\mathcal{H}}(F_1(x)G_1(x))\right) \\
&\quad = {\det}_{\mathcal{H}_1\oplus\mathcal{H}_2}(U(a;\alpha))\exp\left(\alpha\int_a^b dx\, \operatorname{tr}_{\mathcal{H}}(F_1(x)G_1(x))\right).
\end{aligned}$$

In the above calculation, (2.116) is an application of (2.106), noting that $ST \in \mathcal{B}_1(L^2((a,b);\mathcal{H}))$ by Lemma 2.11, while (2.117) makes use of the determinant property in (2.100).

To prove ${\det}_{2,L^2((a,b);\mathcal{H})}(\boldsymbol{I} - \alpha\boldsymbol{K})$ coincides with the expressions in (2.111)–(2.113), we apply (2.114) and carry out the analogous computation,

$$\begin{aligned}
&{\det}_{2,L^2((a,b);\mathcal{H})}(\boldsymbol{I} - \alpha\boldsymbol{K}) \\
&\quad = {\det}_{2,L^2((a,b);\mathcal{H})}\big((\boldsymbol{I} - \alpha\boldsymbol{H}_a)\big[\boldsymbol{I} - \alpha(\boldsymbol{I} - \boldsymbol{H}_a)^{-1}QR\big]\big) \\
&\quad = {\det}_{2,L^2((a,b);\mathcal{H})}(\boldsymbol{I} - \alpha\boldsymbol{H}_a)\,{\det}_{2,L^2((a,b);\mathcal{H})}\big(\boldsymbol{I} - \alpha(\boldsymbol{I} - \boldsymbol{H}_a)^{-1}QR\big) \\
&\qquad \times \exp\big(-\operatorname{tr}_{L^2((a,b);\mathcal{H})}\big(\alpha^2\boldsymbol{H}_a(\boldsymbol{I} - \boldsymbol{H}_a)^{-1}QR\big)\big) \\
&\quad = {\det}_{L^2((a,b);\mathcal{H})}\big(\boldsymbol{I} - \alpha(\boldsymbol{I} - \boldsymbol{H}_a)^{-1}QR\big)\exp\big(\operatorname{tr}_{L^2((a,b);\mathcal{H})}\big(\alpha(\boldsymbol{I} - \boldsymbol{H}_a)^{-1}QR\big)\big) \\
&\qquad \times \exp\big(-\operatorname{tr}_{L^2((a,b);\mathcal{H})}\big(\alpha^2\boldsymbol{H}_a(\boldsymbol{I} - \boldsymbol{H}_a)^{-1}QR\big)\big) \\
&\quad = {\det}_{L^2((a,b);\mathcal{H})}\big(\boldsymbol{I} - \alpha(\boldsymbol{I} - \boldsymbol{H}_a)^{-1}QR\big)\exp\big(\alpha\operatorname{tr}_{L^2((a,b);\mathcal{H})}(QR)\big) \\
&\quad = {\det}_{\mathcal{H}_2}\big(I_{\mathcal{H}_2} - \alpha R(\boldsymbol{I} - \boldsymbol{H}_a)^{-1}Q\big)\exp\big(\alpha\operatorname{tr}_{L^2((a,b);\mathcal{H})}(QR)\big) \\
&\quad = {\det}_{\mathcal{H}_2}\left(I_{\mathcal{H}_2} - \alpha\int_a^b dx\, G_2(x)\widehat{F}_2(x;\alpha)\right)\exp\left(\alpha\int_a^b dx\, \operatorname{tr}_{\mathcal{H}}(F_2(x)G_2(x))\right) \\
&\quad = {\det}_{\mathcal{H}_1\oplus\mathcal{H}_2}(U(b;\alpha))\exp\left(\alpha\int_a^b dx\, \operatorname{tr}_{\mathcal{H}}(F_2(x)G_2(x))\right). \qquad (2.118)
\end{aligned}$$

□

As a consequence of Theorem 2.12, we recover the following result in the case where $\boldsymbol{K}$ is trace class $\boldsymbol{K} \in \mathcal{B}_1(L^2((a,b);\mathcal{H}))$, not just $\boldsymbol{K} \in \mathcal{B}_2(L^2((a,b);\mathcal{H}))$, first proved in [5].

Corollary 2.13 ([5]). *Assume Hypothesis* 2.10*, let* $\alpha \in \mathbb{C}$*, and suppose that* $\boldsymbol{K}$ *belongs to the trace class,* $\boldsymbol{K} \in \mathcal{B}_1(L^2((a,b);\mathcal{H}))$*. Then,* $\boldsymbol{H}_a, \boldsymbol{H}_b \in \mathcal{B}_1(L^2((a,b);\mathcal{H}))$ *and*

$$\operatorname{tr}_{L^2((a,b);\mathcal{H})}(\boldsymbol{H}_a) = \operatorname{tr}_{L^2((a,b);\mathcal{H})}(\boldsymbol{H}_b) = 0, \tag{2.119}$$

$$\operatorname{det}_{L^2((a,b);\mathcal{H})}(\boldsymbol{I} - \alpha\boldsymbol{H}_a) = \operatorname{det}_{L^2((a,b);\mathcal{H})}(\boldsymbol{I} - \alpha\boldsymbol{H}_b) = 1, \tag{2.120}$$

$$\operatorname{tr}_{L^2((a,b);\mathcal{H})}(\boldsymbol{K}) = \int_a^b dx\, \operatorname{tr}_{\mathcal{H}_1}(G_1(x)F_1(x)) = \int_a^b dx\, \operatorname{tr}_{\mathcal{H}}(F_1(x)G_1(x)) \tag{2.121}$$

$$= \int_a^b dx\, \operatorname{tr}_{\mathcal{H}_2}(G_2(x)F_2(x)) = \int_a^b dx\, \operatorname{tr}_{\mathcal{H}}(F_2(x)G_2(x)). \tag{2.122}$$

Assume in addition that U *is given by* (2.69)*. Then,*

$$\operatorname{det}_{L^2((a,b);\mathcal{H})}(\boldsymbol{I} - \alpha\boldsymbol{K}) = \operatorname{det}_{\mathcal{H}_1}\big(I_{\mathcal{H}_1} - \alpha T(\boldsymbol{I} - \alpha\boldsymbol{H}_b)^{-1}S\big) \tag{2.123}$$

$$= \operatorname{det}_{\mathcal{H}_1}\bigg(I_{\mathcal{H}_1} - \alpha \int_a^b dx\, G_1(x)\widehat{F}_1(x;\alpha)\bigg) \tag{2.124}$$

$$= \operatorname{det}_{\mathcal{H}_1 \oplus \mathcal{H}_2}(U(a;\alpha)) \tag{2.125}$$

$$= \operatorname{det}_{\mathcal{H}_2}\big(I_{\mathcal{H}_2} - \alpha R(\boldsymbol{I} - \alpha\boldsymbol{H}_a)^{-1}Q\big) \tag{2.126}$$

$$= \operatorname{det}_{\mathcal{H}_2}\bigg(I_{\mathcal{H}_2} - \alpha \int_a^b dx\, G_2(x)\widehat{F}_2(x;\alpha)\bigg) \tag{2.127}$$

$$= \operatorname{det}_{\mathcal{H}_1 \oplus \mathcal{H}_2}(U(b;\alpha)). \tag{2.128}$$

Proof. If $\boldsymbol{K} \in \mathcal{B}_1(L^2((a,b);\mathcal{H}))$, then $\boldsymbol{H}_a, \boldsymbol{H}_b \in \mathcal{B}_1(L^2((a,b);\mathcal{H}))$ is a consequence of (2.29) and (2.30), since $QR, ST \in \mathcal{B}_1(L^2((a,b);\mathcal{H}))$ by Lemma 2.11 (cf. (2.90)). Since $\boldsymbol{H}_a$ and $\boldsymbol{H}_b$ are quasi-nilpotent, they have no non-zero eigenvalues. Thus, relations (2.119) are clear from Lidskii's theorem (cf., e.g., [14, Theorem VII.6.1], [19, Sect. III.8, Sect. IV.1], [37, Theorem 3.7]), and the relations (2.120) follow from (2.98). Subsequently, (2.29), (2.30), and cyclicity of the trace (i.e., the second equality in (2.100)) imply

$$\begin{aligned} \operatorname{tr}_{L^2((a,b);\mathcal{H})}(\boldsymbol{K}) &= \operatorname{tr}_{L^2((a,b);\mathcal{H})}(QR) = \operatorname{tr}_{\mathcal{H}_2}(RQ) \\ &= \operatorname{tr}_{L^2((a,b);\mathcal{H})}(ST) = \operatorname{tr}_{\mathcal{H}_1}(TS). \end{aligned} \tag{2.129}$$

The equalities throughout (2.121) and (2.122) then follow from (2.92) and (2.93). Finally, relations (2.123)–(2.128) follow from those throughout (2.118), (2.121), and (2.122), noting that (cf. (2.106))

$$\operatorname{det}_{L^2((a,b);\mathcal{H})}(\boldsymbol{I}-\alpha\boldsymbol{K}) = \operatorname{det}_{2,L^2((a,b);\mathcal{H})}(\boldsymbol{I}-\alpha\boldsymbol{K}) \exp(-\alpha \operatorname{tr}_{L^2((a,b);\mathcal{H})}(\boldsymbol{K})). \tag{2.130}$$

□

The results (2.119)–(2.123), (2.125), (2.126), (2.128) can be found in the finite-dimensional context ($\dim(\mathcal{H}) < \infty$ and $\dim(\mathcal{H}_j) < \infty$, $j = 1, 2$) in Gohberg, Goldberg, and Kaashoek [14, Theorem 3.2] and in Gohberg, Goldberg, and Krupnik [17, Sects. XIII.5, XIII.6] under the additional assumptions that a, b are finite. The more general case where $(a,b) \subseteq \mathbb{R}$ is an arbitrary interval, as well as (2.124) and

(2.127), still in the case where $\mathcal{H}$ and $\mathcal{H}_j$, $j = 1, 2$, are finite-dimensional, was derived in [11].

3. Some Applications to Schrödinger Operators with Operator-Valued Potentials

To illustrate the potential of the theory developed in Section 2, we now briefly discuss some applications to Schrödinger operators with operator-valued potentials.

We start with some necessary notation: Let $(a, b) \subseteq \mathbb{R}$ be a finite or infinite interval and $\mathcal{H}$ a complex, separable Hilbert space. Integration of $\mathcal{H}$-valued functions on (a, b) will always be understood in the sense of Bochner, in particular, if $p \geqslant 1$, the Banach space $L^p((a, b); dx; \mathcal{H})$ denotes the set of equivalence classes of strongly measurable $\mathcal{H}$-valued functions which differ at most on sets of Lebesgue measure zero, such that $\|f(\cdot)\|_{\mathcal{H}}^p \in L^1((a, b); dx)$. The corresponding norm in $L^p((a, b); dx; \mathcal{H})$ is given by

$$\|f\|_{L^p((a,b);dx;\mathcal{H})} = \left(\int_{(a,b)} dx \, \|f(x)\|_{\mathcal{H}}^p \right)^{1/p}. \tag{3.1}$$

In the case $p = 2$, $L^2((a, b); dx; \mathcal{H})$ is a separable Hilbert space. One recalls that by a result of Pettis [28], weak measurability of $\mathcal{H}$-valued functions implies their strong measurability.

Sobolev spaces $W^{n,p}((a, b); dx; \mathcal{H})$ for $n \in \mathbb{N}$ and $p \geqslant 1$ are defined as follows: $W^{1,p}((a, b); dx; \mathcal{H})$ is the set of all $f \in L^p((a, b); dx; \mathcal{H})$ such that there exists a $g \in L^p((a, b); dx; \mathcal{H})$ and an $x_0 \in (a, b)$ such that

$$f(x) = f(x_0) + \int_{x_0}^{x} dx' \, g(x') \ \text{ for a.e. } x \in (a, b). \tag{3.2}$$

In this case g is the strong derivative of f, $g = f'$. Similarly, $W^{n,p}((a, b); dx; \mathcal{H})$ is the set of all $f \in L^p((a, b); dx; \mathcal{H})$ so that the first n strong derivatives of f are in $L^p((a, b); dx; \mathcal{H})$.

For simplicity of notation, from this point on we will omit the Lebesgue measure whenever no confusion can occur and henceforth simply write $L^p((a, b); \mathcal{H})$ for $L^p((a, b); dx; \mathcal{H})$. Moreover, in the special case where $\mathcal{H} = \mathbb{C}$, we omit $\mathcal{H}$ and typically (but not always) the Lebesgue measure and just write $L^p((a, b))$.

We begin with some applications recently considered in [5] which illustrate Theorem 2.12 and Corollary 2.13. We closely follow the treatment in [5] and refer to [12], [13] for background on Schrödinger operators with operator-valued potentials.

We start with the following basic assumptions.

Hypothesis 3.1. *Suppose that* $V : \mathbb{R} \to \mathcal{B}_1(\mathcal{H})$ *is a weakly measurable operator-valued function with* $\|V(\cdot)\|_{\mathcal{B}_1(\mathcal{H})} \in L^1(\mathbb{R})$.

We note that no self-adjointness condition $V(x) = V(x)^*$ for a.e. $x \in \mathbb{R}$ is assumed to hold in $\mathcal{H}$.

We introduce the densely defined, closed, linear operators in $L^2(\mathbb{R}; \mathcal{H})$ defined by

$$\boldsymbol{H}_0 f = -f'', \quad f \in \operatorname{dom}(\boldsymbol{H}_0) = W^{2,2}(\mathbb{R}; \mathcal{H}), \tag{3.3}$$

$$\boldsymbol{H} f = \tau f, \tag{3.4}$$

$$f \in \operatorname{dom}(\boldsymbol{H}) = \{g \in L^2(\mathbb{R}; \mathcal{H}) \,|\, g, g' \in AC_{\mathrm{loc}}(\mathbb{R}; \mathcal{H});\ \tau g \in L^2(\mathbb{R}; \mathcal{H})\},$$

where we denoted

$$(\tau f)(x) = -f''(x) + V(x)f(x) \text{ for a.e. } x \in \mathbb{R}. \tag{3.5}$$

In addition, we introduce the densely defined, closed, linear operator $\boldsymbol{V}$ in $L^2(\mathbb{R};\mathcal{H})$ by

$$\begin{aligned} &(\boldsymbol{V}f)(x) = V(x)f(x), \\ &f \in \operatorname{dom}(\boldsymbol{V}) = \bigg\{g \in L^2(\mathbb{R};\mathcal{H}) \,\bigg|\, g(x) \in \operatorname{dom}(V(x)) \text{ for a.e. } x \in \mathbb{R}, \\ &\qquad x \mapsto V(x)g(x) \text{ is (weakly) measurable, } \int_{\mathbb{R}} dx\, \|V(x)g(x)\|_{\mathcal{H}}^2 < \infty\bigg\}. \end{aligned} \tag{3.6}$$

Next we turn to the $\mathcal{B}(\mathcal{H})$-valued Jost solutions $f_\pm(z,\cdot)$ of

$$-\psi''(z,x) + V(x)\psi(z,x) = z\psi(z,x), \quad z \in \mathbb{C},\ x \in \mathbb{R}, \tag{3.7}$$

(i.e., $f_\pm(z,\cdot)h \in W^{2,1}_{\rm loc}((a,b);\mathcal{H})$ for every $h \in \mathcal{H}$) defined by

$$\begin{aligned} f_\pm(z,x) = e^{\pm iz^{1/2}x} I_{\mathcal{H}} - \int_x^{\pm\infty} dx'\, g_0(z,x,x')V(x')f_\pm(z,x'), \\ z \in \mathbb{C},\ \operatorname{Im}(z^{1/2}) \geqslant 0,\ x \in \mathbb{R}, \end{aligned} \tag{3.8}$$

where $g_0(z,\cdot,\cdot)$ is the $\mathcal{B}(\mathcal{H})$-valued Volterra Green's function of $\boldsymbol{H}_0$ given by

$$g_0(z,x,x') = z^{-1/2}\sin(z^{1/2}(x-x'))I_{\mathcal{H}}, \quad z \in \mathbb{C},\ x,x' \in \mathbb{R}. \tag{3.9}$$

We also recall the $\mathcal{B}(\mathcal{H})$-valued Green's function of $\boldsymbol{H}_0$,

$$\begin{aligned} G_0(z,x,x') = \big(\boldsymbol{H}_0 - z\boldsymbol{I}\big)^{-1}(x,x') = \frac{i}{2z^{1/2}} e^{iz^{1/2}|x-x'|} I_{\mathcal{H}}, \\ z \in \mathbb{C}\backslash[0,\infty),\ \operatorname{Im}(z^{1/2}) > 0,\ x,x' \in \mathbb{R}, \end{aligned} \tag{3.10}$$

with $\boldsymbol{I}$ representing the identity operator in $L^2(\mathbb{R};\mathcal{H})$.

The $\mathcal{B}(\mathcal{H})$-valued Jost function $\mathcal{F}$ associated with the pair of self-adjoint operators $(\boldsymbol{H},\boldsymbol{H}_0)$ is then given by

$$\mathcal{F}(z) = \frac{1}{2iz^{1/2}} W(f_-(\overline{z})^*, f_+(z)) \tag{3.11}$$

$$= I_{\mathcal{H}} - \frac{1}{2iz^{1/2}} \int_{\mathbb{R}} dx\, e^{-iz^{1/2}x} V(x) f_+(z,x), \tag{3.12}$$

$$= I_{\mathcal{H}} - \frac{1}{2iz^{1/2}} \int_{\mathbb{R}} dx\, f_-(\overline{z},x)^* V(x) e^{iz^{1/2}x}, \tag{3.13}$$

$$z \in \mathbb{C}\backslash\{0\},\ \operatorname{Im}(z^{1/2}) \geqslant 0.$$

Here $W(\cdot,\cdot)$ denotes the Wronskian defined by

$$W(F_1,F_2)(x) = F_1(x)F_2'(x) - F_1'(x)F_2(x), \quad x \in (a,b), \tag{3.14}$$

for F_1, F_2 strongly continuously differentiable $\mathcal{B}(\mathcal{H})$-valued functions.

Next, we recall the polar decomposition of a densely defined, closed, linear operator S in a complex separable Hilbert space $\mathcal{K}$

$$S = |S|U_S = U_S|S|, \tag{3.15}$$

where U_S is a partial isometry in $\mathcal{K}$ and $|S| = (S^*S)^{1/2}$, Introducing the factorization of $\boldsymbol{V} = \boldsymbol{u}\boldsymbol{v}$, where

$$\boldsymbol{u} = |\boldsymbol{V}|^{1/2}\boldsymbol{U_V} = \boldsymbol{U_V}|\boldsymbol{V}|^{1/2}, \quad \boldsymbol{v} = |\boldsymbol{V}|^{1/2}, \quad \boldsymbol{V} = |\boldsymbol{V}|\boldsymbol{U_V} = \boldsymbol{U_V}|\boldsymbol{V}| = \boldsymbol{u}\boldsymbol{v} = \boldsymbol{v}\boldsymbol{u}, \tag{3.16}$$

one verifies one verifies (see, e.g., [10], [24] and the references cited therein) that

$$\begin{aligned} &(\boldsymbol{H} - z\boldsymbol{I})^{-1} - (\boldsymbol{H}_0 - z\boldsymbol{I})^{-1} \\ &\quad = (\boldsymbol{H}_0 - z\boldsymbol{I})^{-1}\boldsymbol{v}\big[\boldsymbol{I} + \overline{\boldsymbol{u}(\boldsymbol{H}_0 - z\boldsymbol{I})^{-1}\boldsymbol{v}}\big]^{-1}\boldsymbol{u}(\boldsymbol{H}_0 - z\boldsymbol{I})^{-1}, \quad z \in \mathbb{C}\backslash\sigma(\boldsymbol{H}). \end{aligned} \tag{3.17}$$

Next, to make contact with the notation used in Section 2, we now introduce the operator $\boldsymbol{K}(z)$ in $L^2(\mathbb{R};\mathcal{H})$ by

$$\boldsymbol{K}(z) = -\overline{\boldsymbol{u}(\boldsymbol{H}_0 - z\boldsymbol{I})^{-1}\boldsymbol{v}}, \quad z \in \mathbb{C}\backslash[0,\infty), \tag{3.18}$$

with integral kernel

$$K(z,x,x') = -u(x)G_0(z,x,x')v(x'), \quad z \in \mathbb{C}\backslash[0,\infty),\ \mathrm{Im}(z^{1/2}) > 0,\ x,x' \in \mathbb{R}, \tag{3.19}$$

and the Volterra operators $\boldsymbol{H}_{-\infty}(z)$, $\boldsymbol{H}_{\infty}(z)$ (cf. (2.13), (2.14)) in $L^2(\mathbb{R};\mathcal{H})$, with integral kernel

$$H(z,x,x') = u(x)g^{(0)}(z,x,x')v(x'). \tag{3.20}$$

Here we used the abbreviations,

$$\begin{aligned} &u(x) = |V(x)|^{1/2}U_{V(x)}, \quad v(x) = |V(x)|^{1/2}, \\ &V(x) = |V(x)|U_{V(x)} = U_{V(x)}|V(x)| = u(x)v(x) \ \text{ for a.e. } x \in \mathbb{R}. \end{aligned} \tag{3.21}$$

Moreover, we introduce for a.e. $x \in \mathbb{R}$,

$$\begin{aligned} &f_1(z,x) = -u(x)e^{iz^{1/2}x}, \qquad g_1(z,x) = (i/2)z^{-1/2}v(x)e^{-iz^{1/2}x}, \\ &f_2(z,x) = -u(x)e^{-iz^{1/2}x}, \quad g_2(z,x) = (i/2)z^{-1/2}v(x)e^{iz^{1/2}x}. \end{aligned} \tag{3.22}$$

Assuming temporarily that

$$\mathrm{supp}(\|V(\cdot)\|_{\mathcal{B}(\mathcal{H})}) \text{ is compact} \tag{3.23}$$

(employing the notion of support for regular distributions on $\mathbb{R}$) in addition to Hypothesis 3.1, identifying $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, and introducing $\hat{f}_j(z,\cdot)$, $j = 1,2$, by

$$\hat{f}_1(z,x) = f_1(z,x) - \int_x^{\infty} dx'\, H(z,x,x')\hat{f}_1(z,x'), \tag{3.24}$$

$$\hat{f}_2(z,x) = f_2(z,x) + \int_{-\infty}^{x} dx'\, H(z,x,x')\hat{f}_2(z,x'), \tag{3.25}$$

$$z \in \mathbb{C}\backslash[0,\infty),\ \mathrm{Im}(z^{1/2}) > 0,\ \text{a.e. } x \in \mathbb{R},$$

yields $\hat{f}_j(z,\cdot) \in L^2(\mathbb{R};\mathcal{H})$, $j = 1,2$, upon a standard iteration of the Volterra integral equations (3.24), (3.25). In fact, $\hat{f}_j(z,\cdot) \in L^2(\mathbb{R};\mathcal{H})$, $j = 1,2$, have compact support as long as (3.23) holds. By comparison with (3.8), one then identifies for all $z \in \mathbb{C}\backslash[0,\infty)$, $\mathrm{Im}(z^{1/2}) > 0$, and a.e. $x \in \mathbb{R}$,

$$\hat{f}_1(z,x) = -u(x)f_+(z,x), \quad \hat{f}_2(z,x) = -u(x)f_-(z,x). \tag{3.26}$$

We note that the temporary compact support assumption (3.23) on $\|V(\cdot)\|_{\mathcal{B}(\mathcal{H})}$ has only been introduced to guarantee that $f_j(z,\cdot), \hat{f}_j(z,\cdot) \in L^2(\mathbb{R};\mathcal{H})$, $j = 1,2$ for all $z \in \mathbb{C}\backslash[0,\infty)$, $\mathrm{Im}(z^{1/2}) > 0$. This extra hypothesis can be removed by a standard approximation argument (see, [5], [11]).

Recalling the following basic fact (cf. [5]),

$$\boldsymbol{K}(z) \in \mathcal{B}_1\big(L^2(\mathbb{R};\mathcal{H})\big), \quad z \in \mathbb{C}\backslash[0,\infty), \tag{3.27}$$

still assuming Hypothesis 3.1, an application of Lemma 2.8 and Corollary 2.13 then yields the following Fredholm determinant reduction result, identifying the Fredholm determinant of $\boldsymbol{I}-\boldsymbol{K}(z)$ and that of the $\mathcal{B}(\mathcal{H})$-valued Jost function $\mathcal{F}(z)$ (the inverse transmission coefficient).

Theorem 3.2 ([5]). *Assume Hypothesis* 3.1*, then*

$$\det_{L^2(\mathbb{R};\mathcal{H})}(\boldsymbol{I}-\boldsymbol{K}(z)) = \det_{\mathcal{H}}(\mathcal{F}(z)), \quad z \in \mathbb{C}\backslash[0,\infty). \tag{3.28}$$

Relation (3.28) represents the infinite-dimensional version of the celebrated Jost–Pais-type reduction of Fredholm determinants [22] (see also [7], [11], [26], [36]).

Next, we revisit the second-order equation (3.7) from a different perspective. We intend to rederive the result analogous to (3.28) in the context of 2-modified determinants $\det_2(\cdot)$ by rewriting the second-order Schrödinger equation as a first-order 2×2 block operator system, taking the latter as our point of departure. (In the special case where $\mathcal{H}$ is finite-dimensional, this was considered in [8], [9], [11], [23].)

Assuming Hypothesis 3.1 for the rest of this example, the Schrödinger equation with the operator-valued potential $V(\cdot)$,

$$-\psi''(z,x) + V(x)\psi(z,x) = z\psi(z,x), \tag{3.29}$$

is equivalent to the first-order 2×2 block operator system

$$\Psi'(z,x) = \begin{pmatrix} 0 & I_{\mathcal{H}} \\ V(x)-z & 0 \end{pmatrix} \Psi(z,x), \quad \Psi(z,x) = \begin{pmatrix} \psi(z,x) \\ \psi'(z,x) \end{pmatrix}. \tag{3.30}$$

Since $\Phi^{(0)}$ defined by

$$\Phi^{(0)}(z,x) = \frac{1}{2iz^{1/2}} \begin{pmatrix} \exp(-iz^{1/2}x)I_{\mathcal{H}} & \exp(iz^{1/2}x)I_{\mathcal{H}} \\ -iz^{1/2}\exp(-iz^{1/2}x)I_{\mathcal{H}} & iz^{1/2}\exp(iz^{1/2}x)I_{\mathcal{H}} \end{pmatrix}, \quad \operatorname{Im}(z^{1/2}) \geqslant 0, \tag{3.31}$$

is a fundamental block operator matrix of the system (3.30) in the case $V=0$ a.e., and since

$$\Phi^{(0)}(z,x)\Phi^{(0)}(z,x')^{-1} = \begin{pmatrix} \cos(z^{1/2}(x-x'))I_{\mathcal{H}} & z^{-1/2}\sin(z^{1/2}(x-x'))I_{\mathcal{H}} \\ -z^{1/2}\sin(z^{1/2}(x-x'))I_{\mathcal{H}} & \cos(z^{1/2}(x-x'))I_{\mathcal{H}} \end{pmatrix}, \tag{3.32}$$

the system (3.30) has the following pair of linearly independent solutions for $z \neq 0$,

$$\begin{aligned} F_\pm(z,x) &= F_\pm^{(0)}(z,x) \\ &\quad - \int_x^{\pm\infty} dx' \begin{pmatrix} \cos(z^{1/2}(x-x'))I_{\mathcal{H}} & z^{-1/2}\sin(z^{1/2}(x-x'))I_{\mathcal{H}} \\ -z^{1/2}\sin(z^{1/2}(x-x'))I_{\mathcal{H}} & \cos(z^{1/2}(x-x'))I_{\mathcal{H}} \end{pmatrix} \\ &\qquad \times \begin{pmatrix} 0 & 0 \\ V(x') & 0 \end{pmatrix} F_\pm(z,x') \\ &= F_\pm^{(0)}(z,x) - \int_x^{\pm\infty} dx' \begin{pmatrix} z^{-1/2}\sin(z^{1/2}(x-x'))I_{\mathcal{H}} & 0 \\ \cos(z^{1/2}(x-x'))I_{\mathcal{H}} & 0 \end{pmatrix} V(x')F_\pm(z,x'), \\ &\qquad \operatorname{Im}(z^{1/2}) \geqslant 0,\ z \neq 0,\ x \in \mathbb{R}, \end{aligned} \tag{3.33}$$

where we abbreviated

$$F_{\pm}^{(0)}(z,x) = \begin{pmatrix} I_{\mathcal{H}} \\ \pm i z^{1/2} I_{\mathcal{H}} \end{pmatrix} \exp(\pm i z^{1/2} x). \tag{3.34}$$

By inspection, one has

$$F_{\pm}(z,x) = \begin{pmatrix} f_{\pm}(z,x) \\ f'_{\pm}(z,x) \end{pmatrix}, \quad \operatorname{Im}(z^{1/2}) \geqslant 0,\ z \neq 0,\ x \in \mathbb{R}, \tag{3.35}$$

with $f_{\pm}(z,\cdot)$ given by (3.8). Next, one introduces

$$\begin{aligned} f_1(z,x) &= -u(x) \begin{pmatrix} I_{\mathcal{H}} \\ i z^{1/2} I_{\mathcal{H}} \end{pmatrix} \exp(i z^{1/2} x), \\ f_2(z,x) &= -u(x) \begin{pmatrix} I_{\mathcal{H}} \\ -i z^{1/2} I_{\mathcal{H}} \end{pmatrix} \exp(-i z^{1/2} x), \\ g_1(z,x) &= v(x) \bigg(\frac{i}{2z^{1/2}} \exp(-i z^{1/2} x) I_{\mathcal{H}} \quad 0 \bigg), \\ g_2(z,x) &= v(x) \bigg(\frac{i}{2z^{1/2}} \exp(i z^{1/2} x) I_{\mathcal{H}} \quad 0 \bigg), \end{aligned} \tag{3.36}$$

and hence

$$\begin{aligned} H(z,x,x') &= f_1(z,x) g_1(z,x') - f_2(z,x) g_2(z,x') \\ &= u(x) \begin{pmatrix} z^{-1/2} \sin(z^{1/2}(x-x')) I_{\mathcal{H}} & 0 \\ \cos(z^{1/2}(x-x')) I_{\mathcal{H}} & 0 \end{pmatrix} v(x'), \end{aligned} \tag{3.37}$$

and we introduce

$$\widetilde{K}(z,x,x') = \begin{cases} f_1(z,x) g_1(z,x'), & x' < x, \\ f_2(z,x) g_2(z,x'), & x < x', \end{cases} \tag{3.38}$$

$$= \begin{cases} -u(x) \frac{1}{2} \exp(i z^{1/2}(x-x')) \begin{pmatrix} i z^{-1/2} I_{\mathcal{H}} & 0 \\ -I_{\mathcal{H}} & 0 \end{pmatrix} v(x'), & x' < x, \\ -u(x) \frac{1}{2} \exp(-i z^{1/2}(x-x')) \begin{pmatrix} i z^{-1/2} I_{\mathcal{H}} & 0 \\ I_{\mathcal{H}} & 0 \end{pmatrix} v(x'), & x < x', \end{cases}$$

$$\operatorname{Im}(z^{1/2}) \geqslant 0,\ z \neq 0,\ x, x' \in \mathbb{R}. \tag{3.39}$$

One notes that $\widetilde{K}(z,\cdot,\cdot)$ is discontinuous on the diagonal $x = x'$. Since

$$\widetilde{K}(z,\cdot,\cdot) \in L^2(\mathbb{R}^2; dx\,dx'; \mathcal{H})^{2\times 2}, \quad \operatorname{Im}(z^{1/2}) \geqslant 0,\ z \neq 0, \tag{3.40}$$

the associated operator $\widetilde{\boldsymbol{K}}(z)$ with integral kernel (3.39) is Hilbert–Schmidt,

$$\widetilde{\boldsymbol{K}}(z) \in \mathcal{B}_2\big(L^2(\mathbb{R};\mathcal{H})^2\big), \quad \operatorname{Im}(z^{1/2}) \geqslant 0,\ z \neq 0. \tag{3.41}$$

Next, assuming again temporarily (3.23), the integral equations defining $\hat{f}_j(z,x)$, $j = 1,2$,

$$\hat{f}_1(z,x) = f_1(z,x) - \int_x^\infty dx'\, H(z,x,x') \hat{f}_1(z,x'), \tag{3.42}$$

$$\hat{f}_2(z,x) = f_2(z,x) + \int_{-\infty}^x dx'\, H(z,x,x') \hat{f}_2(z,x'), \tag{3.43}$$

$$\operatorname{Im}(z^{1/2}) \geqslant 0,\ z \neq 0,\ x \in \mathbb{R},$$

yield solutions $\hat f_j(z,\cdot)\in L^2(\mathbb{R};\mathcal{H})^2$, $j=1,2$. By comparison with (3.33), one then identifies

$$\hat f_1(z,x) = -u(x)F_+(z,x), \quad \hat f_2(z,x) = -u(x)F_-(z,x). \tag{3.44}$$

We note that the temporary compact support assumption on V has only been invoked to guarantee that $f_j(z,\cdot), \hat f_j(z,\cdot) \in L^2(\mathbb{R};\mathcal{H})^2$, $j=1,2$. This extra hypothesis can be removed along a standard approximation method as detailed in [11].

An application of Lemma 2.11 and Theorem 2.12 then yields the following result (with $\boldsymbol{K}(\cdot)$ defined in (3.18) and $\boldsymbol{I}_2$ denoting the unit operator in $L^2(\mathbb{R};\mathcal{H})^2$).

Theorem 3.3. *Assume Hypothesis* 3.1*, then*

$$\begin{aligned}{\det}_{2,L^2(\mathbb{R};\mathcal{H})^2}\big(\boldsymbol{I}_2-\widetilde{\boldsymbol{K}}(z)\big) &= \mathcal{F}(z)\exp\bigg(-\frac{i}{2z^{1/2}}\int_{\mathbb{R}}dx\ \operatorname{tr}_{\mathcal{H}}(V(x))\bigg) &(3.45)\\ &= {\det}_{2,L^2(\mathbb{R};\mathcal{H})}(\boldsymbol{I}-\boldsymbol{K}(z)), \quad z\in\mathbb{C}\backslash[0,\infty). &(3.46)\end{aligned}$$

Thus, equation (3.29) and the first-order system (3.30) share the same 2-modified Fredholm determinant.

While we focused on Schrödinger operators and associated first-order systems with operator-valued potentials on $\mathbb{R}$, completely analogous results can be derived on the half-line $(0,\infty)$. Rather than repeating such applications for the half-line, we turn to a slightly different application involving semi-separable integral operators in $L^2((0,\infty);\mathcal{H})$ analogous to (3.18).

We introduce the following basic assumptions.

Hypothesis 3.4. *Let $V:(0,\infty)\to\mathcal{B}_1(\mathcal{H})$ be a weakly measurable operator-valued function with $\|V(\cdot)\|_{\mathcal{B}_1(\mathcal{H})}\in L^1((0,\infty);(1+x)dx)$.*

Again we note that $V(x)$ is not necessarily assumed to be self-adjoint in $\mathcal{H}$ for a.e. $x\geqslant 0$.

In analogy to (3.3), (3.4), we introduce the densely defined, closed, Dirichlet-type operators in $L^2((0,\infty);\mathcal{H})$ defined by

$$\begin{aligned}&\boldsymbol{H}_{0,+}f=-f'',\\ &f\in\operatorname{dom}\big(\boldsymbol{H}_{0,+}\big)=\{g\in L^2((0,\infty);\mathcal{H})\,|\,g,g'\in AC([0,R];\mathcal{H}) \text{ for all } R>0,\\ &\qquad\qquad f(0_+)=0,\ f''\in L^2((0,\infty);\mathcal{H})\}, &(3.47)\\ &\boldsymbol{H}_{+}f=-f''+Vf,\\ &f\in\operatorname{dom}(\boldsymbol{H}_{+})=\{g\in L^2((0,\infty);\mathcal{H})\,|\,g,g'\in AC([0,R];\mathcal{H}) \text{ for all } R>0, &(3.48)\\ &\qquad\qquad f(0_+)=0,\ (-f''+Vf)\in L^2((0,\infty);\mathcal{H})\}.\end{aligned}$$

We also introduce the $\mathcal{B}(\mathcal{H})$-valued Green's function of $\boldsymbol{H}_{0,+}$,

$$\begin{aligned}G_{0,+}(z,x,x') &= \big(\boldsymbol{H}_{0,+}-z\boldsymbol{I}_+\big)^{-1}(x,x') = \begin{cases} z^{-1/2}\sin(z^{1/2}x)e^{iz^{1/2}x'}I_{\mathcal{H}}, & x\leqslant x',\\ z^{-1/2}\sin(z^{1/2}x')e^{iz^{1/2}x}I_{\mathcal{H}}, & x\geqslant x',\end{cases}\\ &= \frac{i}{2z^{1/2}}\Big[e^{iz^{1/2}|x-x'|}-e^{iz^{1/2}(x+x')}\Big]I_{\mathcal{H}}, \quad z\in\mathbb{C}\backslash\sigma(\boldsymbol{H}_{0,+}),\ x,x'\geqslant 0, \end{aligned} \tag{3.49}$$

with $\boldsymbol{I}_+$ denoting the identity operator in $L^2((0,\infty);\mathcal{H})$. Introducing the factorization analogous to (3.16), (3.21) (for $x\geqslant 0$), one verifies as in (3.17),

$$(\boldsymbol{H}_+-z\boldsymbol{I}_+)^{-1} = \big(\boldsymbol{H}_{0,+}-z\boldsymbol{I}_+\big)^{-1}$$

$$- \big(\boldsymbol{H}_{0,+} - z\boldsymbol{I}_+\big)^{-1}\boldsymbol{v}\Big[I + \overline{\boldsymbol{u}\big(\boldsymbol{H}_{0,+} - z\boldsymbol{I}_+\big)^{-1}\boldsymbol{v}}\Big]^{-1}\boldsymbol{u}\big(\boldsymbol{H}_{0,+} - z\boldsymbol{I}_+\big)^{-1}, \quad z \in \mathbb{C}\backslash\sigma(\boldsymbol{H}_+), \tag{3.50}$$

and hence also introduces the operator $\boldsymbol{K}_+(z)$ in $L^2((0,\infty);\mathcal{H})$ by

$$\boldsymbol{K}_+(z) = -\overline{\boldsymbol{u}\big(\boldsymbol{H}_{0,+} - z\boldsymbol{I}_+\big)^{-1}\boldsymbol{v}}, \quad z \in \mathbb{C}\backslash\sigma\big(\boldsymbol{H}_{0,+}\big), \tag{3.51}$$

with $\mathcal{B}(\mathcal{H})$-valued integral kernel

$$K_+(z,x,x') = -u(x)G_{0,+}(z,x,x')v(x'), \quad \operatorname{Im}(z^{1/2}) \geqslant 0,\ x,x' > 0. \tag{3.52}$$

Assuming $V(x)$ to be self-adjoint for a.e. $x > 0$, we introduce its negative part $V_-(\cdot)$ (using the spectral theorem) by $V_-(\cdot) = [|V(\cdot)| - V(\cdot)]/2$ a.e. on $(0,\infty)$. We also use the notation $N(\lambda; A)$, $\lambda < \inf(\sigma_{ess}(A))$ to denote the number of discrete eigenvalues (counting multiplicity) of the self-adjoint operator A less than or equal to λ.

Then the well-known Bargmann bound [3] on the number of negative eigenvalues for Dirichtlet-type half-line Schrödinger operators reads as follows in the current context of operator-valued potentials:

Theorem 3.5. *Assume Hypothesis* 3.4 *and suppose that $V(x)$ is self-adjoint in $\mathcal{H}$ for a.e. $x > 0$. Then the number of negative eigenvalues of $\boldsymbol{H}_+$, denoted by $N(\boldsymbol{H}_+)$, satisfies the bound,*

$$N(\boldsymbol{H}_+) \leqslant \int_{(0,\infty)} dx\, x \operatorname{tr}_{\mathcal{H}}(V_-(x)). \tag{3.53}$$

Proof. As usual we may replace $V(\cdot)$ consistently by $V_-(\cdot)$. The Birman–Schwinger principle then implies

$$\begin{aligned} N(-\lambda; \boldsymbol{H}_+) &\leqslant \operatorname{tr}_{L^2((0,\infty);\mathcal{H})}\big(\boldsymbol{v}_-(\boldsymbol{H}_{0,+} + \lambda\boldsymbol{I}_+)^{-1}\boldsymbol{v}_-\big) \\ &= \frac{1}{2\lambda^{1/2}} \int_{(0,\infty)} dx\, \big[1 - e^{-2\lambda^{1/2}x}\big] \operatorname{tr}_{\mathcal{H}}(V_-(x)) \\ &\leqslant \int_{(0,\infty)} dx\, x \operatorname{tr}_{\mathcal{H}}(V_-(x)). \end{aligned} \tag{3.54}$$

Here, in obvious notation, $\boldsymbol{v}_-$ is defined as $\boldsymbol{v}$ in (3.16), (3.21), but with $V(\cdot)$ replaced by $V_-(\cdot)$, and we employed the well-known inequality $\big[1 - e^{-r}\big] \leqslant r$, $r \geqslant 0$ (cf., e.g., [1, 4.2.29, p. 70]). To complete the proof it suffices to let $\lambda \downarrow 0$. □

This proof was kindly communicated to us by A. Laptev [25] in the context of matrix-valued potentials $V(\cdot)$. The proof is clearly of a canonical nature and independent of the dimension of $\mathcal{H}$.

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