

ON FACTORIZATIONS OF ANALYTIC OPERATOR-VALUED FUNCTIONS AND EIGENVALUE MULTIPLICITY QUESTIONS

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ABSTRACT. We study several natural multiplicity questions that arise in the context of the Birman–Schwinger principle applied to non-self-adjoint operators. In particular, we re-prove (and extend) a recent result by Latushkin and Sukhtyaev by employing a different technique based on factorizations of analytic operator-valued functions due to Howland. Factorizations of analytic operator-valued functions are of particular interest in themselves and again we re-derive Howland’s results and subsequently extend them.

Considering algebraic multiplicities of finitely meromorphic operator-valued functions, we recall the notion of the index of a finitely meromorphic operator-valued function and use that to prove an analog of the well-known Weinstein–Aronszajn formula relating algebraic multiplicities of the underlying unperturbed and perturbed operators.

Finally, we consider pairs of projections for which the difference belongs to the trace class and relate their Fredholm index to the index of the naturally underlying Birman–Schwinger operator.

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1. INTRODUCTION

One of the principal aims of this paper is to investigate factorizations of analytic operator-valued functions and their connection to eigenvalue multiplicity questions for perturbations of a non-self-adjoint densely defined, closed linear operator H_0 in a separable Hilbert space \mathcal{H} by a non-self-adjoint additive perturbation term which formally factors into a product $V_2^*V_1$ over an auxiliary Hilbert space, \mathcal{K} :

$$V_j : \text{dom}(V_j) \subseteq \mathcal{H} \rightarrow \mathcal{K}, \quad j \in \{1, 2\}, \tag{1.1}$$

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with V_j densely defined, closed linear operators. Under appropriate assumptions on V_j , $j = 1, 2$, Kato [24] introduced a technique to define the “sum,” $H = H_0 + V_2^*V_1$ indirectly in terms of its resolvent $R(z) = (H - zI_{\mathcal{H}})^{-1}$ and the free resolvent $R_0(z) = (H_0 - zI_{\mathcal{H}})^{-1}$, according to the equation,

$$R(z) = R_0(z) - \overline{R_0(z)V_2^*} [I_{\mathcal{K}} - \overline{V_1R_0(z)V_2^*}]^{-1} V_1R_0(z), \quad (1.2)$$

$$z \in \{\zeta \in \rho(H_0) \mid 1 \in \rho(K(\zeta))\},$$

where $K(\cdot)$ represents an abstract Birman–Schwinger-type operator defined by

$$K(z) = -\overline{V_1R_0(z)V_2^*}, \quad z \in \rho(H_0). \quad (1.3)$$

Kato assumed H_0 to be self-adjoint and shows that under appropriate hypotheses on V_j , $j = 1, 2$, (1.2) defines the resolvent of a densely defined, closed operator H in \mathcal{H} .

Shortly thereafter, assuming $K(z)$ to be compact for each $z \in \rho(H_0)$, Konno and Kuroda [28] established an abstract variant of the Birman–Schwinger principle by proving that $z_0 \in \rho(H_0)$ is an eigenvalue of H if and only if 1 is an eigenvalue of the Birman–Schwinger operator $K(z_0)$. Moreover, one has equality of the corresponding *geometric multiplicities* (i.e., the dimensions of the corresponding eigenspaces): the geometric multiplicity of z_0 as an eigenvalue of H coincides with the geometric multiplicity of 1 as an eigenvalue of $K(z_0)$. Actually, Konno and Kuroda [28] assume that H_0 is self-adjoint and

$$(V_1f, V_2g)_{\mathcal{K}} = (V_2f, V_1g)_{\mathcal{K}}, \quad \text{for all } f, g \in \text{dom}(V_1) \cap \text{dom}(V_2), \quad (1.4)$$

which guarantees that H is self-adjoint as well. However, as shown in [14], the construction of H by Kato [24] via (1.2) and the geometric multiplicity results by Konno and Kuroda [28] extend to the case where H_0 and H are non-self-adjoint, and, moreover, the compactness assumption on $K(z)$ may be relaxed and replaced by the assumption that $I_{\mathcal{K}} - K(z)$ is a Fredholm operator in \mathcal{K} for each $z \in \rho(H_0)$.

The abstract formulation of the Birman–Schwinger principle set forth by Konno and Kuroda [28] (and its extension in [14]) yields equality of the geometric multiplicities of a point $z_0 \in \rho(H_0)$ as an eigenvalue of H and 1 as an eigenvalue of the Birman–Schwinger operator $K(z_0)$. Thus, if the operators H and $K(z_0)$ are non-self-adjoint, the algebraic and geometric multiplicities of z_0 (resp., 1) as an eigenvalue of H (resp., $K(z_0)$) will differ in general and hence it is entirely natural to inquire about the status of the associated *algebraic multiplicities* (defined in terms of the dimension of the range of the associated Riesz projections).

At first, one might be tempted to conjecture that the algebraic multiplicities of z_0 as an eigenvalue of H and 1 as an eigenvalue of $K(z_0)$ coincide, as with the corresponding geometric multiplicities. However, this is easily dismissed by explicit counterexamples. Therefore, one is forced to search for alternatives. It turns out that in lieu of the algebraic multiplicity of 1 as an eigenvalue of $K(z_0)$, one instead should consider the algebraic multiplicity of z_0 as a zero of finite-type of the analytic operator-valued function $I_{\mathcal{K}} - K(\cdot)$, denoted by $m_a(z_0; I_{\mathcal{K}} - K(\cdot))$. Indeed, Latushkin and Sukhtyaev [30] have shown, under appropriate assumptions, that the algebraic multiplicity of z_0 as an eigenvalue of H coincides with the algebraic multiplicity of z_0 as a zero of finite-type of the analytic operator-valued function $I_{\mathcal{K}} - K(\cdot)$.

Latushkin and Sukhtyaev rely on a Gohberg–Sigal Rouché-type Theorem to arrive at their result. We recover and slightly extend their result by entirely different

means. The approach employed in this paper relies on a factorization technique for analytic operator-valued functions originally due to Howland [21] (of interest in its own right), which shows that, given an analytic family $A(\cdot)$ of compact operators in \mathcal{H} defined on a domain Ω with $z_0 \in \Omega$ a weak zero (i.e., $A(z_0)$ is not invertible), if P is any projection onto $\text{ran}(A(z_0))$, and $Q = I_{\mathcal{H}} - P$, then $A(\cdot)$ may be factored according to

$$A(z) = [Q - (z - z_0)P]A_1(z), \quad z \in \Omega, \quad (1.5)$$

where $A_1(z)$ is analytic in Ω , $A_1(z) - I_{\mathcal{H}}$ is compact, and

$$\dim(\text{ran}(A_1(z_0))^\perp) \leq \dim(\text{ran}(A(z_0))^\perp). \quad (1.6)$$

Howland's motivation for arriving at such a factorization was to determine necessary and sufficient conditions for a pole of $A(z)^{-1}$ to be simple. In this paper, we extend Howland's factorization result to analytic families of Fredholm operators, thus relaxing the compactness assumption on $A(\cdot)$, and apply the factorization to algebraic multiplicities within the context of the abstract Birman–Schwinger principle. We emphasize that these considerations, combined with Evans function methods (cf. [4], [13], [15] and the extensive literature cited therein), have immediate applications in the area of linear stability theory for nonlinear evolution equations.

Next, we briefly turn to a summary of the contents of this paper: In Section 2, we recall the method introduced by Kato [24], and extended in [14], to define additive perturbations $H = H_0 + W$ of a non-self-adjoint operator H_0 by a non-self-adjoint perturbation term which formally factors according to $W = V_2^*V_1$. More specifically, the sum $H = H_0 + W$ is defined indirectly through a resolvent formalism, with the resolvent of H given by (1.2). Introducing the Birman–Schwinger operator $K(\cdot)$ in (1.3) and assuming $I_{\mathcal{H}} - K(z)$ is a Fredholm operator for all $z \in \rho(H_0)$, we recall in Theorem 2.7 an abstract version (of a variant) of the Birman–Schwinger principle due to Konno and Kuroda [28] in the case where H_0 and H are self-adjoint and $K(z)$ is compact, that states $z_0 \in \rho(H_0)$ is an eigenvalue of H if and only if 1 is an eigenvalue of $K(z_0)$, and the geometric multiplicity of z_0 as an eigenvalue of H is finite and coincides with the geometric multiplicity of 1 as an eigenvalue of $K(z_0)$. In Section 3, we recall the notion of a finitely meromorphic family of operators and the analytic Fredholm theorem in Theorem 3.3. In Theorems 3.4 and 3.5, we extend Howland's factorization to the case of an analytic family $A(\cdot)$ of Fredholm operators and recover Howland's necessary and sufficient condition for a pole of $A(\cdot)^{-1}$ to be simple in Corollary 3.6. Theorems 3.7 and 3.8 provide factorizations analogous to those in Theorems 3.4 and 3.5 but with the orders of the factors reversed (this appears to be a new result). In Section 4, we consider algebraic multiplicities of zeros of analytic operator-valued functions and study their application to the abstract Birman–Schwinger operator $K(\cdot)$. In Theorem 4.5, we use the extension of Howland's factorization to prove and slightly extend the algebraic multiplicity result of Latushkin and Sukhtyaev [30], proving that any $z_0 \in \rho(H_0) \cap \sigma(H)$ is a discrete eigenvalue of H if z_0 is isolated in $\sigma(H)$. In this case, z_0 is a zero of finite algebraic multiplicity of $I_{\mathcal{K}} - K(\cdot)$, and the algebraic multiplicity of z_0 as an eigenvalue of H equals the algebraic multiplicity of z_0 as a zero of $I_{\mathcal{K}} - A(\cdot)$. Example 4.7 shows that the algebraic multiplicity of z_0 as an eigenvalue of H need not equal the multiplicity of 1 as an eigenvalue of $K(z_0)$ in general. In Section 5, we extend Theorem 4.5 to the case where $K(\cdot)$ is finitely meromorphic and recover an analog of the Weinstein–Aronszajn formula for the case when H_0 and H have

common discrete eigenvalues. In our final Section 6, we apply some of the results from Sections 4 and 5 to ordered pairs of projections (P, Q) for which the difference $P - Q$ belongs to the trace class.

We will use the following notation in this paper. Let \mathcal{H} and \mathcal{K} be separable complex Hilbert spaces, $(\cdot, \cdot)_{\mathcal{H}}$ and $(\cdot, \cdot)_{\mathcal{K}}$ the scalar products in \mathcal{H} and \mathcal{K} (linear in the second factor), and $I_{\mathcal{H}}$ and $I_{\mathcal{K}}$ the identity operators in \mathcal{H} and \mathcal{K} , respectively. Next, let T be a closed linear operator from $\text{dom}(T) \subseteq \mathcal{H}$ to $\text{ran}(T) \subseteq \mathcal{K}$, with $\text{dom}(T)$ and $\text{ran}(T)$ denoting the domain and range of T . The closure of a closable operator S is denoted by \bar{S} . The kernel (null space) of T is denoted by $\ker(T)$. The spectrum, point spectrum (i.e., the set of eigenvalues), and resolvent set of a closed linear operator in \mathcal{H} will be denoted by $\sigma(\cdot)$, $\sigma_p(\cdot)$, and $\rho(\cdot)$; the discrete spectrum of T (i.e., points in $\sigma_p(T)$ which are isolated from the rest of $\sigma(T)$, and which are eigenvalues of T of finite algebraic multiplicity) is abbreviated by $\sigma_d(T)$.

The Banach spaces of bounded and compact linear operators in \mathcal{H} are denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_{\infty}(\mathcal{H})$, respectively. Similarly, the Schatten–von Neumann (trace) ideals will subsequently be denoted by $\mathcal{B}_p(\mathcal{H})$, $p \in [1, \infty)$, and the subspace of all finite rank operators in $\mathcal{B}_1(\mathcal{H})$ will be abbreviated by $\mathcal{F}(\mathcal{H})$. Analogous notation $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $\mathcal{B}_{\infty}(\mathcal{H}_1, \mathcal{H}_2)$, etc., will be used for bounded, compact, etc., operators between two Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 . In addition, $\text{tr}_{\mathcal{H}}(T)$ denotes the trace of a trace class operator $T \in \mathcal{B}_1(\mathcal{H})$ and $\det_{p, \mathcal{H}}(I_{\mathcal{H}} + S)$ represents the (modified) Fredholm determinant associated with an operator $S \in \mathcal{B}_p(\mathcal{H})$, $p \in \mathbb{N}$ (for $p = 1$ we omit the subscript 1). In addition, $\Phi(\mathcal{H})$ denotes the set of bounded Fredholm operators on \mathcal{H} (i.e., the set of operators $T \in \mathcal{B}(\mathcal{H})$ such that $\dim(\ker(T)) < \infty$, $\text{ran}(T)$ is closed in \mathcal{H} , and $\dim(\ker(T^*)) < \infty$). The corresponding (Fredholm) index of $T \in \Phi(\mathcal{H})$ is then given by $\text{ind}(T) = \dim(\ker(T)) - \dim(\ker(T^*))$.

The symbol $\dot{+}$ denotes a direct (but not necessary orthogonal direct) decomposition in connection with subspaces of Banach spaces.

Finally, we denote by $D(z_0; r_0) \subset \mathbb{C}$ the open disk with center z_0 and radius $r_0 > 0$, and by $C(z_0; r_0) = \partial D(z_0; r_0)$ the corresponding circle.

2. ABSTRACT PERTURBATION THEORY

In this introductory section, following Kato [24], Konno and Kuroda [28], and Howland [20], we consider a class of factorable non-self-adjoint perturbations of a given unperturbed non-self-adjoint operator. We closely follow the treatment in [14] (in which H_0 is explicitly permitted to be non-self-adjoint, cf. Hypothesis 2.1 (i) below) and refer to the latter for detailed proofs.

We start with our first set of hypotheses.

Hypothesis 2.1. (i) Suppose that $H_0: \text{dom}(H_0) \rightarrow \mathcal{H}$, $\text{dom}(H_0) \subseteq \mathcal{H}$ is a densely defined, closed, linear operator in \mathcal{H} with nonempty resolvent set,

$$\rho(H_0) \neq \emptyset, \quad (2.1)$$

$V_1: \text{dom}(V_1) \rightarrow \mathcal{K}$, $\text{dom}(V_1) \subseteq \mathcal{H}$ a densely defined, closed, linear operator from \mathcal{H} to \mathcal{K} , and $V_2: \text{dom}(V_2) \rightarrow \mathcal{K}$, $\text{dom}(V_2) \subseteq \mathcal{H}$ a densely defined, closed, linear operator from \mathcal{H} to \mathcal{K} such that

$$\text{dom}(V_1) \supseteq \text{dom}(H_0), \quad \text{dom}(V_2) \supseteq \text{dom}(H_0^*). \quad (2.2)$$

In the following we denote

$$R_0(z) = (H_0 - zI_{\mathcal{H}})^{-1}, \quad z \in \rho(H_0). \quad (2.3)$$

(ii) For some (and hence for all) $z \in \rho(H_0)$, the operator $-V_1 R_0(z) V_2^*$, defined on $\text{dom}(V_2^*)$, has a bounded extension in \mathcal{K} , denoted by $K(z)$,

$$K(z) = -\overline{V_1 R_0(z) V_2^*} \in \mathcal{B}(\mathcal{K}). \quad (2.4)$$

(iii) $1 \in \rho(K(z_0))$ for some $z_0 \in \rho(H_0)$.

That $K(z_0) \in \mathcal{B}(\mathcal{K})$ for some $z_0 \in \rho(H_0)$ implies $K(z) \in \mathcal{B}(\mathcal{K})$ for all $z \in \rho(H_0)$ (as mentioned in Hypothesis 2.1 (ii)) is an immediate consequence of (2.2) and the resolvent equation for H_0 .

We emphasize that in the case where H_0 is self-adjoint, the following results in Lemma 2.2, Theorem 2.3, and Remark 2.4 are due to Kato [24] (see also [20], [28]). The more general case we consider here requires only minor modifications, but for the convenience of the reader we will sketch most of the proofs.

Lemma 2.2. *Let $z, z_1, z_2 \in \rho(H_0)$. Then Hypothesis 2.1 implies the following facts:*

$$V_1 R_0(z) \in \mathcal{B}(\mathcal{H}, \mathcal{K}), \quad \overline{R_0(z) V_2^*} = [V_2(H_0^* - \bar{z}I_{\mathcal{H}})^{-1}]^* \in \mathcal{B}(\mathcal{K}, \mathcal{H}), \quad (2.5)$$

$$\overline{R_0(z_1) V_2^*} - \overline{R_0(z_2) V_2^*} = (z_1 - z_2) R_0(z_1) \overline{R_0(z_2) V_2^*} \quad (2.6)$$

$$= (z_1 - z_2) R_0(z_2) \overline{R_0(z_1) V_2^*}, \quad (2.7)$$

$$K(z) = -V_1 \overline{[R_0(z) V_2^*]}, \quad K(\bar{z})^* = -V_2 \overline{[R_0(\bar{z})^* V_1^*]}, \quad (2.8)$$

$$\text{ran}(\overline{R_0(z) V_2^*}) \subseteq \text{dom}(V_1), \quad \text{ran}(\overline{R_0(\bar{z})^* V_1^*}) \subseteq \text{dom}(V_2), \quad (2.9)$$

$$K(z_1) - K(z_2) = (z_2 - z_1) V_1 R_0(z_1) \overline{R_0(z_2) V_2^*} \quad (2.10)$$

$$= (z_2 - z_1) V_1 R_0(z_2) \overline{R_0(z_1) V_2^*}. \quad (2.11)$$

Next, following Kato [24], one introduces

$$R(z) = R_0(z) - \overline{R_0(z) V_2^*} [I_{\mathcal{K}} - K(z)]^{-1} V_1 R_0(z), \quad (2.12)$$

$$z \in \{\zeta \in \rho(H_0) \mid 1 \in \rho(K(\zeta))\}.$$

Theorem 2.3. *Assume Hypothesis 2.1 and suppose $z \in \{\zeta \in \rho(H_0) \mid 1 \in \rho(K(\zeta))\}$. Then, $R(z)$ defined in (2.12) defines a densely defined, closed, linear operator H in \mathcal{H} by*

$$R(z) = (H - zI_{\mathcal{H}})^{-1}. \quad (2.13)$$

Moreover,

$$V_1 R(z), V_2 R(z)^* \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \quad (2.14)$$

and

$$R(z) = R_0(z) - \overline{R(z) V_2^*} V_1 R_0(z) \quad (2.15)$$

$$= R_0(z) - \overline{R_0(z) V_2^*} V_1 R(z). \quad (2.16)$$

Finally, H is an extension of $(H_0 + V_2^* V_1)|_{\text{dom}(H_0) \cap \text{dom}(V_2^* V_1)}$ (the latter intersection domain may consist of $\{0\}$ only),

$$H \supseteq (H_0 + V_2^* V_1)|_{\text{dom}(H_0) \cap \text{dom}(V_2^* V_1)}. \quad (2.17)$$

Remark 2.4. (i) Assume that H_0 is self-adjoint in \mathcal{H} . Then H is also self-adjoint if

$$(V_1 f, V_2 g)_{\mathcal{K}} = (V_2 f, V_1 g)_{\mathcal{K}} \quad \text{for all } f, g \in \text{dom}(V_1) \cap \text{dom}(V_2). \quad (2.18)$$

(ii) The formalism is symmetric with respect to H_0 and H in the following sense: The densely defined operator $-V_1R(z)V_2^*$ has a bounded extension to all of \mathcal{K} for all $z \in \{\zeta \in \rho(H_0) \mid 1 \in \rho(K(\zeta))\}$, in particular,

$$I_{\mathcal{K}} - \overline{V_1R(z)V_2^*} = [I_{\mathcal{K}} - K(z)]^{-1}, \quad z \in \{\zeta \in \rho(H_0) \mid 1 \in \rho(K(\zeta))\}. \quad (2.19)$$

Moreover,

$$R_0(z) = R(z) + \overline{R(z)V_2^*} [I_{\mathcal{K}} - \overline{V_1R(z)V_2^*}]^{-1} V_1R(z), \quad (2.20)$$

$$z \in \{\zeta \in \rho(H_0) \mid 1 \in \rho(K(\zeta))\},$$

and

$$H_0 \supseteq (H - V_2^*V_1)|_{\text{dom}(H) \cap \text{dom}(V_2^*V_1)}. \quad (2.21)$$

(iii) The basic hypotheses (2.2) which amount to

$$V_1R_0(z) \in \mathcal{B}(\mathcal{H}, \mathcal{K}), \quad \overline{R_0(z)V_2^*} = [V_2(H_0^* - \bar{z}I_{\mathcal{H}})^{-1}]^* \in \mathcal{B}(\mathcal{K}, \mathcal{H}), \quad z \in \rho(H_0). \quad (2.22)$$

(cf. (2.5)) are more general than a quadratic form perturbation approach which would result in conditions of the form

$$V_1R_0(z)^{1/2} \in \mathcal{B}(\mathcal{H}, \mathcal{K}), \quad \overline{R_0(z)^{1/2}V_2^*} = [V_2(H_0^* - \bar{z}I_{\mathcal{H}})^{-1/2}]^* \in \mathcal{B}(\mathcal{K}, \mathcal{H}), \quad (2.23)$$

$$z \in \rho(H_0),$$

or even an operator perturbation approach which would involve conditions of the form

$$[V_2^*V_1]R_0(z) \in \mathcal{B}(\mathcal{H}), \quad z \in \rho(H_0). \quad (2.24)$$

The next result represents an abstract version of (a variant of) the Birman–Schwinger principle due to Birman [5] and Schwinger [43] (cf. also [6], [12], [26], [27], [36], [38], [44], [45, Ch. III], and [46]). We will focus on geometric multiplicities and again follow [14] closely.

We need to strengthen our hypotheses a bit and hence introduce the following assumption:

Hypothesis 2.5. In addition to Hypothesis 2.1 we suppose the condition:

(iv) $[I_{\mathcal{K}} - K(z)] \in \Phi(\mathcal{K})$ for all $z \in \rho(H_0)$.

Remark 2.6. In concrete applications, say, to Schrödinger-type operators, condition (iv) in Hypothesis 2.5 is frequently replaced by the stronger assumption:

(iv') $K(z) \in \mathcal{B}_{\infty}(\mathcal{K})$ for all $z \in \rho(H_0)$.

In this case $[I_{\mathcal{K}} - K(z)]$ is a Fredholm operator with index zero for all $z \in \rho(H_0)$.

An elementary example illustrating that condition (iv') is stronger than (iv) is easily constructed as follows: Choose

$$\begin{aligned} \mathcal{K} &= \mathcal{H}, \quad V_j = I_{\mathcal{H}}, \quad j = 1, 2, \quad H_0 = H_0^*, \quad H = H_0 + I_{\mathcal{H}}, \\ \sigma_p(H_0) &= \emptyset, \quad \sigma_{ess}(H_0) \neq \emptyset. \end{aligned} \quad (2.25)$$

Next, choose $z \in \mathbb{C} \setminus \mathbb{R}$. Then, $K(z) = (H_0 - zI_{\mathcal{H}})^{-1}$, and $I_{\mathcal{H}} - (H_0 - zI_{\mathcal{H}})^{-1} = (H_0 - (z+1)I_{\mathcal{H}})(H_0 - zI_{\mathcal{H}})^{-1}$, implying that $\text{ran}(I_{\mathcal{H}} - K(z)) = \mathcal{H}$ and $\ker(I_{\mathcal{H}} - K(z)) = \ker(I_{\mathcal{H}} - K(z)^*) = \{0\}$. Hence, $[I_{\mathcal{K}} - K(z)] \in \Phi(\mathcal{K})$ with $\text{ind}(I_{\mathcal{K}} - K(z)) = 0$. However, since $\sigma_{ess}(K(z)) \not\supseteq \{0\}$, one concludes that $K(z) \notin \mathcal{B}_{\infty}(\mathcal{H})$.

Since by (2.19),

$$-\overline{V_1R(z)V_2^*} = [I_{\mathcal{K}} - K(z)]^{-1}K(z) = -I_{\mathcal{K}} + [I_{\mathcal{K}} - K(z)]^{-1}, \quad (2.26)$$

Hypothesis 2.5 implies that $I_{\mathcal{K}} - \overline{V_1 R(\cdot) V_2^*}$ extends to a Fredholm operator in $\Phi(\mathcal{K})$ as long as the right-hand side of (2.26) exists. Similarly, under condition (iv') in Remark 2.6, $\overline{V_1 R(\cdot) V_2^*}$ extends to a compact operator in \mathcal{K} as long as the right-hand side of (2.26) exists.

Regarding eigenvalues, we recall that if T is a densely defined, closed linear operator in \mathcal{H} , then the *geometric multiplicity*, $m_g(z_0; T)$, of an eigenvalue $z_0 \in \sigma_p(T)$ of T is given by

$$m_g(z_0; T) = \dim(\ker(T - z_0 I_{\mathcal{H}})). \quad (2.27)$$

The following general result is due to Konno and Kuroda [28] in the case where H_0 is self-adjoint and condition (iv') in Remark 2.6 is assumed.

Theorem 2.7 ([28]). *Assume Hypothesis 2.5 and let $z_0 \in \rho(H_0)$. Then,*

$$Hf = z_0 f, \quad 0 \neq f \in \text{dom}(H) \text{ implies } K(z_0)g = g \quad (2.28)$$

where, for fixed $z_1 \in \{\zeta \in \rho(H_0) \mid 1 \in \rho(K(\zeta))\}$, $z_1 \neq z_0$,

$$0 \neq g = [I_{\mathcal{K}} - K(z_1)]^{-1} V_1 R_0(z_1) f \quad (2.29)$$

$$= (z_0 - z_1)^{-1} V_1 f. \quad (2.30)$$

Conversely,

$$K(z_0)g = g, \quad 0 \neq g \in \mathcal{K} \text{ implies } Hf = z_0 f, \quad (2.31)$$

where

$$0 \neq f = -\overline{R_0(z_0) V_2^*} g \in \text{dom}(H). \quad (2.32)$$

Moreover,

$$m_g(z_0; H) = \dim(\ker(H - z_0 I_{\mathcal{H}})) = \dim(\ker(I_{\mathcal{K}} - K(z_0))) = m_g(1; K(z_0)) < \infty. \quad (2.33)$$

In particular, let $z \in \rho(H_0)$, then

$$z \in \rho(H) \text{ if and only if } 1 \in \rho(K(z)). \quad (2.34)$$

It is possible to avoid the Fredholm operator (resp., compactness) assumption in condition (iv) in Hypothesis 2.5 (resp., condition (iv') in Remark 2.6) in Theorem 2.7 provided that (2.33) is replaced by the statement:

$$\text{The subspaces } \ker(H - z_0 I_{\mathcal{H}}) \text{ and } \ker(I_{\mathcal{K}} - K(z_0)) \text{ are isomorphic} \quad (2.35)$$

(cf. [14]). Of course, (2.33) follows from (2.35) provided $\ker(I_{\mathcal{K}} - K(z_0))$ is finite-dimensional, which in turn follows from Hypothesis 2.5.

3. ON FACTORIZATIONS OF ANALYTIC OPERATOR-VALUED FUNCTIONS

In this section, we consider factorizations of analytic operator-valued functions. We recall and extend a factorization result due to Howland [21].

Assuming $\Omega \subseteq \mathbb{C}$ to be open and $M(\cdot)$ to be a $\mathcal{B}(\mathcal{H})$ -valued meromorphic function on Ω that has the norm convergent Laurent expansion around $z_0 \in \Omega$ of the type

$$M(z) = \sum_{k=-N_0}^{\infty} (z - z_0)^k M_k(z_0), \quad M_k(z_0) \in \mathcal{B}(\mathcal{H}), \quad k \in \mathbb{Z}, \quad k \geq -N_0, \quad (3.1)$$

$$0 < |z - z_0| < \varepsilon_0,$$

for some $N_0 = N_0(z_0) \in \mathbb{N}$ and some $0 < \varepsilon_0 = \varepsilon_0(z_0)$ sufficiently small, we denote the principal part, $\text{pp}_{z_0} \{M(\cdot)\}$, of $M(\cdot)$ at z_0 by

$$\text{pp}_{z_0} \{M(z)\} = \sum_{k=-N_0}^{-1} (z - z_0)^k M_k(z_0), \quad M_{-k}(z_0) \in \mathcal{B}(\mathcal{H}), \quad 1 \leq k \leq N_0, \quad (3.2)$$

$$0 < |z - z_0| < \varepsilon_0.$$

Given the notation (3.2), we start with the following definition.

Definition 3.1. Let $\Omega \subseteq \mathbb{C}$ be open and connected. Suppose that $M(\cdot)$ is a $\mathcal{B}(\mathcal{H})$ -valued analytic function on Ω except for isolated singularities. Then $M(\cdot)$ is called *finitely meromorphic at $z_0 \in \Omega$* if $M(\cdot)$ is analytic on the punctured disk $D(z_0; \varepsilon_0) \setminus \{z_0\} \subset \Omega$ centered at z_0 with sufficiently small $\varepsilon_0 > 0$, and the principal part of $M(\cdot)$ at z_0 is of finite rank, that is, if the principal part of $M(\cdot)$ is of the type (3.2), and one has

$$M_{-k}(z_0) \in \mathcal{F}(\mathcal{H}), \quad 1 \leq k \leq N_0. \quad (3.3)$$

In addition, $M(\cdot)$ is called *finitely meromorphic on Ω* if it is meromorphic on Ω and finitely meromorphic at each of its poles.

In using the term *finitely meromorphic* we closely follow the convention in [18] (see also [16, Sect. XI.9] and [17, Sect. 4.1]). We also note that the notions *completely meromorphic* (cf. [20], adopted in [14]) and *essentially meromorphic* (cf. [41]) have been used instead in the literature.

Throughout this section we make the following assumptions:

Hypothesis 3.2. Let $\Omega \subseteq \mathbb{C}$ be open and connected, and suppose that $A : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ is analytic and that

$$A(z) \in \Phi(\mathcal{H}) \quad \text{for all } z \in \Omega. \quad (3.4)$$

One then recalls the the analytic Fredholm theorem in the following form:

Theorem 3.3 ([17], Sect. 4.1, [18], [20], [39, Theorem VI.14], [48]).

Assume that $A : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ satisfies Hypothesis 3.2. Then either

(i) $A(z)$ is not boundedly invertible for any $z \in \Omega$,

or else,

(ii) $A(\cdot)^{-1}$ is finitely meromorphic on Ω . More precisely, there exists a discrete subset $\mathcal{D}_1 \subset \Omega$ (possibly, $\mathcal{D}_1 = \emptyset$) such that $A(z)^{-1} \in \mathcal{B}(\mathcal{H})$ for all $z \in \Omega \setminus \mathcal{D}_1$, $A(\cdot)^{-1}$ is analytic on $\Omega \setminus \mathcal{D}_1$, and meromorphic on Ω . In addition,

$$A(z)^{-1} \in \Phi(\mathcal{H}) \quad \text{for all } z \in \Omega \setminus \mathcal{D}_1, \quad (3.5)$$

and if $z_1 \in \mathcal{D}_1$ then

$$A(z)^{-1} = \sum_{k=-N_0(z_1)}^{\infty} (z - z_1)^k C_k(z_1), \quad 0 < |z - z_1| < \varepsilon_0(z_1), \quad (3.6)$$

with

$$\begin{aligned} C_{-k}(z_1) &\in \mathcal{F}(\mathcal{H}), \quad 1 \leq k \leq N_0(z_1), \quad C_0(z_1) \in \Phi(\mathcal{H}) \cap \mathcal{B}(\mathcal{H}), \\ C_k(z_1) &\in \mathcal{B}(\mathcal{H}), \quad k \in \mathbb{N}. \end{aligned} \quad (3.7)$$

In addition, if $[I_{\mathcal{H}} - A(z)] \in \mathcal{B}_{\infty}(\mathcal{H})$ for all $z \in \Omega$, then

$$[I_{\mathcal{H}} - A(z)^{-1}] \in \mathcal{B}_{\infty}(\mathcal{H}), \quad z \in \Omega \setminus \mathcal{D}_1, \quad [I_{\mathcal{H}} - C_0(z_1)] \in \mathcal{B}_{\infty}(\mathcal{H}), \quad z_1 \in \mathcal{D}_1. \quad (3.8)$$

For an interesting extension of the analytic Fredholm theorem in connection with Hahn holomorphic functions we refer to [35].

For a linear operator S in \mathcal{H} with closed range one defines the *defect of S* , denoted by $\text{def}(S)$, by the codimension of $\text{ran}(S)$ in \mathcal{H} , that is,

$$\text{def}(S) = \dim(\text{ran}(S)^\perp). \quad (3.9)$$

In addition, we recall the notion of linear independence with respect to a linear subspace of \mathcal{H} : Let $\mathcal{D} \subseteq \mathcal{H}$ be a linear subspace of \mathcal{H} . Then vectors $f_k \in \mathcal{H}$, $1 \leq k \leq N$, $N \in \mathbb{N}$, are called *linearly independent (mod \mathcal{D})*, if

$$\begin{aligned} \sum_{k=1}^N c_k f_k \in \mathcal{D} \text{ for some coefficients } c_k \in \mathbb{C}, 1 \leq k \leq N, \\ \text{implies } c_k = 0, 1 \leq k \leq N. \end{aligned} \quad (3.10)$$

In addition, with \mathcal{D} and \mathcal{E} linear subspaces of \mathcal{H} with $\mathcal{D} \subseteq \mathcal{E}$, the quotient subspace \mathcal{E}/\mathcal{D} consists of equivalence classes $[f]$ such that $g \in [f]$ if and only if $(f - g) \in \mathcal{D}$, in particular, $f = g \pmod{\mathcal{D}}$ is equivalent to $(f - g) \in \mathcal{D}$. Moreover, the dimension of $\mathcal{E} \pmod{\mathcal{D}}$, denoted by $\dim_{\mathcal{D}}(\mathcal{E})$, equals $n \in \mathbb{N}$, if there are n , but not more than n , linearly independent vectors in \mathcal{E} , such that no linear combination (except, the trivial one) belongs to \mathcal{D} . If no such finite $n \in \mathbb{N}$ exists, one defines $\dim_{\mathcal{D}}(\mathcal{E}) = \infty$.

The following three results due to Howland [21] are fundamental for the remainder of this section and for convenience of the reader we include their proof under slightly more general hypotheses, replacing Howland's assumption that $[A(\cdot) - I_{\mathcal{H}}] \in \mathcal{B}_\infty(\mathcal{H})$ by the assumption that $A(\cdot)$ is Fredholm. In addition, we occasionally offer a few additional details in the proofs of these results.

Theorem 3.4 ([21]). *Assume that $A : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ satisfies Hypothesis 3.2, suppose that $A(z)$ is boundedly invertible for some $z \in \Omega$ (i.e., case (ii) in Theorem 3.3 applies), and let $z_0 \in \Omega$ be a pole of $A(\cdot)^{-1}$. Denote by Q_1 any projection onto $\text{ran}(A(z_0))$ and let $P_1 = I_{\mathcal{H}} - Q_1$. Then,*

$$A(z) = [Q_1 - (z - z_0)P_1]A_1(z), \quad z \in \Omega, \quad (3.11)$$

where

$$A_1(\cdot) \text{ is analytic on } \Omega, \quad (3.12)$$

$$A_1(z) \in \Phi(\mathcal{H}), \quad z \in \Omega, \quad (3.13)$$

$$\text{ind}(A(z)) = \text{ind}(A_1(z)) = 0, \quad z \in \Omega, \quad |z - z_0| \text{ sufficiently small}, \quad (3.14)$$

$$\text{def}(A_1(z_0)) \leq \text{def}(A(z_0)). \quad (3.15)$$

If z_0 is a pole of $A(\cdot)^{-1}$ of order $n_0 \in \mathbb{N}$, then z_0 is a pole of $A_1(\cdot)^{-1}$ of order $n_0 - 1$. Finally,

$$[I_{\mathcal{H}} - A(\cdot)] \in \mathcal{F}(\mathcal{H}) \text{ (resp., } \mathcal{B}_p(\mathcal{H}) \text{ for some } 1 \leq p \leq \infty) \quad (3.16)$$

if and only if $[I_{\mathcal{H}} - A_1(\cdot)] \in \mathcal{F}(\mathcal{H})$ (resp., $\mathcal{B}_p(\mathcal{H})$ for some $1 \leq p \leq \infty$).

Proof. In the following let $z \in \Omega$. Since by hypothesis z_0 is a pole of $A(\cdot)^{-1}$ and hence an isolated singularity of $A(\cdot)^{-1}$, the second alternative of the analytic Fredholm theorem, Theorem 3.3, is realized. Due to assumption (3.4), $\text{ran}(A(z_0))$ is closed in \mathcal{H} . With respect to the decomposition $\mathcal{H} = P_1\mathcal{H} \dot{+} Q_1\mathcal{H}$ one infers that

$$Q_1 - (z - z_0)P_1 = \begin{pmatrix} -(z - z_0)P_1 & 0 \\ 0 & Q_1 \end{pmatrix}. \quad (3.17)$$

In addition, the projection P_1 is finite-dimensional which may be seen as follows. By [25, p. 156, 267], the adjoint P_1^* is a projection onto $\text{ran}(A(z_0))^\perp = \ker(A(z_0)^*)$. However, $A(z_0)^* \in \Phi(\mathcal{H})$ since $A(z_0) \in \Phi(\mathcal{H})$, so that P_1^* is a finite-dimensional projection,

$$\dim(\text{ran}((P_1^*)) = \dim(\ker(A(z_0)^*)) < \infty. \quad (3.18)$$

Evidently, (3.18) also implies (cf., e.g., [49, Theorem 6.1])

$$\dim(\text{ran}((P_1)) = \dim(\ker(A(z_0)^*)) = \dim(\text{ran}(P_1^*)) < \infty. \quad (3.19)$$

Next, the representation in (3.17) implies

$$\begin{aligned} [Q_1 - (z - z_0)P_1]^{-1} &= \begin{pmatrix} -(z - z_0)^{-1}P_1 & 0 \\ 0 & Q_1 \end{pmatrix} \\ &= Q_1 - (z - z_0)^{-1}P_1, \quad z \in \Omega \setminus \{z_0\}, \end{aligned} \quad (3.20)$$

$$\det_{\mathcal{H}}(Q_1 - (z - z_0)P_1) = (z_0 - z)^{p_1}, \quad p_1 = \dim(\text{ran}(P_1)). \quad (3.21)$$

Thus,

$$A_1(z) = [Q_1 - (z - z_0)P_1]^{-1}A(z) = Q_1A(z) - (z - z_0)^{-1}P_1[A(z) - A(z_0)] \quad (3.22)$$

is analytic on Ω since $P_1A(z_0) = P_1Q_1A(z_0) = 0$ as Q_1 acts on $\text{ran}(A(z_0))$ as the identity operator by hypothesis. In particular,

$$A_1(z_0) = Q_1A(z_0) - P_1A'(z_0). \quad (3.23)$$

Moreover, using once more that

$$A_1(z) = [Q_1 - (z - z_0)^{-1}P_1]A(z), \quad z \in \Omega \setminus \{z_0\}, \quad (3.24)$$

one notices that by hypothesis, $A(\cdot) \in \Phi(\mathcal{H})$ on Ω , and that by (3.20),

$$[Q_1 - (z - z_0)^{-1}P_1]^{-1} = [Q_1 - (z - z_0)P_1] \in \mathcal{B}(\mathcal{H}), \quad z \in \Omega \setminus \{z_0\}, \quad (3.25)$$

and analogously for its adjoint. In particular, $\text{ran}([Q_1 - (z - z_0)P_1]^{-1}) = \mathcal{H}$, and hence one also concludes that $[Q_1 - (z - z_0)^{-1}P_1] \in \Phi(\mathcal{H})$ for $z \in \Omega \setminus \{z_0\}$, and hence $A_1(z) \in \Phi(\mathcal{H})$ for $z \in \Omega \setminus \{z_0\}$. The remaining case $z = z_0$ now follows from (3.23), since P_1 and hence $P_1A'(z_0)$ is of finite rank (thus, compact), $Q_1 = [I_{\mathcal{H}} - P_1] \in \Phi(\mathcal{H})$ and $A(z_0) \in \Phi(\mathcal{H})$, implying $Q_1A(z_0) \in \Phi(\mathcal{H})$, and the fact that a Fredholm operator plus a compact operator is again Fredholm (cf., e.g., [42, Theorem 5.10]).

Invariance of the Fredholm index as recorded in (3.14) is shown as follows. First, by Hypothesis 3.2 and alternative (ii) in Theorem 3.3, $\text{ind}(A(z)) = 0$ whenever $A(z)$ is boundedly invertible, in particular, this holds for a sufficiently small, punctured disk $D(z_0; \varepsilon) \setminus \{z_0\} \subset \Omega$ with center z_0 , that is, for $0 < |z - z_0| < \varepsilon_0$ for $0 < \varepsilon_0$ sufficiently small. Since $A(\cdot)$ is analytic in Ω , a perturbation argument, writing $A(z_0) = A(z) + [A(z_0) - A(z)]$ then yields (cf. [42, Theorem 5.11])

$$\text{ind}(A(z_0)) = \text{ind}(A(z)) = 0, \quad (3.26)$$

choosing $0 < \varepsilon_0$ sufficiently small. Precisely the same arguments apply to $\text{ind}(A_1(\cdot))$ and hence yield (3.14).

Next, since $A(z_0) = Q_1A_1(z_0)$, Q_1 maps $\text{ran}(A_1(z_0))$ onto $\text{ran}(A(z_0))$. Consequently, if $Q_1^*f \in \text{ran}(A_1(z_0))^\perp$ for some $f \in \mathcal{H}$, then

$$(f, A(z_0)g)_{\mathcal{H}} = (f, Q_1A_1(z_0)g)_{\mathcal{H}} = (Q_1^*f, A_1(z_0)g)_{\mathcal{H}} = 0 \text{ for all } g \in \mathcal{H} \quad (3.27)$$

implies $f \in \text{ran}(A(z_0))^\perp$. Hence, one concludes that $Q_1^* f = 0$ since $\ker(Q_1^*) = \text{ran}(Q_1)^\perp = \text{ran}(A(z_0))^\perp$ and thus,

$$\text{ran}(Q_1^*) \cap \text{ran}(A_1(z_0))^\perp = \{0\} \quad (3.28)$$

yields the existence of a finite-dimensional (hence, closed) linear subspace $\mathcal{H}_1 \subset \mathcal{H}$ such that

$$\mathcal{H} = \text{ran}(Q_1^*) \dot{+} \text{ran}(A_1(z_0))^\perp \dot{+} \mathcal{H}_1. \quad (3.29)$$

Using the fact that

$$\mathcal{H} = \text{ran}(Q_1^*) \dot{+} \ker(Q_1^*) = \text{ran}(Q_1^*) \dot{+} \text{ran}(Q_1)^\perp = \text{ran}(Q_1^*) \dot{+} \text{ran}(A(z_0))^\perp, \quad (3.30)$$

one infers from (3.29), (3.30), and the paragraph following (3.10), that

$$\begin{aligned} \text{def}(A_1(z_0)) &= \dim(\text{ran}(A_1(z_0))^\perp) = \dim_{[\text{ran}(Q_1^*) \dot{+} \mathcal{H}_1]^\perp}(\mathcal{H}) \\ &\leq \dim_{[\text{ran}(Q_1^*)]^\perp}(\mathcal{H}) = \dim(\text{ran}(A(z_0))^\perp) = \text{def}(A(z_0)). \end{aligned} \quad (3.31)$$

Finally, suppose that $A(\cdot)^{-1}$ has a pole of order $n_0 \in \mathbb{N}$ at z_0 . Since $A(z)^{-1} = A_1(z)^{-1}[Q_1 - (z - z_0)^{-1}P_1]$, $A_1(z)^{-1}$ must have a pole at z_0 of order at least $n_0 - 1$. Since

$$A_1(z)^{-1} = -(z - z_0)A(z)^{-1}P_1 + A(z)^{-1}Q_1, \quad (3.32)$$

the order of the pole of the first term on the right-hand side of (3.32) cannot exceed $n_0 - 1$. If $Q_1 f = A(z_0)g$, for some $f, g \in \mathcal{H}$, one obtains

$$\begin{aligned} (z_0 - z)^{n_0-1}A(z)^{-1}Q_1 f &= (z_0 - z)^{n_0-1}A(z)^{-1}A(z_0)g \\ &= (z_0 - z)^{n_0-1}g + (z_0 - z)^{n_0}A(z)^{-1}(z - z_0)^{-1}[A(z)g - A(z_0)g]. \end{aligned} \quad (3.33)$$

For each fixed $f \in \mathcal{H}$, the latter expression is uniformly bounded with respect to z near z_0 , and hence the uniform boundedness principle guarantees that also the pole of $A(z)^{-1}Q_1$ at z_0 cannot exceed $n_0 - 1$. Thus, $A_1(\cdot)^{-1}$ has precisely a pole of order $n_0 - 1$ at z_0 . To prove (3.16) for $z \neq z_0$, it suffices to note the pair of formulas

$$I_{\mathcal{H}} - A(z) = [1 + (z - z_0)]P_1 + [Q_1 - (z - z_0)P_1][I_{\mathcal{H}} - A_1(z)], \quad z \in \Omega, \quad (3.34)$$

$$I_{\mathcal{H}} - A_1(z) = [1 + (z - z_0)^{-1}]P_1 + [Q_1 - (z - z_0)^{-1}P_1][I_{\mathcal{H}} - A(z)], \quad z \in \Omega \setminus \{z_0\}, \quad (3.35)$$

and use the following facts: $P_1 \in \mathcal{F}(\mathcal{H}) \subset \mathcal{B}_p(\mathcal{H})$, both $\mathcal{F}(\mathcal{H})$ and $\mathcal{B}_p(\mathcal{H})$ are closed under addition, and $\mathcal{B}_p(\mathcal{H})$ is a two-sided ideal of $\mathcal{B}(\mathcal{H})$, $1 \leq p \leq \infty$. To settle the case $z = z_0$, one uses (3.34) to conclude that $[I_{\mathcal{H}} - A(z_0)] \in \mathcal{F}(\mathcal{H})$ (resp., $[I_{\mathcal{H}} - A(z_0)] \in \mathcal{B}_p(\mathcal{H})$) if $[I_{\mathcal{H}} - A_1(z_0)] \in \mathcal{F}(\mathcal{H})$ (resp., $[I_{\mathcal{H}} - A_1(z_0)] \in \mathcal{B}_p(\mathcal{H})$). To arrive at the converse, one applies (3.11) with $z = z_0$ to obtain

$$I_{\mathcal{H}} - A_1(z_0) = I - (P_1 + Q_1)A_1(z_0) = -P_1A_1(z_0) + [I_{\mathcal{H}} - A(z_0)], \quad (3.36)$$

noting that $P_1A_1(z_0) \in \mathcal{F}(\mathcal{H})$. \square

Still assuming that $A : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ satisfies Hypothesis 3.2 and that $A(\cdot)^{-1}$ has a pole at $z_0 \in \Omega$, we now decompose \mathcal{H} as follows. Introducing the Riesz projection $P(z)$ associated with $A(z)$, $z \in \mathcal{N}(z_0)$ (cf., e.g., [25, Sect. III.6]), with $\mathcal{N}(z_0) \subset \Omega$ a sufficiently small neighborhood of z_0

$$P(z) = \frac{-1}{2\pi i} \oint_{\mathcal{C}(0; \varepsilon_0)} d\zeta (A(z) - \zeta I_{\mathcal{H}})^{-1}, \quad z \in \mathcal{N}(z_0), \quad (3.37)$$

then $P(\cdot)$ is analytic on $\mathcal{N}(z_0)$ and

$$\dim(\text{ran}(P(z))) < \infty, \quad z \in \mathcal{N}(z_0). \quad (3.38)$$

In addition, introduce the projections

$$Q(z) = I_{\mathcal{H}} - P(z), \quad z \in \mathcal{N}(z_0). \quad (3.39)$$

Next, following Wolf [51] one introduces the transformation

$$T(z) = P(z_0)P(z) + Q(z_0)Q(z) = P(z_0)P(z) + [I_{\mathcal{H}} - P(z_0)][I_{\mathcal{H}} - P(z)], \quad z \in \mathcal{N}(z_0), \quad (3.40)$$

such that

$$P(z_0)T(z) = T(z)P(z), \quad Q(z_0)T(z) = T(z)Q(z), \quad z \in \mathcal{N}(z_0). \quad (3.41)$$

In addition, for $|z - z_0|$ sufficiently small, also $T(\cdot)^{-1}$ is analytic,

$$T(z) = I_{\mathcal{H}} + O(z - z_0), \quad |z - z_0| \text{ sufficiently small}, \quad (3.42)$$

and without loss of generality we may assume in the following that $T(\cdot)$ and $T(\cdot)^{-1}$ are analytic on $\mathcal{N}(z_0)$. This yields the decomposition of \mathcal{H} into

$$\mathcal{H} = P(z_0)\mathcal{H} \dot{+} Q(z_0)\mathcal{H} \quad (3.43)$$

and the associated 2×2 block operator decomposition of $T(z)A(z)T(z)^{-1}$ into

$$T(z)A(z)T(z)^{-1} = \begin{pmatrix} F(z) & 0 \\ 0 & G(z) \end{pmatrix}, \quad z \in \mathcal{N}(z_0), \quad (3.44)$$

where $F(\cdot)$ and $G(\cdot)$ are analytic on $\mathcal{N}(z_0)$, and, again without loss of generality, $G(\cdot)$ is boundedly invertible on $\mathcal{N}(z_0)$,

$$G(z)^{-1} \in \mathcal{B}(\text{ran}(Q(z_0))), \quad z \in \mathcal{N}(z_0). \quad (3.45)$$

Next, we introduce more notation: Let $\Omega_0 \subseteq \mathbb{C}$ be open and connected and $f: \Omega_0 \rightarrow \mathbb{C} \cup \{\infty\}$ be meromorphic and not identically vanishing on Ω_0 . The multiplicity function $m(z; f)$, $z \in \Omega_0$, is then defined by

$$m(z; f) = \begin{cases} k, & \text{if } z \text{ is a zero of } f \text{ of order } k, \\ -k, & \text{if } z \text{ is a pole of order } k, \\ 0, & \text{otherwise} \end{cases} \quad (3.46)$$

$$= \frac{1}{2\pi i} \oint_{C(z; \varepsilon)} d\zeta \frac{f'(\zeta)}{f(\zeta)}, \quad z \in \Omega_0, \quad (3.47)$$

for $\varepsilon > 0$ sufficiently small. Here the circle $C(z; \varepsilon)$ is chosen sufficiently small such that $C(z; \varepsilon)$ contains no other singularities or zeros of f except, possibly, z . If f vanishes identically on Ω_0 , one defines

$$m(z; f) = \infty, \quad z \in \Omega_0. \quad (3.48)$$

Given the block decomposition (3.44), we follow Howland in introducing the quantity $\nu(z_0; A(\cdot))$ by

$$\nu(z_0; A(\cdot)) = \begin{cases} m(z_0; \det_{\text{ran}(P(z_0))}(F(\cdot))), & \text{if } \det_{\text{ran}(P(z_0))}(F(\cdot)) \not\equiv 0 \text{ on } \mathcal{N}(z_0), \\ \infty, & \text{if } \det_{\text{ran}(P(z_0))}(F(\cdot)) \equiv 0 \text{ on } \mathcal{N}(z_0). \end{cases} \quad (3.49)$$

We also recall the abbreviation

$$m_g(0; A(z_0)) = \dim(\ker(A(z_0))). \quad (3.50)$$

Repeated applications of Theorem 3.4 then yields the following principal factorization result of [21] (again, we extend it to the case of Fredholm operators $A(\cdot)$):

Theorem 3.5 ([21]). *Assume that $A : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ satisfies Hypothesis 3.2 and let $z_0 \in \Omega$ be a pole of $A(\cdot)^{-1}$ of order $n_0 \in \mathbb{N}$. Then there exist projections P_j and $Q_j = I_{\mathcal{H}} - P_j$ in \mathcal{H} such that with $p_j = \dim(\text{ran}(P_j))$, $1 \leq j \leq n_0$, one infers that*

$$A(z) = [Q_1 - (z - z_0)P_1][Q_2 - (z - z_0)P_2] \cdots [Q_{n_0} - (z - z_0)P_{n_0}]A_{n_0}(z), \quad z \in \Omega, \quad (3.51)$$

and

$$1 \leq p_{n_0} \leq p_{n_0-1} \leq \cdots \leq p_2 \leq p_1 < \infty, \quad (3.52)$$

where

$$A_{n_0}(\cdot) \text{ is analytic on } \Omega, \quad (3.53)$$

$$A_{n_0}(z) \in \Phi(\mathcal{H}), \quad z \in \Omega, \quad (3.54)$$

$$\text{ind}(A(z)) = \text{ind}(A_{n_0}(z)) = 0, \quad z \in \Omega, \quad |z - z_0| \text{ sufficiently small}, \quad (3.55)$$

$$A_{n_0}(z)^{-1} \in \mathcal{B}(\mathcal{H}), \quad z \in \Omega, \quad |z - z_0| \text{ sufficiently small}. \quad (3.56)$$

In addition,

$$p_1 = \dim(\ker(A(z_0))) = m_g(0; A(z_0)), \quad (3.57)$$

and hence

$$\nu(z_0; A(\cdot)) = \sum_{j=1}^{n_0} p_j \geq m_g(0; A(z_0)), \quad \nu(z_0; A(\cdot)) \geq n_0. \quad (3.58)$$

Finally,

$$[I_{\mathcal{H}} - A(\cdot)] \in \mathcal{F}(\mathcal{H}) \text{ (resp., } \mathcal{B}_p(\mathcal{H}) \text{ for some } 1 \leq p \leq \infty) \quad (3.59)$$

if and only if $[I_{\mathcal{H}} - A_{n_0}(\cdot)] \in \mathcal{F}(\mathcal{H})$ (resp., $\mathcal{B}_p(\mathcal{H})$ for some $1 \leq p \leq \infty$).

Proof. Since by hypothesis z_0 is a pole of $A(\cdot)^{-1}$ and hence an isolated singularity of $A(\cdot)^{-1}$, the second alternative of the analytic Fredholm theorem, Theorem 3.3, is realized. Applying Theorem 3.4 n_0 times, one obtains the n_0 factors $[Q_j - (z - z_0)P_j]$ and ends up with the facts (3.53)–(3.55). In addition, z_0 is *not* a pole of $A_{n_0}(\cdot)^{-1}$.

The bounded invertibility of $A_{n_0}(\cdot)$ in a sufficiently small punctured disk centered at z_0 is clear from that of $A(\cdot)$, (3.20), and (3.51). To prove that also $A_{n_0}(z_0)^{-1} \in \mathcal{B}(\mathcal{H})$ one can argue as follows. Since $A_{n_0}(\cdot)^{-1}$ has no pole at $z = z_0$, $A_{n_0}(z_0)$ is injective, $\ker(A_{n_0}(z_0)) = \{0\}$. In addition, since $A_{n_0}(z_0)$ is bounded and hence closed, $A_{n_0}(z_0)^{-1}$ is closed as well (cf., e.g., [25, p. 165]). As $\text{ind}(A_{n_0}(z_0)) = \dim(\ker(A_{n_0}(z_0))) = 0$, also $\dim(\ker(A_{n_0}(z_0)^*)) = 0$, and hence $\text{ran}(A_{n_0}(z_0)) = \mathcal{H}$, since $\text{ran}(A_{n_0}(z_0))$ is closed in \mathcal{H} . Hence, $A_{n_0}(z_0)^{-1}$ is defined on all of \mathcal{H} and an application of the closed graph theorem (see, e.g., [25, Theorem III.5.20]) then yields $A_{n_0}(z_0)^{-1} \in \mathcal{B}(\mathcal{H})$ and hence proves (3.56).

Equation (3.57) is clear from the decomposition $\mathcal{H} = P_1\mathcal{H} \dot{+} Q_1\mathcal{H}$ with $Q_1\mathcal{H} = \text{ran}(A(z_0))$. In light of the second alternative of the analytic Fredholm theorem, in particular (3.6) and (3.7), $A(\cdot)^{-1}$ has a finite-dimensional residue at z_0 . In addition,

$$\dim(\ker(A(z_0)^\sharp)) = \dim(\text{ran}(P_1^\sharp)) = m_g(z_0; A(z_0)^\sharp) = p_1 < \infty, \quad (3.60)$$

where T^\sharp represents T or T^* . The inequality (3.15) for the defects then yields the inequalities (3.52).

Next, writing $[Q_j - (z - z_0)P_j] = I_{\mathcal{H}} - [1 + (z - z_0)]P_j$, $1 \leq j \leq n_0$, one can rewrite (3.51) in the form

$$A(z) = [I_{\mathcal{H}} - F_0(z)]A_{n_0}(z), \quad z \in \Omega, \quad (3.61)$$

with $F_0(\cdot) \in \mathcal{F}(\mathcal{H})$ analytic on Ω . Similarly, writing $[Q_j - (z - z_0)P_j]^{-1} = Q_j - (z - z_0)^{-1}P_j = I_{\mathcal{H}} - [1 + (z - z_0)^{-1}]P_j$, $1 \leq j \leq n_0$, one obtains that

$$[I_{\mathcal{H}} - F_0(z)]^{-1} = I_{\mathcal{H}} + F_1(z), \quad z \in \mathcal{N}(z_0), \quad (3.62)$$

for a sufficiently small neighborhood $\mathcal{N}(z_0) \subset \Omega$ of z_0 , and with $F_1(\cdot) \in \mathcal{F}(\mathcal{H})$ meromorphic on $\mathcal{N}(z_0)$ and analytic on $\mathcal{N}(z_0) \setminus \{z_0\}$. Thus, one computes using (3.44) and (3.61)

$$\begin{aligned} T(z)A(z)T(z)^{-1} &= \begin{pmatrix} F(z) & 0 \\ 0 & G(z) \end{pmatrix} \\ &= [I_{\mathcal{H}} - T(z)F_0(z)T(z)^{-1}]T(z)A_{n_0}(z)T(z)^{-1}, \quad z \in \mathcal{N}(z_0), \end{aligned} \quad (3.63)$$

and hence

$$\begin{aligned} \begin{pmatrix} F(z) & 0 \\ 0 & I_{\text{ran}(Q(z_0))} \end{pmatrix} &= T(z)A(z)T(z)^{-1} \begin{pmatrix} I_{\text{ran}(P(z_0))} & 0 \\ 0 & G(z)^{-1} \end{pmatrix} \\ &= [I_{\mathcal{H}} - T(z)F_0(z)T(z)^{-1}]T(z)A_{n_0}(z)T(z)^{-1} \begin{pmatrix} I_{\text{ran}(P(z_0))} & 0 \\ 0 & G(z)^{-1} \end{pmatrix}, \quad (3.64) \\ & \quad z \in \mathcal{N}(z_0), \end{aligned}$$

implying

$$\begin{aligned} T(z)A_{n_0}(z)T(z)^{-1} &\begin{pmatrix} I_{\text{ran}(P(z_0))} & 0 \\ 0 & G(z)^{-1} \end{pmatrix} \\ &= [I_{\mathcal{H}} - T(z)F_0(z)T(z)^{-1}]^{-1} \begin{pmatrix} F(z) & 0 \\ 0 & I_{\text{ran}(Q(z_0))} \end{pmatrix} \\ &= [I_{\mathcal{H}} + T(z)F_1(z)T(z)^{-1}] \begin{pmatrix} I_{\text{ran}(P(z_0))} + [F(z) - I_{\text{ran}(P(z_0))}] & 0 \\ 0 & I_{\text{ran}(Q(z_0))} \end{pmatrix} \\ &= [I_{\mathcal{H}} + F_2(z)][I_{\mathcal{H}} - F_3(z)] \\ &= [I_{\mathcal{H}} - F_4(z)], \quad z \in \mathcal{N}(z_0). \end{aligned} \quad (3.65)$$

In (3.65), we have set

$$F_2(z) = T(z)F_1(z)T(z)^{-1}, \quad (3.66)$$

$$F_3(z) = I_{\mathcal{H}} - \begin{pmatrix} F(z) & 0 \\ 0 & I_{\text{ran}(Q(z_0))} \end{pmatrix}, \quad (3.67)$$

$$F_4(z) = -F_2(z) + F_2(z)F_3(z) + F_3(z), \quad z \in \mathcal{N}(z_0), \quad (3.68)$$

where $F_k(\cdot) \in \mathcal{F}(\mathcal{H})$, $2 \leq k \leq 4$, $F_3(\cdot)$ is analytic on $\mathcal{N}(z_0)$, $F_2(\cdot), F_4(\cdot)$ are meromorphic on $\mathcal{N}(z_0)$ and analytic on $\mathcal{N}(z_0) \setminus \{z_0\}$. In fact, since the left-hand side of (3.65) is analytic and boundedly invertible on $\mathcal{N}(z_0)$, $F_4(\cdot)$ is analytic on $\mathcal{N}(z_0)$ and

$$[I_{\mathcal{H}} - F_4(z)]^{-1} \in \mathcal{B}(\mathcal{H}), \quad \det_{\mathcal{H}}(I_{\mathcal{H}} - F_4(z)) \neq 0, \quad z \in \mathcal{N}(z_0). \quad (3.69)$$

Combining (3.64) and (3.65) then yields

$$\begin{aligned} \begin{pmatrix} F(z) & 0 \\ 0 & I_{\text{ran}(Q(z_0))} \end{pmatrix} &= [I_{\mathcal{H}} - T(z)F_0(z)T(z)^{-1}][I_{\mathcal{H}} - F_4(z)] \\ &= T(z)[Q_1 - (z - z_0)P_1][Q_2 - (z - z_0)P_2] \cdots [Q_{n_0} - (z - z_0)P_{n_0}]T(z)^{-1} \\ & \quad \times [I_{\mathcal{H}} - F_4(z)], \quad z \in \mathcal{N}(z_0). \end{aligned} \quad (3.70)$$

Thus, by (3.21),

$$\det_{\mathcal{H}} \left(\begin{pmatrix} F(z) & 0 \\ 0 & I_{\text{ran}(Q(z_0))} \end{pmatrix} \right) = \left(\prod_{j=1}^{n_0} (z_0 - z)^{p_j} \right) \det_{\mathcal{H}}(I_{\mathcal{H}} - F_4(z)), \quad z \in \mathcal{N}(z_0), \quad (3.71)$$

and hence (3.58) holds.

Relations (3.59) are clear from (3.16). \square

Corollary 3.6 ([21]). *Assume that $A : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ satisfies Hypothesis 3.2 and let $z_0 \in \Omega$ be a pole of $A(\cdot)^{-1}$. Then z_0 is a simple pole of $A(\cdot)^{-1}$ if and only if $\nu(z_0; A(\cdot)) = m_g(0; A(z_0))$.*

In the remainder of this section we briefly derive the analogous factorizations in Theorems 3.4 and 3.5 but with the order of factors in (3.11) and (3.51) interchanged. This appears to be a new result.

Theorem 3.7. *Assume that $A : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ satisfies Hypothesis 3.2 and let $z_0 \in \Omega$ be a pole of $A(\cdot)^{-1}$. Denote by \tilde{P}_1 any projection onto $\ker(A(z_0))$ and let $\tilde{Q}_1 = I_{\mathcal{H}} - \tilde{P}_1$. Then,*

$$A(z) = \tilde{A}_1(z) [\tilde{Q}_1 - (z - z_0) \tilde{P}_1], \quad z \in \Omega, \quad (3.72)$$

where

$$\tilde{A}_1(\cdot) \text{ is analytic on } \Omega, \quad (3.73)$$

$$\tilde{A}_1(z) \in \Phi(\mathcal{H}), \quad z \in \Omega, \quad (3.74)$$

$$\text{def}(\tilde{A}_1(z_0)) \leq \text{def}(A(z_0)), \quad (3.75)$$

$$\text{ind}(\tilde{A}(z)) = \text{ind}(\tilde{A}_1(z)) = 0, \quad z \in \Omega, \quad |z - z_0| \text{ sufficiently small.} \quad (3.76)$$

If z_0 is a pole of $A(\cdot)^{-1}$ of order $n_0 \in \mathbb{N}$, then z_0 is a pole of $(\tilde{A}_1(\cdot))^{-1}$ of order $n_0 - 1$. Finally,

$$[I_{\mathcal{H}} - A(\cdot)] \in \mathcal{F}(\mathcal{H}) \text{ (resp., } \mathcal{B}_p(\mathcal{H}) \text{ for some } 1 \leq p \leq \infty) \quad (3.77)$$

if and only if $[I_{\mathcal{H}} - \tilde{A}_1(\cdot)] \in \mathcal{F}(\mathcal{H})$ (resp., $\mathcal{B}_p(\mathcal{H})$ for some $1 \leq p \leq \infty$).

Proof. Set

$$\underline{\Omega} = \{\bar{z} \in \mathbb{C} \mid z \in \Omega\}, \quad (3.78)$$

and define $B : \underline{\Omega} \rightarrow \mathcal{B}(\mathcal{H})$ by

$$B(\zeta) = A(\bar{\zeta})^*, \quad \zeta \in \underline{\Omega}. \quad (3.79)$$

Evidently, $B(\zeta) \in \Phi(\mathcal{H})$, $\zeta \in \underline{\Omega}$, and \bar{z}_0 is a pole of $B(\cdot)^{-1}$. If \tilde{P}_1 is any projection onto $\ker(A(z_0))$ and $\tilde{Q}_1 = I_{\mathcal{H}} - \tilde{P}_1$, then

$$\tilde{Q}_1^* = I_{\mathcal{H}} - \tilde{P}_1^* \quad (3.80)$$

projects onto (cf., e.g., [25, p. 155–156, 267])

$$[\text{ran}(\tilde{P}_1)]^{\perp} = [\ker(A(z_0))]^{\perp} = \text{ran}(A(z_0)^*) = \text{ran}(B(\bar{z}_0)), \quad (3.81)$$

employing the closed range property of $A(z_0)$ and hence of $A(z_0)^*$ due to the Fredholm hypothesis on $A(\cdot)$ in (3.4). Applying Theorem 3.4 to $B(\cdot)$ and \tilde{Q}_1^* , one obtains an $A_1(\cdot) : \underline{\Omega} \rightarrow \Phi(\mathcal{H})$ with the properties listed in Theorem 3.4. In particular,

$$B(\zeta) = [\tilde{Q}_1^* - (\zeta - \bar{z}_0) \tilde{P}_1^*] A_1(\zeta), \quad \zeta \in \underline{\Omega}. \quad (3.82)$$

Taking adjoints in (3.82) results in

$$A(\bar{\zeta}) = B(\zeta)^* = A_1(\zeta)^* [\tilde{Q}_1 - (\bar{\zeta} - z_0)\tilde{P}_1], \quad \zeta \in \underline{\Omega}, \quad (3.83)$$

and since $\zeta \in \underline{\Omega}$ is equivalent to $\zeta = \bar{z}$ for some $z \in \Omega$, one obtains

$$A(z) = A_1(\bar{z})^* [\tilde{Q}_1 - (z - z_0)\tilde{P}_1], \quad z \in \Omega. \quad (3.84)$$

Therefore, (3.72) holds with

$$\tilde{A}_1(z) := A_1(\bar{z})^*, \quad z \in \Omega, \quad (3.85)$$

and $A_1(\cdot)$ inherits the properties (3.73) and (3.75) from the corresponding properties (3.12) and (3.13). Employing $A(z_0) = \tilde{A}_1(z_0)\tilde{Q}_1$, one concludes that

$$\text{ran}(A(z_0)) \subseteq \text{ran}(\tilde{A}_1(z_0)), \quad \text{ran}(\tilde{A}_1(z_0))^\perp \subseteq \text{ran}(A(z_0))^\perp, \quad (3.86)$$

proving (3.75). Invariance of the Fredholm index as recorded in (3.76) follows precisely as in the proof of Theorem 3.5. The statement about orders of poles of $A(\cdot)^{-1}$ and $\tilde{A}_1(\cdot)^{-1}$ is also clear. If z_0 is a pole of $A(\cdot)^{-1}$ of order $n_0 \in \mathbb{N}$, then \bar{z}_0 is a pole of $B(\cdot)^{-1}$ of order n_0 . Hence, \bar{z}_0 is a pole of $A_1(\cdot)^{-1}$ of order $n_0 - 1$ (by Theorem 3.4), and z_0 is a pole of $\tilde{A}_1(\cdot)^{-1}$ of order $n_0 - 1$. Finally, (3.77) follows from the corresponding statement (3.16) for $B(\cdot)$ and $A_1(\cdot)$ by taking adjoints. \square

Applying Theorem 3.5 to $B(\cdot)$ as defined in (3.79), one obtains the following analog of Theorem 3.5 with the order of factors reversed.

Theorem 3.8. *Assume that $A : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ satisfies Hypothesis 3.2, and let $z_0 \in \Omega$ be a pole of $A(\cdot)^{-1}$ of order $n_0 \in \mathbb{N}$. Then there exist projections \tilde{P}_j and $\tilde{Q}_j = I_{\mathcal{H}} - \tilde{P}_j$ in \mathcal{H} such that with $\tilde{p}_j = \dim(\text{ran}(\tilde{P}_j))$, $1 \leq j \leq n_0$, one infers that*

$$A(z) = \tilde{A}_{n_0}(z) [\tilde{Q}_{n_0} - (z - z_0)\tilde{P}_{n_0}] \cdots [\tilde{Q}_2 - (z - z_0)\tilde{P}_2] [\tilde{Q}_1 - (z - z_0)\tilde{P}_1], \quad z \in \Omega, \quad (3.87)$$

and

$$1 \leq \tilde{p}_{n_0} \leq \tilde{p}_{n_0-1} \leq \cdots \leq \tilde{p}_2 \leq \tilde{p}_1 < \infty, \quad (3.88)$$

where

$$\tilde{A}_{n_0}(\cdot) \text{ is analytic on } \Omega, \quad (3.89)$$

$$\tilde{A}_{n_0}(z) \in \Phi(\mathcal{H}), \quad z \in \Omega, \quad (3.90)$$

$$\text{ind}(\tilde{A}(z)) = \text{ind}(\tilde{A}_{n_0}(z)) = 0, \quad z \in \Omega, \quad |z - z_0| \text{ sufficiently small}, \quad (3.91)$$

$$[\tilde{A}_{n_0}(z)]^{-1} \in \mathcal{B}(\mathcal{H}), \quad z \in \Omega, \quad |z - z_0| \text{ sufficiently small}. \quad (3.92)$$

In addition,

$$\tilde{p}_1 = \dim(\ker(A(z_0))) = m_g(0; A(z_0)), \quad (3.93)$$

and, hence,

$$\tilde{\nu}(z_0; A(\cdot)) = \sum_{j=1}^{n_0} \tilde{p}_j \geq m_g(0; A(z_0)), \quad \tilde{\nu}(z_0; A(\cdot)) \geq n_0. \quad (3.94)$$

Finally,

$$[I_{\mathcal{H}} - A(\cdot)] \in \mathcal{F}(\mathcal{H}) \text{ (resp., } \mathcal{B}_p(\mathcal{H}) \text{ for some } 1 \leq p \leq \infty) \quad (3.95)$$

if and only if $[I_{\mathcal{H}} - \tilde{A}_{n_0}(\cdot)] \in \mathcal{F}(\mathcal{H})$ (resp., $\mathcal{B}_p(\mathcal{H})$ for some $1 \leq p \leq \infty$).

Proof. Applying Theorem 3.4 to $B(\cdot)$ defined in (3.79), one obtains the existence of $A_{n_0}(\cdot)$ with the properties in Theorem 3.4 and projections P_j and $Q_j = I_{\mathcal{H}} - P_j$ in \mathcal{H} such that, with $\dim(\text{ran}(P_j)) < \infty$, $1 \leq j \leq n_0$, the factorization

$$B(\zeta) = [Q_1 - (\zeta - \bar{z}_0)P_1][Q_2 - (\zeta - \bar{z}_0)P_2] \cdots [Q_{n_0} - (\zeta - \bar{z}_0)P_{n_0}]A_{n_0}(\zeta), \quad \zeta \in \underline{\Omega}, \quad (3.96)$$

holds. Taking adjoints in (3.96) implies

$$\begin{aligned} A(\bar{\zeta}) &= B(\zeta)^* \\ &= A_{n_0}(\zeta)^*[Q_{n_0}^* - (\bar{\zeta} - z_0)P_{n_0}^*] \cdots [Q_2^* - (\bar{\zeta} - z_0)P_2^*][Q_1^* - (\bar{\zeta} - z_0)P_1^*], \\ &\quad \zeta \in \underline{\Omega}. \end{aligned} \quad (3.97)$$

Writing $\bar{\zeta} = z \in \Omega$, (3.97) takes the form

$$A(z) = A_{n_0}(\bar{z})^*[Q_{n_0}^* - (z - z_0)P_{n_0}^*] \cdots [Q_2^* - (z - z_0)P_2^*][Q_1^* - (z - z_0)P_1^*], \quad z \in \Omega. \quad (3.98)$$

Defining $\widetilde{P}_j = P_j^*$, $1 \leq j \leq n_0$, and

$$\widetilde{A}_{n_0}(z) = A_{n_0}(\bar{z})^*, \quad z \in \Omega, \quad (3.99)$$

yields the factorization (3.87) with $\widetilde{p}_j = \dim(\text{ran}(P_j)^*) = \dim(\text{ran}(P_j))$, $1 \leq j \leq n_0$. The properties (3.89)–(3.95) follow immediately from the corresponding properties of $A_{n_0}(\cdot)$ and $B(\cdot)$. We omit further details at this point, and only note that

$$\widetilde{p}_1 = \dim(\ker(B(\bar{z}_0))) = \dim(\ker(A(z_0))). \quad (3.100)$$

□

Remark 3.9. Since $A(\cdot), A_1(\cdot), \dots, A_{n_0}(\cdot)$ in Theorems 3.4 and 3.5 all have index zero at $z_0 \in \Omega$, the dimensions p_j in Theorem 3.5 and \widetilde{p}_j in Theorem 3.8 satisfy $p_j = \widetilde{p}_j$, $1 \leq j \leq n_0$, repeatedly applying equalities of the type $p_1 = \dim(\ker(A(z_0)^*)) = \dim(\ker(A(z_0))) = \widetilde{p}_1$, etc. In particular, $\nu(z; A(\cdot)) = \widetilde{\nu}(z_0; A(\cdot))$.

4. ALGEBRAIC MULTIPLICITIES OF ZEROS OF OPERATOR-VALUED FUNCTIONS: THE ANALYTIC CASE

In this section we consider algebraic multiplicities of zeros of analytic operator-valued functions and then study applications to the Birman–Schwinger operator $K(\cdot)$ in (2.4) in search of the analog of Theorem 2.7 for algebraic multiplicities.

The pertinent facts in this context can be found in [18] (see also, [16, Sects. XI.8, XI.9], [17, Ch. 4], and [34, Sect. 11]). Let $\Omega \subseteq \mathbb{C}$ be open and connected, $z_0 \in \Omega$, suppose that $A : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ is analytic on Ω , and that $A(z_0)$ is a Fredholm operator (i.e., $\dim(\ker(A(z_0))) < \infty$, $\text{ran}(A(z_0))$ is closed in \mathcal{H} , and $\dim(\ker(A(z_0)^*)) < \infty$) of index zero, that is,

$$\text{ind}(A(z_0)) = \dim(\ker(A(z_0))) - \dim(\ker(A(z_0)^*)) = 0. \quad (4.1)$$

By [18] (or by [16, Theorem XI.8.1]) there exists a neighborhood $\mathcal{N}(z_0) \subset \Omega$ and analytic and boundedly invertible operator-valued functions $E_j : \Omega \rightarrow \mathcal{B}(\mathcal{H})$, $j = 1, 2$, such that

$$A(z) = E_1(z)\widetilde{A}(z)E_2(z), \quad z \in \mathcal{N}(z_0), \quad (4.2)$$

where $\widetilde{A}(\cdot)$ is of the particular form

$$\widetilde{A}(z) = \widetilde{P}_0 + \sum_{j=1}^r (z - z_0)^{n_j} \widetilde{P}_j, \quad z \in \mathcal{N}(z_0), \quad (4.3)$$

with

$$\begin{aligned} & \tilde{P}_k, \quad 0 \leq k \leq r, \quad \text{mutually disjoint projections in } \mathcal{H}, \\ & [I_{\mathcal{H}} - \tilde{P}_0] \in \mathcal{F}(\mathcal{H}), \quad \dim(\text{ran}(\tilde{P}_j)) = 1, \quad 1 \leq j \leq r, \\ & n_1 \leq n_2 \leq \cdots \leq n_r, \quad n_j \in \mathbb{N}, \quad 1 \leq j \leq r. \end{aligned} \quad (4.4)$$

Moreover (cf. [16, Sect. XI.8], [18]), the integers n_j , $1 \leq j \leq r$, are uniquely determined by $A(\cdot)$, and the geometric multiplicity $m_g(0; A(z_0))$ of $A(z_0)$ is given by

$$m_g(0; A(z_0)) = \dim(\ker(A(z_0))) = \dim(\text{ran}(I_{\mathcal{H}} - \tilde{P}_0)). \quad (4.5)$$

Definition 4.1. Let $\Omega \subseteq \mathbb{C}$ be open and connected, $z_0 \in \Omega$, suppose that $A : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ is analytic on Ω . Then z_0 is called a *zero of finite-type of $A(\cdot)$* if $A(z_0)$ is a Fredholm operator, $\ker(A(z_0)) \neq \{0\}$, and $A(\cdot)$ is boundedly invertible on $D(z_0; \varepsilon_0) \setminus \{z_0\}$, for sufficiently small $\varepsilon_0 > 0$.

Our choice of notation calling z_0 a *zero* of $A(\cdot)$ is close to Howland's notation of a *weak zero* of an operator-valued function in [21]. In Gohberg and Sigal [18] (and in related literature in the former Soviet Union) the notion of a *characteristic value* (or *eigenvalue*) is used instead in this connection.

If z_0 is a zero of finite-type of $A(\cdot)$, then (since $A(\cdot)$ is boundedly invertible on $D(z_0; \varepsilon_0) \setminus \{z_0\}$, for sufficiently small $\varepsilon_0 > 0$),

$$\text{ind}(A(z_0)) = 0, \quad (4.6)$$

and one has

$$\sum_{k=0}^r \tilde{P}_k = I_{\mathcal{H}}, \quad (4.7)$$

and

$$A(\cdot)^{-1} \text{ is finitely meromorphic at } z_0 \quad (4.8)$$

(cf. [16, Sect. XI.9], [18] for these facts).

Definition 4.2. Let $\Omega \subseteq \mathbb{C}$ be open and connected, $z_0 \in \Omega$, suppose that $A : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ is analytic on Ω , and assume that z_0 is a zero of finite-type of $A(\cdot)$. Then $m_a(z_0; A(\cdot))$, the *algebraic multiplicity of the zero of $A(\cdot)$ at z_0* , is defined to be (cf. [16, Sect. XI.9])

$$m_a(z_0; A(\cdot)) = \sum_{j=1}^r n_j, \quad (4.9)$$

with n_j , $1 \leq j \leq r$, introduced in (4.3)–(4.4).

Under the assumptions in Definition 4.2, one also has an extension of the argument principle for scalar analytic functions to the operator-valued case (cf. [16, Theorem XI.9.1], [18]) in the form

$$\begin{aligned} m_a(z_0; A(\cdot)) &= \text{tr}_{\mathcal{H}} \left(\frac{1}{2\pi i} \oint_{C(z_0; \varepsilon)} d\zeta A'(\zeta) A(\zeta)^{-1} \right) \\ &= \text{tr}_{\mathcal{H}} \left(\frac{1}{2\pi i} \oint_{C(z_0; \varepsilon)} d\zeta A(\zeta)^{-1} A'(\zeta) \right), \quad 0 < \varepsilon < \varepsilon_0. \end{aligned} \quad (4.10)$$

Since $A(\cdot)^{-1}$ is finitely meromorphic, the integral in (4.10) is a finite rank operator (the analytic and non-finite-rank part under the integral in (4.10) yielding a zero

contribution when integrated over $C(z_0; \varepsilon)$ and hence the trace in (4.10) is well-defined.

Next, recalling our notation of the principal part of an operator-valued meromorphic function in (3.2), one also obtains

$$\begin{aligned} m_a(z_0; A(\cdot)) &= \operatorname{tr}_{\mathcal{H}} \left(\frac{1}{2\pi i} \oint_{C(z_0; \varepsilon)} d\zeta \operatorname{pp}_{z_0} \{A'(\zeta)A(\zeta)^{-1}\} \right) \\ &= \operatorname{tr}_{\mathcal{H}} \left(\frac{1}{2\pi i} \oint_{C(z_0; \varepsilon)} d\zeta \operatorname{pp}_{z_0} \{A(\zeta)^{-1}A'(\zeta)\} \right), \quad 0 < \varepsilon < \varepsilon_0. \end{aligned} \quad (4.11)$$

Moreover, we mention the following useful result.

Lemma 4.3 ([16, Lemma 9.3], [17, Proposition 4.2.2]). *Let $\Omega \subseteq \mathbb{C}$ be open and connected and $M_j(\cdot)$, $j = 1, 2$, be finitely meromorphic at $z_0 \in \Omega$. Then $M_1(\cdot)M_2(\cdot)$ and $M_2(\cdot)M_1(\cdot)$ are finitely meromorphic at $z_0 \in \Omega$, and for $0 < \varepsilon_0$ sufficiently small,*

$$\oint_{C(z_0; \varepsilon)} d\zeta M_1(\zeta)M_2(\zeta), \quad \oint_{C(z_0; \varepsilon)} d\zeta M_2(\zeta)M_1(\zeta) \in \mathcal{F}(\mathcal{H}), \quad (4.12)$$

$$\operatorname{tr}_{\mathcal{H}} \left(\oint_{C(z_0; \varepsilon)} d\zeta M_1(\zeta)M_2(\zeta) \right) = \operatorname{tr}_{\mathcal{H}} \left(\oint_{C(z_0; \varepsilon)} d\zeta M_2(\zeta)M_1(\zeta) \right), \quad (4.13)$$

$$\operatorname{tr}_{\mathcal{H}} (\operatorname{pp}_{z_0} \{M_1(z)M_2(z)\}) = \operatorname{tr}_{\mathcal{H}} (\operatorname{pp}_{z_0} \{M_2(z)M_1(z)\}), \quad 0 < |z - z_0| < \varepsilon_0. \quad (4.14)$$

One notes that $m_a(z_0; A(\cdot))$ must be distinguished from $m_a(0; A(z_0))$. However, in the special case where $A(z) = A - zI_{\mathcal{H}}$, $z \in \Omega$, one has the following fact:

Remark 4.4. Let $\Omega \subseteq \mathbb{C}$ be open and connected, $z_0 \in \Omega$, and suppose that the particular function $A(z) = A - zI_{\mathcal{H}}$, $z \in \Omega$, with $A \in \mathcal{B}(\mathcal{H})$, satisfies the conditions of Definition 4.2. Then $m_a(z_0; A(\cdot))$, the *algebraic multiplicity of the zero of $A(\cdot)$ at z_0* equals the algebraic multiplicity $m_a(z_0; A)$ of the eigenvalue z_0 of A ,

$$m_a(z_0; A(\cdot)) = m_a(z_0; A), \quad (4.15)$$

since the right-hand side of (4.10) equals

$$\operatorname{tr}_{\mathcal{H}} \left(\frac{-1}{2\pi i} \oint_{C(z_0; \varepsilon)} d\zeta (A - \zeta I_{\mathcal{H}})^{-1} \right) = \operatorname{tr}_{\mathcal{H}} (P(z_0; A)) = m_a(z_0; A), \quad 0 < \varepsilon < \varepsilon_0, \quad (4.16)$$

with $P(z_0; A)$ the Riesz projection associated with A and its isolated eigenvalue z_0 .

Following standard practice, we now introduce the *discrete spectrum* of a densely defined, closed, linear operator T in \mathcal{H} by

$$\sigma_d(T) = \{z \in \sigma_p(T) \mid z \text{ is an isolated point of } \sigma(T), \text{ with } m_a(z; T) < \infty\}, \quad (4.17)$$

and denote its *essential spectrum* by

$$\sigma_{ess}(T) = \mathbb{C} \setminus \sigma_d(T). \quad (4.18)$$

Since T is not assumed to be self-adjoint, one notes that several inequivalent definitions of $\sigma_{ess}(T)$ are in use in the literature (cf., e.g., [10, Ch. IX] for a detailed discussion), but in this paper we will only use the one in (4.18).

Given this background material, we now apply it to reprove and slightly extend a multiplicity result due to Latushkin and Sukhtyaev [30].

Theorem 4.5. *Assume Hypothesis 2.5 and suppose that $z_0 \in \rho(H_0) \cap \sigma(H)$ with $D(z_0; \varepsilon_0) \cap \sigma(H) = \{z_0\}$ for some $\varepsilon_0 > 0$. Then z_0 is a discrete eigenvalue of H ,*

$$z_0 \in \sigma_d(H). \quad (4.19)$$

In addition, z_0 is a zero of finite-type of $I_{\mathcal{K}} - K(\cdot)$ and

$$m_a(z_0; H) = m_a(z_0; I_{\mathcal{K}} - K(\cdot)) = \nu(z_0; I_{\mathcal{K}} - K(\cdot)). \quad (4.20)$$

Proof. By (2.12), any singularity $z_0 \in \rho(H_0)$ of $R(z) = (H - zI_{\mathcal{H}})^{-1}$ must be a singularity of $[I_{\mathcal{K}} - K(\cdot)]^{-1}$. Since by Hypothesis 2.5, $[I_{\mathcal{K}} - K(\cdot)] \in \Phi(\mathcal{K})$ on $\rho(H_0)$, and $[I_{\mathcal{K}} - K(z_1)]$ is boundedly invertible for some $z_1 \in \rho(H_0)$, alternative (ii) in Theorem 3.3, with $\Omega = \rho(H_0)$, applies to $[I_{\mathcal{K}} - K(\cdot)]$. In particular, $[I_{\mathcal{K}} - K(\cdot)]^{-1}$ is finitely meromorphic on $\rho(H_0)$ and hence so is $R(\cdot)$. By [25, Sect. III.6.5], this implies that $z_0 \in \sigma_p(H)$ and then again by the finitely meromorphic property of $R(\cdot)$ on $\rho(H_0)$, the Riesz projection associated with z_0 ,

$$P(z_0; H) = \frac{-1}{2\pi i} \oint_{C(z_0; \varepsilon)} dz (H - zI_{\mathcal{H}})^{-1}, \quad 0 < \varepsilon < \varepsilon_0, \quad (4.21)$$

is finite-dimensional, which in turn is equivalent to the eigenvalue z_0 having finite algebraic multiplicity and hence to (4.19).

Without loss of generality we may assume that $z_0 = 0$ for the remainder of the proof of Theorem 4.5. Identifying $A(\cdot) = I_{\mathcal{K}} - K(\cdot)$, an application of Theorem 3.5 (using the notation employed in the latter) yields for $0 < |z| < \varepsilon_0$,

$$A(z) = [Q_1 - zP_1][Q_2 - zP_2] \cdots [Q_{n_0} - zP_{n_0}]A_{n_0}(z), \quad (4.22)$$

and

$$\nu(0; A(\cdot)) = \sum_{j=1}^{n_0} p_j, \quad p_j = \dim(\text{ran}(P_j)), \quad Q_j = I_{\mathcal{H}} - P_j, \quad 1 \leq j \leq n_0. \quad (4.23)$$

Thus, one computes

$$\begin{aligned} A(z)^{-1}A'(z) &= [A_{n_0}(z)]^{-1}[Q_{n_0} - zP_{n_0}]^{-1} \cdots [Q_1 - zP_1]^{-1} \\ &\quad \times \{ [-P_1][Q_2 - zP_2] \cdots [Q_{n_0} - zP_{n_0}]A_{n_0}(z) \\ &\quad + [Q_1 - zP_1][-P_2] \cdots [Q_{n_0} - zP_{n_0}]A_{n_0}(z) \\ &\quad + \cdots + [Q_1 - zP_1][Q_2 - zP_2] \cdots [Q_{n_0-1} - zP_{n_0-1}][-P_{n_0}]A_{n_0}(z) \\ &\quad + [Q_1 - zP_1][Q_2 - zP_2] \cdots [Q_{n_0} - zP_{n_0}]A'_{n_0}(z) \}. \end{aligned} \quad (4.24)$$

Next, continuing the computation in (4.24), one infers

$$\begin{aligned} &A(z)^{-1}A'(z) \\ &= [A_{n_0}(z)]^{-1}A'_{n_0}(z) \\ &\quad + [A_{n_0}(z)]^{-1}[Q_{n_0} - zP_{n_0}]^{-1} \cdots [Q_1 - zP_1]^{-1} \\ &\quad \times \{ [-P_1][Q_2 - zP_2] \cdots [Q_{n_0} - zP_{n_0}]A_{n_0}(z) \\ &\quad + [Q_1 - zP_1][-P_2] \cdots [Q_{n_0} - zP_{n_0}]A_{n_0}(z) \\ &\quad + \cdots + [Q_1 - zP_1][Q_2 - zP_2] \cdots [Q_{n_0-1} - zP_{n_0-1}][-P_{n_0}]A_{n_0}(z) \}. \end{aligned} \quad (4.25)$$

Since the first term on the right-hand side of (4.25) is analytic at $z_0 = 0$, its contour integral over $C(0; \varepsilon_0)$, $0 < \varepsilon < \varepsilon_0$, vanishes and one obtains upon repeatedly

applying cyclicity of the trace (i.e., $\text{tr}_{\mathcal{H}}(CD) = \text{tr}_{\mathcal{H}}(DC)$ for $C, D \in \mathcal{B}(\mathcal{H})$, with $CD, DC \in \mathcal{B}_1(\mathcal{H})$),

$$\begin{aligned}
m_a(0; A(\cdot)) &= \text{tr}_{\mathcal{K}} \left(\frac{1}{2\pi i} \oint_{C(0; \varepsilon_0)} d\zeta A(\zeta)^{-1} A'(\zeta) \right) \\
&= \text{tr}_{\mathcal{K}} \left(\frac{1}{2\pi i} \oint_{C(0; \varepsilon_0)} d\zeta [A_{n_0}(\zeta)]^{-1} [Q_{n_0} - \zeta P_{n_0}]^{-1} \cdots [Q_1 - \zeta P_1]^{-1} \right. \\
&\quad \times \{ [-P_1][Q_2 - \zeta P_2] \cdots [Q_{n_0} - \zeta P_{n_0}] A_{n_0}(\zeta) \\
&\quad + [Q_1 - \zeta P_1][-P_2] \cdots [Q_{n_0} - \zeta P_{n_0}] A_{n_0}(\zeta) \\
&\quad + \cdots + [Q_1 - \zeta P_1][Q_2 - \zeta P_2] \cdots [Q_{n_0-1} - \zeta P_{n_0-1}][-P_{n_0}] A_{n_0}(\zeta) \} \Big) \\
&= \frac{1}{2\pi i} \oint_{C(0; \varepsilon_0)} d\zeta \text{tr}_{\mathcal{K}} \left([A_{n_0}(\zeta)]^{-1} [Q_{n_0} - \zeta P_{n_0}]^{-1} \cdots [Q_1 - \zeta P_1]^{-1} \right. \\
&\quad \times \{ [-P_1][Q_2 - \zeta P_2] \cdots [Q_{n_0} - \zeta P_{n_0}] A_{n_0}(\zeta) \\
&\quad + [Q_1 - \zeta P_1][-P_2] \cdots [Q_{n_0} - \zeta P_{n_0}] A_{n_0}(\zeta) \\
&\quad + \cdots + [Q_1 - \zeta P_1][Q_2 - \zeta P_2] \cdots [Q_{n_0-1} - \zeta P_{n_0-1}][-P_{n_0}] A_{n_0}(\zeta) \} \Big) \\
&= \frac{1}{2\pi i} \oint_{C(0; \varepsilon_0)} d\zeta \text{tr}_{\mathcal{K}} \left([Q_{n_0} - \zeta P_{n_0}]^{-1} \cdots [Q_1 - \zeta P_1]^{-1} \right. \\
&\quad \times \{ [-P_1][Q_2 - \zeta P_2] \cdots [Q_{n_0} - \zeta P_{n_0}] + [Q_1 - \zeta P_1][-P_2] \cdots [Q_{n_0} - \zeta P_{n_0}] \\
&\quad + \cdots + [Q_1 - \zeta P_1][Q_2 - \zeta P_2] \cdots [Q_{n_0-1} - \zeta P_{n_0-1}][-P_{n_0}] \} \Big) \\
&= \frac{1}{2\pi i} \oint_{C(0; \varepsilon_0)} d\zeta \sum_{j=1}^{n_0} \text{tr}_{\mathcal{K}} \left([Q_j - \zeta P_j]^{-1} [-P_j] \right) \\
&= \frac{1}{2\pi i} \oint_{C(0; \varepsilon_0)} d\zeta \sum_{j=1}^{n_0} \text{tr}_{\mathcal{K}} \left([Q_j - \zeta^{-1} P_j] [-P_j] \right) \\
&= \frac{1}{2\pi i} \oint_{C(0; \varepsilon_0)} d\zeta \left(\sum_{j=1}^{n_0} \text{tr}_{\mathcal{K}}(P_j) \right) \zeta^{-1} \\
&= \sum_{j=1}^{n_0} p_j = \nu(0; A(\cdot)). \tag{4.26}
\end{aligned}$$

Next, one computes

$$\begin{aligned}
m_a(0; I_{\mathcal{K}} - K(\cdot)) &= \frac{1}{2\pi i} \text{tr}_{\mathcal{K}} \left(\oint_{C(0; \varepsilon_0)} d\zeta [I_{\mathcal{K}} - K(\zeta)]^{-1} [-K'(\zeta)] \right) \\
&= \frac{1}{2\pi i} \text{tr}_{\mathcal{K}} \left(\oint_{C(0; \varepsilon_0)} d\zeta [I_{\mathcal{K}} - K(\zeta)]^{-1} V_1 R_0(\zeta) \overline{R_0(\zeta)} \overline{V_2^*} \right) \\
&= \frac{1}{2\pi i} \text{tr}_{\mathcal{H}} \left(\oint_{C(0; \varepsilon_0)} d\zeta \overline{R_0(\zeta)} \overline{V_2^*} [I_{\mathcal{K}} - K(\zeta)]^{-1} V_1 R_0(\zeta) \right), \tag{4.27}
\end{aligned}$$

where we used (4.13) in the last step in (4.27). Similarly, using (2.12),

$$\begin{aligned}
m_a(0; H) &= \frac{-1}{2\pi i} \operatorname{tr}_{\mathcal{H}} \left(\oint_{C(0; \varepsilon_0)} d\zeta (H - \zeta I_{\mathcal{H}})^{-1} \right) \\
&= \frac{-1}{2\pi i} \operatorname{tr}_{\mathcal{H}} \left(\oint_{C(0; \varepsilon_0)} d\zeta [(H - \zeta I_{\mathcal{H}})^{-1} - (H_0 - \zeta I_{\mathcal{H}})^{-1}] \right) \\
&= \frac{1}{2\pi i} \operatorname{tr}_{\mathcal{H}} \left(\oint_{C(0; \varepsilon_0)} d\zeta \overline{R_0(\zeta)} V_2^* [I_{\mathcal{K}} - K(\zeta)]^{-1} V_1 R_0(\zeta) \right). \tag{4.28}
\end{aligned}$$

Combining (4.26)–(4.28) proves (4.20). \square

We note that the first equality in (4.20) is due to Latushkin and Sukhtyaev [30] (our proof seems slightly shorter); the second equality in (4.20) is new.

Remark 4.6. It is amusing to note that in the special finite-dimensional case, $\mathcal{H} = \mathcal{K}$, $\dim(\mathcal{H}) < \infty$, the following special case of (4.20), namely,

$$m_a(z_0; H) = \nu(z_0; I_{\mathcal{H}} - K(\cdot)), \quad z_0 \in \rho(H_0) \cap \sigma(H), \tag{4.29}$$

has basically a one-line proof! Indeed, in the matrix-valued context, the careful symmetrization in (2.12) becomes unnecessary and so abbreviating

$$V = V_2^* V_1, \quad K(z) = -V(H_0 - zI_{\mathcal{H}})^{-1}, \quad z \in \rho(H_0), \tag{4.30}$$

one obtains

$$\begin{aligned}
(H - zI_{\mathcal{H}})(H_0 - zI_{\mathcal{H}})^{-1} &= (H_0 + V - zI_{\mathcal{H}})(H_0 - zI_{\mathcal{H}})^{-1} \\
&= I_{\mathcal{H}} + V(H_0 - zI_{\mathcal{H}})^{-1}, \quad z \in \rho(H_0). \tag{4.31}
\end{aligned}$$

Thus, the underlying perturbation determinant becomes a ratio of determinants,

$$\det_{\mathcal{H}}(I_{\mathcal{H}} + V(H_0 - zI_{\mathcal{H}})^{-1}) = \frac{\det_{\mathcal{H}}(H - zI_{\mathcal{H}})}{\det_{\mathcal{H}}(H_0 - zI_{\mathcal{H}})}, \quad z \in \rho(H_0), \tag{4.32}$$

implying

$$\begin{aligned}
\nu(z_0; I_{\mathcal{H}} - K(\cdot)) &= m(z_0; \det_{\mathcal{H}}(I_{\mathcal{H}} - K(\cdot))) \\
&= m_a(z_0; H) - m_a(z_0; H_0) \\
&= m_a(z_0; H), \quad z_0 \in \rho(H_0) \cap \sigma(H). \tag{4.33}
\end{aligned}$$

At first sight, (2.33) together with (4.20) appear to describe an incomplete picture since we did not yet invoke a discussion of the quantity $m_a(1; K(z_0))$. However, the following elementary two-dimensional examples show that $m_a(1; K(z_0))$ in general differs from the value determined in (4.20):

Example 4.7. Let $\mathcal{H} = \mathcal{K} = \mathbb{C}^2$.

(i) Introduce

$$H_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 \\ -1 & -2 \end{pmatrix}, \quad H = H_0 + V = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}. \tag{4.34}$$

Then, $\sigma(H_0) = \{1\}$, $0 \in \sigma(H)$, and

$$\begin{aligned} K(z) &= -V(H_0 - zI_2)^{-1} = \frac{-1}{(1-z)^2} \begin{pmatrix} 0 & 0 \\ -(1-z) & 1-2(1-z) \end{pmatrix}, \quad z \in \mathbb{C} \setminus \{1\}, \\ K(0) &= \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \\ \det_{\mathbb{C}^2}(H - zI_2) &= z^2, \quad \det_{\mathbb{C}^2}(I_2 - K(z)) = z^2(1-z)^{-2}, \\ \det_{\mathbb{C}^2}(zI_2 - K(0)) &= -(1-z)[1 - (1-z)], \\ m_a(0; H) &= \nu(0; I_2 - K(\cdot)) = 2, \quad m_g(0; H) = m_g(1; K(0)) = m_a(1; K(0)) = 1. \end{aligned} \quad (4.35)$$

(ii) Introduce

$$H_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad H = H_0 + V = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}. \quad (4.36)$$

Then, $\sigma(H_0) = \{1\}$, $0 \in \sigma(H)$, and

$$\begin{aligned} K(z) &= -V(H_0 - zI_2)^{-1} = \frac{-1}{(1-z)^2} \begin{pmatrix} 0 & -(1-z) \\ (1-z) & -(2-z) \end{pmatrix}, \quad z \in \mathbb{C} \setminus \{1\}, \\ K(0) &= \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \\ \det_{\mathbb{C}^2}(H - zI_2) &= -z(1-z), \quad \det_{\mathbb{C}^2}(I_2 - K(z)) = -z(1-z)^{-1}, \\ \det_{\mathbb{C}^2}(zI_2 - K(0)) &= (1-z)^2, \\ m_g(0; H) &= m_a(0; H) = \nu(0; I_2 - K(\cdot)) = m_g(1; K(0)) = 1, \quad m_a(1; K(0)) = 2. \end{aligned} \quad (4.37)$$

We note that everything in this section applies to the Banach space setting, as is clear from the treatments in [17, Ch. 4], [18], [21]. We should also note that notions of algebraic multiplicities of parts of the spectrum of operator-valued functions, differing from the one employed in the present paper, were used in [31, Part II], [32], [33].

5. ALGEBRAIC MULTIPLICITIES OF OPERATOR-VALUED FUNCTIONS: THE MEROMORPHIC CASE

The principal purpose of this section is to properly extend Theorem 4.5 to the case where $K(\cdot)$ is finitely meromorphic and (4.20) represents the analog of the Weinstein–Aronszajn formula in the sense that H and H_0 have common discrete eigenvalues.

Hypothesis 5.1. Let $\Omega \subseteq \mathbb{C}$ be open and connected, and $\mathcal{D}_0 \subset \Omega$ a discrete set (i.e., a set without limit points in Ω). Suppose that $M : \Omega \setminus \mathcal{D}_0 \rightarrow \mathcal{B}(\mathcal{H})$ is analytic and that $M(\cdot)$ is finitely meromorphic on Ω . In addition, suppose that

$$M(z) \in \Phi(\mathcal{H}) \text{ for all } z \in \Omega \setminus \mathcal{D}_0, \quad (5.1)$$

and for all $z_0 \in \mathcal{D}_0$, such that

$$M(z) = \sum_{k=-N_0}^{\infty} (z - z_0)^k M_k(z_0), \quad 0 < |z - z_0| < \varepsilon_0, \quad (5.2)$$

for some $N_0 = N_0(z_0) \in \mathbb{N}$ and some $0 < \varepsilon_0 = \varepsilon_0(z_0)$ sufficiently small, with

$$\begin{aligned} M_{-k}(z_0) &\in \mathcal{F}(\mathcal{H}), \quad 1 \leq k \leq N_0(z_0), \quad M_0(z_0) \in \Phi(\mathcal{H}) \cap \mathcal{B}(\mathcal{H}), \\ M_k(z_0) &\in \mathcal{B}(\mathcal{H}), \quad k \in \mathbb{N}. \end{aligned} \quad (5.3)$$

One then recalls the meromorphic Fredholm theorem in the following form:

Theorem 5.2 ([18], [20], [40, Theorem XIII.13], [41], [48]).

Assume that $M : \Omega \setminus \mathcal{D}_0 \rightarrow \mathcal{B}(\mathcal{H})$ satisfies Hypothesis 5.1. Then either

(i) $M(z)$ is not boundedly invertible for any $z \in \Omega \setminus \mathcal{D}_0$,

or else,

(ii) $M(\cdot)^{-1}$ is finitely meromorphic on Ω . More precisely, there exists a discrete subset $\mathcal{D}_1 \subset \Omega$ (possibly, $\mathcal{D}_1 = \emptyset$) such that $M(z)^{-1} \in \mathcal{B}(\mathcal{H})$ for all $z \in \Omega \setminus \{\mathcal{D}_0 \cup \mathcal{D}_1\}$, $M(\cdot)^{-1}$ extends to an analytic function on $\Omega \setminus \mathcal{D}_1$, meromorphic on Ω . In addition,

$$M(z)^{-1} \in \Phi(\mathcal{H}) \quad \text{for all } z \in \Omega \setminus \mathcal{D}_1, \quad (5.4)$$

and if $z_1 \in \mathcal{D}_1$ then for some $N_0(z_1) \in \mathbb{N}$, and for some $0 < \varepsilon_0(z_1)$ sufficiently small,

$$M(z)^{-1} = \sum_{k=-N_0(z_1)}^{\infty} (z - z_1)^k D_k(z_1), \quad 0 < |z - z_1| < \varepsilon_0(z_1), \quad (5.5)$$

with

$$\begin{aligned} D_{-k}(z_1) &\in \mathcal{F}(\mathcal{H}), \quad 1 \leq k \leq N_0(z_1), \quad D_0(z_1) \in \Phi(\mathcal{H}) \cap \mathcal{B}(\mathcal{H}), \\ D_k(z_1) &\in \mathcal{B}(\mathcal{H}), \quad k \in \mathbb{N}. \end{aligned} \quad (5.6)$$

In addition, if $[I_{\mathcal{H}} - M(z)] \in \mathcal{B}_{\infty}(\mathcal{H})$ for all $z \in \Omega \setminus \mathcal{D}_0$, then

$$[I_{\mathcal{H}} - M(z)^{-1}] \in \mathcal{B}_{\infty}(\mathcal{H}), \quad z \in \Omega \setminus \mathcal{D}_1, \quad [I_{\mathcal{H}} - D_0(z_1)] \in \mathcal{B}_{\infty}(\mathcal{H}), \quad z_1 \in \mathcal{D}_1. \quad (5.7)$$

Next we strengthen Hypothesis 5.1 as follows:

Hypothesis 5.3. Suppose $M(\cdot) : \Omega \setminus \mathcal{D}_0 \rightarrow \mathcal{B}(\mathcal{H})$ satisfies Hypothesis 5.1, let $z_0 \in \mathcal{D}_0$, and assume that $M(\cdot)$ is boundedly invertible on $D(z_0; \varepsilon_0) \setminus \{z_0\}$ for some $0 < \varepsilon_0$ sufficiently small.

Definition 5.4. Assume Hypothesis 5.3. Then the *index of $M(\cdot)$ with respect to the counterclockwise oriented circle $C(z_0; \varepsilon)$* , $\text{ind}_{C(z_0; \varepsilon)}(M(\cdot))$, is defined by

$$\begin{aligned} \text{ind}_{C(z_0; \varepsilon)}(M(\cdot)) &= \text{tr}_{\mathcal{H}} \left(\frac{1}{2\pi i} \oint_{C(z_0; \varepsilon)} d\zeta M'(\zeta) M(\zeta)^{-1} \right) \\ &= \text{tr}_{\mathcal{H}} \left(\frac{1}{2\pi i} \oint_{C(z_0; \varepsilon)} d\zeta M(\zeta)^{-1} M'(\zeta) \right), \quad 0 < \varepsilon < \varepsilon_0. \end{aligned} \quad (5.8)$$

By the operator-valued version of the argument principle proved in [18] (see also [17, Theorem 4.4.1]), one has

$$\text{ind}_{C(z_0; \varepsilon)}(M(\cdot)) \in \mathbb{Z} \quad (5.9)$$

under the conditions of Definition 5.4.

The next result presents our generalization of Theorem 4.5 to the case where $K(\cdot)$ is finitely meromorphic. In particular, we will now prove the analog of the Weinstein–Aronszajn formula (cf., e.g., [2], [20], [25, Sect. IV.6], [29], [50, Sect. 9.3]) in the case where H and H_0 have common discrete eigenvalues.

Theorem 5.5. *Assume Hypothesis 2.5 and suppose that $z_0 \in (\sigma_d(H_0) \cap \sigma(H))$ with $D(z_0; \varepsilon_0) \cap \sigma(H) = \{z_0\}$ for some $0 < \varepsilon_0$ sufficiently small. Moreover, if*

$$K(z) = \sum_{k=-N_0}^{\infty} (z - z_0)^k K_k(z_0), \quad 0 < |z - z_0| < \varepsilon_0, \quad (5.10)$$

for some $N_0 = N_0(z_0) \in \mathbb{N}$ and some $0 < \varepsilon_0 = \varepsilon_0(z_0)$ sufficiently small, with

$$K_{-k}(z_0) \in \mathcal{F}(\mathcal{K}), \quad 1 \leq k \leq N_0(z_0), \quad K_k(z_0) \in \mathcal{B}(\mathcal{K}), \quad k \in \mathbb{N} \cup \{0\}, \quad (5.11)$$

we assume that¹

$$[I_{\mathcal{K}} - K_0(z_0)] \in \Phi(\mathcal{K}). \quad (5.12)$$

Then z_0 is a discrete eigenvalue of H ,

$$z_0 \in \sigma_d(H), \quad (5.13)$$

and $I_{\mathcal{K}} - K(\cdot)$ satisfies the conditions of Hypothesis 5.3 on $\Omega = \mathbb{C} \setminus \sigma_{\text{ess}}(H_0)$. In addition,

$$m_a(z_0; H) = m_a(z_0; H_0) + \text{ind}_{C(z_0; \varepsilon)}(I_{\mathcal{K}} - K(\cdot)), \quad 0 < \varepsilon < \varepsilon_0. \quad (5.14)$$

Proof. That (5.13) holds is shown as in the proof of Theorem 4.5. That $I_{\mathcal{K}} - K(\cdot)$ satisfies the conditions of Hypothesis 5.3 is clear from combining Hypothesis 2.5 and Theorem 5.2. The rest also follows the proof of Theorem 4.5:

$$\begin{aligned} \text{ind}_{C(z_0; \varepsilon)}(I_{\mathcal{K}} - K(\cdot)) &= \frac{1}{2\pi i} \text{tr}_{\mathcal{K}} \left(\oint_{C(z_0; \varepsilon_0)} d\zeta [I_{\mathcal{K}} - K(\zeta)]^{-1} [-K'(\zeta)] \right) \\ &= \frac{1}{2\pi i} \text{tr}_{\mathcal{K}} \left(\oint_{C(z_0; \varepsilon_0)} d\zeta [I_{\mathcal{K}} - K(\zeta)]^{-1} V_1 R_0(\zeta) \overline{R_0(\zeta)} V_2^* \right) \\ &= \frac{1}{2\pi i} \text{tr}_{\mathcal{H}} \left(\oint_{C(z_0; \varepsilon_0)} d\zeta \overline{R_0(\zeta)} V_2^* [I_{\mathcal{K}} - K(\zeta)]^{-1} V_1 R_0(\zeta) \right) \\ &= \frac{-1}{2\pi i} \text{tr}_{\mathcal{H}} \left(\oint_{C(z_0; \varepsilon_0)} d\zeta [(H - \zeta I_{\mathcal{H}})^{-1} - (H_0 - \zeta I_{\mathcal{H}})^{-1}] \right) \\ &= m_a(z_0; H) - m_a(z_0; H_0). \end{aligned} \quad (5.15)$$

□

6. PAIRS OF PROJECTIONS AND AN INDEX COMPUTATION

In our final section we apply some of the principal results of Section 4 and 5 to pairs of projections and their index and make a connection to an underlying Birman–Schwinger-type operator.

One recalls that an ordered pair of projections (P, Q) is called a *Fredholm pair* if

$$QP : \text{ran}(P) \rightarrow \text{ran}(Q) \text{ belongs to } \Phi(\mathcal{H}). \quad (6.1)$$

In this case, the *index* of the pair (P, Q) , denoted $\text{ind}(P, Q)$, is defined to be the index of the Fredholm operator QP :

$$\text{ind}(P, Q) = \dim(\ker(QP)) - \dim(\ker((QP)^*)). \quad (6.2)$$

For pertinent literature related to pairs of projections, we refer to [1], [3], [8], [9], [11], [19], [22], [23], [37], [47], and the references cited therein. The following results are well-known and will be used throughout this section.

¹Condition (5.12) was inadvertently omitted in our paper, *Integral Eq. Operator Theory* **82**, 61–94 (2015), as kindly pointed out by Jussi Behrndt.

Theorem 6.1 ([1], [3]). *If (P, Q) is a pair of orthogonal projections, then the following items hold.*

(i) *If $(P - Q) \in \mathcal{B}_\infty(\mathcal{H})$, then (P, Q) is a Fredholm pair.*

(ii) *If (P, Q) is a Fredholm pair, then*

$$\text{ind}(P, Q) = m_1 - m_{-1}, \quad (6.3)$$

where

$$\begin{aligned} m_1 &:= \dim(\{v \in \mathcal{H} \mid Pv = v, Qv = 0\}), \\ m_{-1} &:= \dim(\{v \in \mathcal{H} \mid Pv = 0, Qv = v\}). \end{aligned} \quad (6.4)$$

(iii) *If (P, Q) is a Fredholm pair, then (Q, P) is a Fredholm pair, and*

$$\text{ind}(Q, P) = -\text{ind}(P, Q). \quad (6.5)$$

(iv) *If (P, Q) is a Fredholm pair with $(P - Q) \in \mathcal{B}_{2n+1}(\mathcal{H})$, for some $n \in \mathbb{N}$, then*

$$\text{tr}_{\mathcal{H}}((P - Q)^{2n+1}) = \text{ind}(P, Q). \quad (6.6)$$

If (P, Q) is a pair of orthogonal projections such that $(P - Q) \in \mathcal{B}_1(\mathcal{H})$, then the associated *perturbation determinant*,

$$\begin{aligned} D_{(P,Q)}(z) &= \det_{\mathcal{H}}(I_{\mathcal{H}} + (P - Q)(Q - zI_{\mathcal{H}})^{-1}) \\ &= \det_{\mathcal{H}}((P - zI_{\mathcal{H}})(Q - zI_{\mathcal{H}})^{-1}), \quad z \in \mathbb{C} \setminus \{0, 1\}, \end{aligned} \quad (6.7)$$

is well-defined. The *Krein–Lifshitz spectral shift function* $\xi_{(P,Q)}(\cdot)$ corresponding to the pair (P, Q) is then given by

$$\xi_{(P,Q)}(\lambda) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \arg[D_{(P,Q)}(\lambda + i\varepsilon)] \quad \text{for a.e. } \lambda \in \mathbb{R} \quad (6.8)$$

(w.r.t. Lebesgue measure), and $\xi_{(P,Q)}$ satisfies:

$$\xi_{(P,Q)} \in L^1(\mathbb{R}), \quad \int_{\mathbb{R}} |\xi_{(P,Q)}(\lambda)| d\lambda \leq \|P - Q\|_{\mathcal{B}_1(\mathcal{H})}. \quad (6.9)$$

Moreover, integrating the spectral shift function allows one to recover the trace of $P - Q$, that is,

$$\int_{\mathbb{R}} \xi_{(P,Q)}(\lambda) d\lambda = \text{tr}_{\mathcal{H}}(P - Q). \quad (6.10)$$

The properties in (6.9) and (6.10) follow from general considerations and do not rely on the fact that P and Q are orthogonal projections. For details on the Krein–Lifshitz spectral shift function within the general context of trace class (and, more generally, resolvent comparable) perturbations of self-adjoint operators, we refer to [7], [52, Ch. 8], [53], [54, Sect. 0.9, Chs. 4, 5, 9]. In the specific case at hand, where P and Q are orthogonal projections, the perturbation determinant and spectral shift function may be computed explicitly.

Theorem 6.2 ([1]). *Suppose (P, Q) is a pair of orthogonal projections with $(P - Q) \in \mathcal{B}_1(\mathcal{H})$. Then the following items hold.*

(i) *The perturbation determinant $D_{(P,Q)}(\cdot)$ is given by*

$$D_{(P,Q)}(z) = \left(\frac{z-1}{z} \right)^{m_1 - m_{-1}}, \quad z \in \mathbb{C} \setminus [0, 1], \quad (6.11)$$

and the spectral shift function is piecewise constant,

$$\xi_{(P,Q)}(\lambda) = \begin{cases} 0, & \lambda \notin [0, 1], \\ m_1 - m_{-1}, & \lambda \in [0, 1], \end{cases} \text{ for a.e. } \lambda \in \mathbb{R}, \quad (6.12)$$

where $m_{\pm 1}$ are defined in (6.4).

(ii) If f is a measurable, complex-valued function which is continuous in neighborhoods of $z = 0$ and $z = 1$, then $[f(P) - f(Q)] \in \mathcal{B}_1(\mathcal{H})$ and

$$\mathrm{tr}_{\mathcal{H}}[f(P) - f(Q)] = [f(1) - f(0)]\mathrm{tr}_{\mathcal{H}}(P - Q) = [f(1) - f(0)](m_1 - m_{-1}). \quad (6.13)$$

Finally, we compute the index of an operator $M_{(P,Q)}(\cdot)$ closely related to the Birman–Schwinger-type operator naturally associated with the pair of projections (P, Q) .

Theorem 6.3. *Suppose that (P, Q) is a pair of orthogonal projections with $(P - Q) \in \mathcal{B}_1(\mathcal{H})$ and that $M_{(P,Q)}(\cdot) : \mathbb{C} \setminus \{0, 1\} \rightarrow \mathcal{B}(\mathcal{H})$ is given by*

$$M_{(P,Q)}(z) := (P - zI_{\mathcal{H}})(Q - zI_{\mathcal{H}})^{-1} = I_{\mathcal{H}} + (P - Q)(Q - zI_{\mathcal{H}})^{-1}, \quad (6.14)$$

$$z \in \mathbb{C} \setminus \{0, 1\}.$$

Then the following items hold.

(i) (P, Q) is a Fredholm pair.

(ii) $M_{(P,Q)}(\cdot)$ is finitely meromorphic, and $M_{(P,Q)}(z) \in \Phi(\mathcal{H})$, $z \in \mathbb{C} \setminus \{0, 1\}$.

(iii) The index of $M_{(P,Q)}(\cdot)$ with respect to the counterclockwise oriented circle $C(z_0; \varepsilon)$, with $\varepsilon > 0$ taken sufficiently small, is given by

$$\mathrm{ind}_{C(z_0; \varepsilon)}(M_{(P,Q)}(\cdot)) = \begin{cases} -\mathrm{ind}(P, Q), & z_0 = 0, \\ 0, & z_0 \in \mathbb{C} \setminus \{0, 1\}, \\ \mathrm{ind}(P, Q), & z_0 = 1. \end{cases} \quad (6.15)$$

Proof. Item (i) follows immediately from Theorem 6.1, owing to the assumption that $(P - Q) \in \mathcal{B}_1(\mathcal{H})$. The claims in item (ii) follow from the representation

$$M_{(P,Q)}(z) = I_{\mathcal{H}} + (P - Q)(Q - zI_{\mathcal{H}})^{-1}, \quad z \in \mathbb{C} \setminus \{0, 1\}, \quad (6.16)$$

which may be obtained by applying a standard resolvent identity in (6.14), and the fact that $(I_{\mathcal{H}} - K) \in \Phi(\mathcal{H})$ if $K \in \mathcal{B}_{\infty}(\mathcal{H})$. It remains to settle item (iii). To this end, one computes

$$M_{(P,Q)}(z)^{-1} = (Q - zI_{\mathcal{H}})(P - zI_{\mathcal{H}})^{-1}, \quad z \in \mathbb{C} \setminus \{0, 1\}. \quad (6.17)$$

and

$$M'_{(P,Q)}(z) = (P - Q)(Q - zI_{\mathcal{H}})^{-2}, \quad z \in \mathbb{C} \setminus \{0, 1\}. \quad (6.18)$$

As a result,

$$M'_{(P,Q)}(z)M_{(P,Q)}(z)^{-1} = (P - Q)(Q - zI_{\mathcal{H}})^{-1}(P - zI_{\mathcal{H}})^{-1}, \quad z \in \mathbb{C} \setminus \{0, 1\}, \quad (6.19)$$

and one computes, for $\varepsilon > 0$ sufficiently small,

$$\begin{aligned}
\operatorname{ind}_{C(z_0; \varepsilon)}(M_{(P, Q)}(\cdot)) &= \operatorname{tr}_{\mathcal{H}} \left\{ \frac{1}{2\pi i} \oint_{C(z_0; \varepsilon)} dz M'_{(P, Q)}(z) M_{(P, Q)}(z)^{-1} \right\} \\
&= \operatorname{tr}_{\mathcal{H}} \left\{ \frac{1}{2\pi i} \oint_{C(z_0; \varepsilon)} dz (P - Q)(Q - zI_{\mathcal{H}})^{-1}(P - zI_{\mathcal{H}})^{-1} \right\} \\
&= -\operatorname{tr}_{\mathcal{H}} \left\{ \frac{1}{2\pi i} \oint_{C(z_0; \varepsilon)} dz (P - zI_{\mathcal{H}})^{-1}(Q - P)(Q - zI_{\mathcal{H}})^{-1} \right\} \\
&= \operatorname{tr}_{\mathcal{H}} \left\{ \frac{1}{2\pi i} \oint_{C(z_0; \varepsilon)} dz [(Q - zI_{\mathcal{H}})^{-1} - (P - zI_{\mathcal{H}})^{-1}] \right\} \\
&= \frac{1}{2\pi i} \oint_{C(z_0; \varepsilon)} dz \operatorname{tr}_{\mathcal{H}} [(Q - zI_{\mathcal{H}})^{-1} - (P - zI_{\mathcal{H}})^{-1}] \\
&= \frac{\operatorname{ind}(P, Q)}{2\pi i} \oint_{C(z_0; \varepsilon)} dz \left(\frac{1}{z-1} - \frac{1}{z} \right). \tag{6.20}
\end{aligned}$$

To obtain the last equality in (6.20) one applies (6.13). Finally, (6.15) follows from (6.20), (6.5), and the residue calculus. \square

We conclude by noting that the actual Birman–Schwinger operator associated with the pair (P, Q) is then given by $K_{(P, Q)}(z) := -(P - Q)(Q - zI_{\mathcal{H}})^{-1}$, and hence

$$M_{(P, Q)}(z) = I_{\mathcal{H}} - K_{(P, Q)}(z) = I_{\mathcal{H}} + (P - Q)(Q - zI_{\mathcal{H}})^{-1}, \quad z \in \mathbb{C} \setminus \{0, 1\}. \tag{6.21}$$

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