INVERSE SPECTRAL PROBLEMS FOR SCHRÖDINGER-TYPE OPERATORS WITH DISTRIBUTIONAL MATRIX-VALUED POTENTIALS

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Abstract. The principal purpose of this note is to provide a reconstruction procedure for distributional matrix-valued potential coefficients of Schrödinger-type operators on a half-line from the underlying Weyl–Titchmarsh function.

1. Introduction

This note should be viewed as an addendum to the paper [13], treating distributional matrix-valued potentials for (generalized) Schrödinger operators based on an intimate connection between such Schrödinger operators and a particular class of supersymmetric Dirac-type operators, and the paper [40] which develops a reconstruction procedure for the potential coefficient of a half-line Dirac operator from the underlying matrix-valued Weyl–Titchmarsh function. As a result, we derive a constructive approach to reconstruct distributional matrix-valued potential coefficients of (generalized) Schrödinger operators on a half-line from the underlying matrix-valued Weyl–Titchmarsh function. The importance of Weyl–Titchmarsh functions in connection with inverse problems for Schrödinger operators, especially, in connection with various uniqueness-type theorems has been well-documented in the literature. For instance, we mention the classical two-spectra uniqueness results due to Borg [4], [5], Levinson [28], Levitan [29], [30, Ch. 3], Levitan and Gasymov [31], Marchenko [35], [36, Ch. 3], (see also [9], [15], [16], [17], [18], [33], [34] and the extensive lists of references therein). The constructive approach to actually reconstruct the potential coefficient goes well beyond uniqueness theorems and now also becomes possible in connection with very singular (distributional) potentials.

For the physical relevance of matrix-valued potentials, we refer, for instance to Chadan and Sabatier [7, Sect. XI.3, XI.4], Newton and Jost [38], and the literature cited therein. The classical reference on inverse scattering for matrix-valued potentials on a half-line is Agranovich and Marchenko [1, Ch. V] (see also [44]).

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More precisely, the half-line Dirac-type operators in $L^2([0,\infty))^{2m}$, $m \in \mathbb{N}$, studied in this note are of the form

$$\begin{aligned}
&(D_+(\alpha)U)(x) = (\mathcal{D}U)(x) \text{ for a.e. } x > 0,\\
&U \in \operatorname{dom}(D_+(\alpha)) = \big\{V \in L^2([0,\infty))^{2m} \,\big|\, V \in AC([0,R])^{2m} \text{ for all } R > 0; \\
&\hspace{6cm} \alpha V(0) = 0;\, \mathcal{D}V \in L^2([0,\infty))^{2m}\big\},
\end{aligned} \tag{1.1}$$

where the $2m \times 2m$ matrix-valued differential expression $\mathcal{D}$ is given by

$$\mathcal{D} = \begin{pmatrix} 0 & -I_m(d/dx) + \phi(x) \\ I_m(d/dx) + \phi(x) & 0 \end{pmatrix}, \tag{1.2}$$

and the boundary condition parameters $\alpha \in \mathbb{C}^{m\times 2m}$ satisfy the conditions

$$\alpha\alpha^* = I_m, \quad \alpha J\alpha^* = 0, \text{ where } J = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}. \tag{1.3}$$

Here the $m \times m$ matrix-valued potential coefficient ϕ is assumed to be locally square integrable on $[0,\infty)$, that is, $\phi \in L^2([0,R])^{m\times m}$ for all $R > 0$, and to satisfy $\phi(\cdot) = \phi(\cdot)^*$ a.e. on $[0,\infty)$.

On the other hand, we define the following two kinds of quasi-derivatives,

$$u^{[1,j]}(x) = u'(x) + (-1)^{j+1}\phi(x)u(x) \text{ for a.e. } x > 0, \quad j = 1,2. \tag{1.4}$$

Thus, introducing the $m \times m$ matrix-valued differential expressions τ_j, $j = 1,2$, by

$$(\tau_j u)(x) = -\big(u^{[1,j]}\big)'(x) + (-1)^{j+1}\phi(x)u^{[1,j]}(x) \text{ for a.e. } x > 0, \quad j = 1,2, \tag{1.5}$$

one infers that formally, τ_j, $j = 1,2$, are of the generalized Schrödinger form

$$\tau_j = -I_m\frac{d^2}{dx^2} + V_j(x), \quad V_j(x) = \phi(x)^2 + (-1)^j\phi'(x), \quad j = 1,2. \tag{1.6}$$

We emphasize that while $\phi^2 \in L^1_{\mathrm{loc}}([0,\infty))^{m\times m}$ represents a standard matrix-valued potential coefficient, in general, ϕ' is now a genuine distribution (unless one assumes in addition that $\phi \in AC_{\mathrm{loc}}([0,\infty))^{m\times m}$). In contrast to these half-line Schrödinger operators, the Dirac-type operators $D_+(\alpha)$ only contain the standard potential coefficient $\phi \in L^2_{\mathrm{loc}}([0,\infty))^{m\times m}$.

The differential expressions τ_j then generate the generalized half-line Schrödinger operators $H_{+,0,j}$, $j = 1,2$, in $L^2([0,\infty))^m$,

$$\begin{aligned}
&(H_{+,0,j}u)(x) = (\tau_j u)(x) = -\big(u^{[1,j]}\big)'(x) + (-1)^{j+1}\phi(x)u^{[1,j]}(x) \text{ for a.e. } x > 0,\\
&u \in \operatorname{dom}(H_{+,0,j}) = \big\{v \in L^2([0,\infty))^m \,\big|\, v, v^{[1,j]} \in AC([0,R])^m \text{ for all } R > 0;\\
&\hspace{3cm} v(0) = 0;\, \big[\big(v^{[1,j]}\big)' + (-1)^j\phi v^{[1,j]}\big] \in L^2([0,\infty))^m\big\}, \quad j = 1,2,
\end{aligned} \tag{1.7}$$

the primary object studied in this note.

Denoting by $M_+^D(\cdot,\alpha)$ and $\widehat{M}_{+,0,j}$, $j = 1,2$, the $m \times m$ matrix-valued Weyl–Titchmarsh functions associated to $D_+(\alpha)$ and $H_{+,0,j}$, $j = 1,2$, respectively, the supersymmetric approach employed in [13] naturally leads to the fundamental identity

$$\widehat{M}_{+,0,1}(z) = \zeta M_+^D(\zeta,\alpha_0) = -z\widehat{M}_{+,0,2}(z)^{-1}, \quad z = \zeta^2,\ \zeta \in \mathbb{C}\backslash\mathbb{R}, \tag{1.8}$$

where $\alpha_0 = (I_m \quad 0)$.

The paper [40], on the other hand, focused on the inverse spectral problem for half-line Dirac-type operators containing $D_+(\alpha_0)$ as a special case, and developed a procedure to reconstruct the matrix-valued potential coefficient from the underlying

$m \times m$ matrix-valued Weyl–Titchmarsh function (i.e., in our particular case at hand, reconstructing ϕ from $M_+^D(\,\cdot\,,\alpha_0)$). The reconstruction of ϕ from $M_+^D(\,\cdot\,,\alpha)$ with an arbitrary α satisfying (1.3) easily follows. The results of [40] generalize earlier results obtained in [39] for the case of locally bounded potentials (see more references, historical remarks and details of the procedure in [41, Ch. 2]).

We note that generalized Schrödinger operators (with measure and distributional potential coefficients) have been studied extensively in the literature. Rather than reviewing the extensive literature here, we refer to [10], [13] which contain detailed historic accounts of this subject.

It remains to briefly describe the content of this paper: Section 2 recalls the basics of Weyl–Titchmarsh theory for half-line Dirac-type operators $D_+(\alpha)$ and the generalized half-line Schrödinger operators $H_{+,0,j}$, $j = 1, 2$. Our principal Section 3 then develops a reconstruction procedure for the $m \times m$ matrix-valued potential coefficient ϕ from the underlying $m \times m$ matrix-valued Weyl–Titchmarsh function $M_+^D(\,\cdot\,,\alpha)$ and hence by (1.8) also for the distributional $m\times m$ matrix-valued potential coefficients $V_j = \phi^2 + (-1)^j\phi'$ in the generalized half-line Schrödinger operators $H_{+,0,j}$ from either one of $\widehat{M}_{+,0,1}$ or $\widehat{M}_{+,0,2}$. For simplicity, we exclusively focus on right half-lines $[0,\infty)$ throughout this note. The case of left half-lines is treated in a completely analogous manner.

Concluding, we briefly summarize some of the notation used in this paper. All $m\times p$ matrices $M \in \mathbb{C}^{m\times p}$ will be considered over the field of complex numbers $\mathbb{C}$. Moreover, I_m denotes the identity matrix in $\mathbb{C}^{m\times m}$, M^* the adjoint (i.e., complex conjugate transpose), and $M^\top$ the transpose of the matrix M.

We denote with $L^2([0,\infty))^m$ the usual space of all square integrable (with respect to the Lebesgue measure) functions on $[0,\infty)$ taking values in $\mathbb{C}^m$, that is,

$$L^2([0,\infty))^m = \left\{ U : [0,\infty) \to \mathbb{C}^m \,\middle|\, \int_0^\infty dx\, \|U(x)\|_{\mathbb{C}^m}^2 < \infty \right\}, \quad m \in \mathbb{N}. \tag{1.9}$$

The set of functions which are only locally square integrable on $[0,\infty)$, that is, belong to $L^2([0,R])^m$ for all $R > 0$, will be referred to as $L^2_{\rm loc}([0,\infty))^m$. The abbreviation "a.e." is employed in the contexts of "(Lebesgue) almost every" as well as "(Lebesgue) almost everywhere" on certain sets.

With $AC_{\rm loc}([0,\infty))^m$ we denote the set of all functions on $[0,\infty)$ which are locally absolutely continuous, that is, belong to $AC([0,R])^m$ for all $R > 0$. The usual Sobolev spaces will be denoted by $H^1([0,R])^m$ and their local counterpart with $H^1_{\rm loc}([0,\infty))^m$. We will also encounter the space $H^{-1}_{\rm loc}([0,\infty))$ of distributions, which is regarded as the dual of the subspace of $H^1_0([0,\infty))$ which consists of functions with compact support in $[0,\infty)$. Note that this space is precisely the space of distributional derivatives of functions in $L^2_{\rm loc}([0,\infty))$.

The symbol $\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$ denotes the Banach space of bounded operators between the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, and $\mathcal{B}(\mathcal{H})$ abbreviates $\mathcal{B}(\mathcal{H},\mathcal{H})$. Finally, the open complex upper half-plane is denoted by $\mathbb{C}_+ = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$.

2. Weyl–Titchmarsh Matrices for Half-Line Dirac and Schrödinger Operators

In this preparatory section, we review a special case of the Weyl–Titchmarsh theory for half-line Dirac-type and Schrödinger operators discussed in detail in [13].

We start by making the following simplified assumption, when compared to [13], dictated by the inverse spectral approach presented in our principal Section 3.

Hypothesis 2.1. *Suppose* $\phi \in L^2_{\mathrm{loc}}([0,\infty))^{m\times m}$, $m \in \mathbb{N}$, *and* $\phi(\cdot) = \phi(\cdot)^*$ *a.e. on* $[0,\infty)$.

Given Hypothesis 2.1, we introduce the $2m \times 2m$ matrix-valued differential expression

$$\mathcal{D} = \begin{pmatrix} 0 & -I_m(d/dx) + \phi(x) \\ I_m(d/dx) + \phi(x) & 0 \end{pmatrix}. \tag{2.1}$$

By [8, Lemma 2.15], $\mathcal{D}$ is in the limit point case at ∞. (For a subsequent and more general result we refer to [27], see also [26] and [32] for such proofs under stronger hypotheses on ϕ).

We emphasize that the special structure of $\mathcal{D}$ in (2.1) is derived from a study of supersymmetric Dirac-type operators in $L^2(\mathbb{R})^{2m}$, and we refer to [13] for a detailed treatment in this context. Furthermore, we also note that [13] was inspired by [24].

In order to discuss $m \times m$ Weyl–Titchmarsh matrices corresponding to self-adjoint realizations of $\mathcal{D}$ in $L^2([0,\infty))^{2m}$, we introduce boundary condition parameters $\alpha = (\alpha_1 \quad \alpha_2) \in \mathbb{C}^{m\times 2m}$ satisfying the conditions

$$\alpha\alpha^* = I_m, \quad \alpha J \alpha^* = 0, \text{ where } J = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}. \tag{2.2}$$

Explicitly, this reads

$$\alpha_1\alpha_1^* + \alpha_2\alpha_2^* = I_m, \quad \alpha_2\alpha_1^* - \alpha_1\alpha_2^* = 0. \tag{2.3}$$

In fact, one also has

$$\alpha_1^*\alpha_1 + \alpha_2^*\alpha_2 = I_m, \quad \alpha_2^*\alpha_1 - \alpha_1^*\alpha_2 = 0, \tag{2.4}$$

as is clear from

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ -\alpha_2 & \alpha_1 \end{pmatrix} \begin{pmatrix} \alpha_1^* & -\alpha_2^* \\ \alpha_2^* & \alpha_1^* \end{pmatrix} = I_{2m} = \begin{pmatrix} \alpha_1^* & -\alpha_2^* \\ \alpha_2^* & \alpha_1^* \end{pmatrix} \begin{pmatrix} \alpha_1 & \alpha_2 \\ -\alpha_2 & \alpha_1 \end{pmatrix}, \tag{2.5}$$

since any left inverse matrix is also a right inverse, and vice versa. Moreover, from (2.4) one obtains

$$\alpha^*\alpha J + J\alpha^*\alpha = J. \tag{2.6}$$

The particular choice where α equals

$$\alpha_0 = (I_m \quad 0), \tag{2.7}$$

will play a fundamental role later on.

The self-adjoint half-line Dirac operators $D_+(\alpha)$ in $L^2([0,\infty))^{2m}$ associated with a self-adjoint boundary condition at $x = 0$ indexed by $\alpha \in \mathbb{C}^{m\times 2m}$ satisfying (2.2), are of the form

$$\begin{aligned} &(D_+(\alpha)U)(x) = (\mathcal{D}U)(x) \text{ for a.e. } x > 0, \\ &U \in \mathrm{dom}(D_+(\alpha)) = \big\{V \in L^2([0,\infty))^{2m} \,\big|\, V \in AC([0,R])^{2m} \text{ for all } R > 0; \\ &\qquad\qquad\qquad\qquad\qquad\qquad \alpha V(0) = 0;\, \mathcal{D}V \in L^2([0,\infty))^{2m}\big\}. \end{aligned} \tag{2.8}$$

Next, we denote by $U_+(\zeta,\,\cdot\,,\alpha)$ the $2m \times m$ matrix-valued Weyl–Titchmarsh solutions of $\mathcal{D}U = \zeta U$, $\zeta \in \mathbb{C}\backslash\mathbb{R}$, satisfying

$$U_+(\zeta,\,\cdot\,,\alpha) \in L^2([0,\infty))^{2m\times m}, \ \zeta \in \mathbb{C}\backslash\mathbb{R}, \tag{2.9}$$

and normalized such that

$$\begin{aligned} U_+(\zeta,x,\alpha) &= \begin{pmatrix} u_{+,1}(\zeta,x,\alpha) \\ u_{+,2}(\zeta,x,\alpha) \end{pmatrix} = \Psi(\zeta,x,\alpha) \begin{pmatrix} I_m \\ M_+^D(\zeta,\alpha) \end{pmatrix} \\ &= \begin{pmatrix} \vartheta_1(\zeta,x,\alpha) & \varphi_1(\zeta,x,\alpha) \\ \vartheta_2(\zeta,x,\alpha) & \varphi_2(\zeta,x,\alpha) \end{pmatrix} \begin{pmatrix} I_m \\ M_+^D(\zeta,\alpha) \end{pmatrix}, \quad x \geqslant 0. \end{aligned} \tag{2.10}$$

In the particular case $\alpha_0 = (I_m \quad 0)$ one obtains

$$U_+(\zeta,0,\alpha_0) = \begin{pmatrix} u_{+,1}(\zeta,0,\alpha_0) \\ u_{+,2}(\zeta,0,\alpha_0) \end{pmatrix} = \begin{pmatrix} I_m \\ M_+^D(\zeta,\alpha_0) \end{pmatrix}. \tag{2.11}$$

Here $M_+^D(\zeta,\alpha)$ represents an $m\times m$ matrix, the superscript "D" indicates the underlying Dirac-type operator, and the functions $\Psi(\zeta,x,\alpha)$, $\vartheta_j(\zeta,x,\alpha)$, and $\varphi_j(\zeta,x,\alpha)$, $j=1,2$, $\zeta\in\mathbb{C}$, are defined as follows: $\Psi(\zeta,\,\cdot\,,\alpha)$ satisfies $\mathcal{D}\Psi = \zeta\Psi$ a.e. on $[0,\infty)$, normalized such that

$$\Psi(\zeta,0,\alpha) = (\alpha^* \;\, J\alpha^*) = \begin{pmatrix} \alpha_1^* & -\alpha_2^* \\ \alpha_2^* & \alpha_1^* \end{pmatrix}. \tag{2.12}$$

Partitioning $\Psi(\zeta,x,\alpha)$ as follows,

$$\Psi(\zeta,x,\alpha) = \begin{pmatrix} \vartheta_1(\zeta,x,\alpha) & \varphi_1(\zeta,x,\alpha) \\ \vartheta_2(\zeta,x,\alpha) & \varphi_2(\zeta,x,\alpha) \end{pmatrix}, \quad \zeta\in\mathbb{C},\ x\geqslant 0, \tag{2.13}$$

defines $\vartheta_j(\zeta,x,\alpha)$ and $\varphi_j(\zeta,x,\alpha)$, $j=1,2$, as $m\times m$ matrices, entire with respect to $\zeta\in\mathbb{C}$, and normalized according to (2.12).

The $m\times m$ matrix-valued spectral function of the Dirac-type operator $D_+(\alpha)$ then generates the measure $\Omega_+^D(\,\cdot\,,\alpha)$ in (2.20) below. In particular, the matrices $M_+^D(\zeta,\alpha)$ represent the sought after half-line Weyl–Titchmarsh matrices associated with $D_+(\alpha)$, whose basic properties can be summarized as follows:

Theorem 2.2 ([2], [3], [6], [8], [19], [20], [21], [22], [23], [25])**.**
Suppose Hypothesis 2.1*, let $\zeta\in\mathbb{C}\backslash\mathbb{R}$, and denote by $\alpha,\delta\in\mathbb{C}^{m\times 2m}$ matrices satisfying* (2.2)*. Then the following hold:*
(i) $M_+^D(\,\cdot\,,\alpha)$ is an $m\times m$ matrix-valued Nevanlinna–Herglotz function of maximal rank m. In particular,

$$\operatorname{Im}(M_+^D(\zeta,\alpha)) \geqslant 0, \quad \zeta\in\mathbb{C}_+, \tag{2.14}$$

$$M_+^D(\overline{\zeta},\alpha) = M_+^D(\zeta,\alpha)^*, \tag{2.15}$$

$$\operatorname{rank}(M_+^D(\zeta,\alpha)) = m, \tag{2.16}$$

$$\lim_{\varepsilon\downarrow 0} M_+^D(\nu+i\varepsilon,\alpha) \text{ exists for a.e. } \nu\in\mathbb{R}, \tag{2.17}$$

$$M_+^D(\zeta,\alpha) = [-\alpha J\delta^* + \alpha\delta^* M_+^D(\zeta,\delta)][\alpha\delta^* + \alpha J\delta^* M_+^D(\zeta,\delta)]^{-1}. \tag{2.18}$$

Local singularities of $M_+^D(\,\cdot\,,\alpha)$ and $M_+^D(\,\cdot\,,\alpha)^{-1}$ are necessarily real and at most of first order in the sense that

$$-\lim_{\epsilon\downarrow 0}\big(i\epsilon\, M_+^D(\nu+i\epsilon,\alpha)\big) \geqslant 0, \quad \lim_{\epsilon\downarrow 0}\big(i\epsilon\, M_+^D(\nu+i\epsilon,\alpha)^{-1}\big) \geqslant 0, \quad \nu\in\mathbb{R}. \tag{2.19}$$

(ii) $M_+^D(\,\cdot\,,\alpha)$ admits the representation

$$M_+^D(\zeta,\alpha) = F_+(\alpha) + \int_{\mathbb{R}} d\Omega_+^D(\nu,\alpha)\,\big[(\nu-\zeta)^{-1} - \nu(1+\nu^2)^{-1}\big], \tag{2.20}$$

where

$$F_+(\alpha) = F_+(\alpha)^*, \qquad \int_{\mathbb{R}} \big\|d\Omega_+^D(\nu,\alpha)\big\|_{\mathbb{C}^{m\times m}} (1+\nu^2)^{-1} < \infty. \tag{2.21}$$

Moreover,

$$\Omega_+^D((\mu,\nu],\alpha) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\mu+\delta}^{\nu+\delta} d\nu' \, \mathrm{Im}\big(M_+^D(\nu'+i\varepsilon,\alpha)\big). \tag{2.22}$$

(iii) $\mathrm{Im}\big(M_+^D(\,\cdot\,,\alpha)\big)$ *satisfies*

$$\begin{aligned}\mathrm{Im}\big(M_+^D(\zeta,\alpha)\big) &= \mathrm{Im}(\zeta)\int_0^\infty dx\, U_+(\zeta,x,\alpha)^* U_+(\zeta,x,\alpha) \\ &= \mathrm{Im}(\zeta)\int_0^\infty dx\, \big[u_{+,1}(\zeta,x,\alpha)^* u_{+,1}(\zeta,x,\alpha) \\ &\qquad\qquad + u_{+,2}(\zeta,x,\alpha)^* u_{+,2}(\zeta,x,\alpha)\big].\end{aligned} \tag{2.23}$$

While $\mathcal{D}$ contains the locally square integrable $m\times m$ matrix-valued coefficient $\phi \in L^2_{\mathrm{loc}}([0,\infty))^{m\times m}$, the associated generalized half-line Schrödinger operators to be discussed next will exhibit distributional potentials and hence are outside the standard Weyl–Titchmarsh theory for Sturm–Liouville operators with locally integrable $m\times m$ matrix-valued potentials on $[0,\infty)$. The supersymmetric approach employed in [13] made the transition from the usual L^1_{loc}-potentials in Schrödinger operators to (matrix-valued) distributional H^{-1}_{loc}-potentials (and more general situations) in an effortless manner. Here, due to our assumption that ϕ belongs to the space $L^2_{\mathrm{loc}}([0,\infty))^{m\times m}$, the corresponding potential belongs to $H^{-1}_{\mathrm{loc}}([0,\infty))^{m\times m}$.

To briefly describe the corresponding generalized half-line Schrödinger operators, we first introduce the following two kinds of quasi-derivatives,

$$\begin{aligned}&u^{[1,1]}(x) = (Au)(x) = u'(x) + \phi(x)u(x) \text{ for a.e. } x>0, \\ &u \in \mathrm{dom}(A) = \big\{v \in L^2([0,\infty))^m \,\big|\, v \in AC([0,R]) \text{ for all } R>0; \\ &\qquad\qquad\qquad\qquad (v'+\phi v) \in L^2([0,\infty))^m\big\},\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}&u^{[1,2]}(x) = -(A^+u)(x) = u'(x) - \phi(x)u(x) \text{ for a.e. } x>0, \\ &u \in \mathrm{dom}(A^+) = \big\{v \in L^2([0,\infty))^m \,\big|\, v \in AC([0,R]) \text{ for all } R>0; \\ &\qquad\qquad\qquad\qquad (v'-\phi v) \in L^2([0,\infty))^m\big\}.\end{aligned} \tag{2.25}$$

Thus, introducing the $m\times m$ matrix-valued differential expressions τ_j, $j=1,2$, by

$$(\tau_1 u)(x) = (A^+Au)(x) = -\big(u^{[1,1]}\big)'(x) + \phi(x)u^{[1,1]}(x) \text{ for a.e. } x>0, \tag{2.26}$$

and

$$(\tau_2 u)(x) = (AA^+u)(x) = -\big(u^{[1,2]}\big)'(x) - \phi(x)u^{[1,2]}(x) \text{ for a.e. } x>0, \tag{2.27}$$

one infers that formally, τ_j, $j=1,2$, are of the generalized Schrödinger form

$$\tau_j = -I_m\frac{d^2}{dx^2} + V_j(x), \quad V_j(x) = \phi(x)^2 + (-1)^j\phi'(x), \quad j=1,2. \tag{2.28}$$

We emphasize that while $\phi^2 \in L^1_{\mathrm{loc}}([0,\infty))^{m\times m}$ represents a standard matrix-valued potential coefficient, in general, ϕ' is now a genuine distribution (unless one assumes in addition that $\phi \in AC_{\mathrm{loc}}([0,\infty))^{m\times m}$). In contrast to these half-line Schrödinger operators, the Dirac-type operators $D_+(\alpha)$ only contain the standard potential coefficient $\phi \in L^2_{\rm loc}([0,\infty))^{m\times m}$.

By inspection, the second-order initial value problems,

$$\begin{aligned}
&((\tau_j - z)f)(x) = g(x) \text{ for a.e. } x > 0,\\
&f, f^{[1,j]} \in AC_{\rm loc}([0,\infty))^m,\ g \in L^1_{\rm loc}([0,\infty))^m,\\
&f(x_0) = c_0,\ \ f^{[1,j]}(x_0) = d_0,\ j = 1,2,
\end{aligned} \tag{2.29}$$

for some $x_0 \geqslant 0$, $c_0, d_0 \in \mathbb{C}$, are equivalent to the first-order initial value problems

$$\begin{aligned}
&\begin{pmatrix} f(x) \\ f^{[1,j]}(x) \end{pmatrix}' = \begin{pmatrix} (-1)^j\phi(x) & 1 \\ -z & (-1)^{j+1}\phi(x) \end{pmatrix} \begin{pmatrix} f(x) \\ f^{[1,j]}(x) \end{pmatrix} - \begin{pmatrix} 0 \\ g(x) \end{pmatrix} \text{ for a.e. } x > x_0,\\
&\begin{pmatrix} f(x_0) \\ f^{[1,j]}(x_0) \end{pmatrix} = \begin{pmatrix} c_0 \\ d_0 \end{pmatrix}, \quad j = 1,2,
\end{aligned} \tag{2.30}$$

respectively. Since by Hypothesis 2.1, $\phi \in L^2_{\rm loc}([0,\infty))^{m\times m}$ (in fact, already $\phi \in L^1_{\rm loc}([0,\infty))^{m\times m}$ would be sufficient), the initial value problems in (2.30) (and hence those in (2.29)) are uniquely solvable by [37, Theorem 16.1] (see also [14, Theorem 10.1] and [37, Theorem 16.2]).

Next, suppose that for some $1 \leqslant p \leqslant m$, $U = (u_1\ u_2)^\top$ is a $2m \times p$ matrix-valued solution of $\mathcal{D}U = \zeta U$, that is,

$$\begin{aligned}
&u_j \in AC_{\rm loc}([0,\infty))^{m\times p},\ j = 1,2, \\
&u_1^{[1,1]} = Au_1 \in L^1_{\rm loc}([0,\infty))^{m\times p}, \quad u_2^{[1,2]} = -A^+u_2 \in L^1_{\rm loc}([0,\infty))^{m\times p}.
\end{aligned} \tag{2.31}$$

Then, if $\zeta \neq 0$, the supersymmetric structure of $\mathcal{D}$ in (2.1) actually implies that also

$$u_1^{[1,1]} = Au_1 = \zeta u_2 \in AC_{\rm loc}([0,\infty))^{m\times p}, \tag{2.32}$$

$$u_2^{[1,2]} = -A^+u_2 = -\zeta u_1 \in AC_{\rm loc}([0,\infty))^{m\times p}, \tag{2.33}$$

and hence that u_j are actually distributional $m \times p$ matrix-valued solutions of $\tau_j u = \zeta^2 u$, $j = 1,2$, that is,

$$\begin{aligned}
&u_j, u_j^{[1,j]} \in AC_{\rm loc}([0,\infty))^{m\times p}, \quad \big(u_j^{[1,j]}\big)' \in L^1_{\rm loc}([0,\infty))^{m\times p},\\
&\tau_j u_j = -\big(u_j^{[1,j]}\big)' + (-1)^{j+1}\phi u_j^{[1,j]} = \zeta^2 u_j, \quad j = 1,2.
\end{aligned} \tag{2.34}$$

Thus, applying the L^2-property (2.23) and (2.31)–(2.34) to the Weyl–Titchmarsh solutions $U_+(\zeta,\,\cdot\,,\alpha)$ associated with the Dirac-type differential expression $\mathcal{D}$, then shows that $u_{+,j}(\zeta,\,\cdot\,,\alpha)$ are Weyl–Titchmarsh solutions associated with τ_j, $j = 1,2$, replacing the complex energy parameter ζ with $z = \zeta^2$. Moreover, introducing the following fundamental system $s_j(z,\,\cdot\,), c_j(z,\,\cdot\,)$, $j = 1,2$, of $m \times m$ matrix-valued solutions of $\tau_j u = zu$, $z \in \mathbb{C}$, $j = 1,2$, normalized for arbitrary $z \in \mathbb{C}$ by

$$s_j(z,0) = 0, \qquad s_j^{[1,j]}(z,0) = I_m, \tag{2.35}$$

$$c_j(z,0) = I_m, \quad c_j^{[1,j]}(z,0) = 0, \qquad j = 1,2, \tag{2.36}$$

one observes as usual that for fixed $x \in \mathbb{R}$, $s_j(\,\cdot\,,x), c_j(\,\cdot\,,x)$, $j = 1,2$ are entire. The connection with the solutions φ_j and ϑ_j, $j = 1,2$, of $\mathcal{D}U = \zeta U$ is given by

$$s_1(z,x) = \zeta^{-1}\varphi_1(\zeta,x,\alpha_0), \quad c_1(z,x) = \vartheta_1(\zeta,x,\alpha_0), \tag{2.37}$$

$$s_2(z,x) = \zeta^{-1}\vartheta_2(\zeta,x,\alpha_0), \quad c_2(z,x) = \varphi_2(\zeta,x,\alpha_0), \quad z = \zeta^2,\ x \geqslant 0. \tag{2.38}$$

In addition, introducing the Weyl–Titchmarsh solutions $\psi_{+,j}(z,\,\cdot\,)$ associated with τ_j, $j=1,2$, via

$$\psi_{+,1}(z,\,\cdot\,) = u_{+,1}(\zeta,\,\cdot\,,\alpha_0), \tag{2.39}$$

$$\psi_{+,2}(z,\,\cdot\,) = u_{+,2}(\zeta,\,\cdot\,,\alpha_0)M_+^D(\zeta,\alpha_0)^{-1}, \quad z=\zeta^2,\ \zeta\in\mathbb{C}\backslash\mathbb{R},\ j=1,2, \tag{2.40}$$

(the right-hand sides being independent of the choice of branch for ζ) and the generalized Dirichlet-type $m\times m$ matrix-valued Weyl–Titchmarsh functions $\widehat{M}_{+,0,j}$ of τ_j,

$$\widehat{M}_{+,0,1}(z) = \zeta M_+^D(\zeta,\alpha_0), \tag{2.41}$$

$$\widehat{M}_{+,0,2}(z) = -\zeta M_+^D(\zeta,\alpha_0)^{-1}, \quad z=\zeta^2,\ \zeta\in\mathbb{C}\backslash\mathbb{R}, \tag{2.42}$$

one infers from (2.11) that

$$\psi_{+,j}(z,\,\cdot\,) = c_j(z,\,\cdot\,) + s_j(z,\,\cdot\,)\widehat{M}_{+,0,j}(z), \quad z\in\mathbb{C}\backslash[0,\infty),\ j=1,2. \tag{2.43}$$

Indeed, (2.43) follows from combining (2.11), (2.32), and (2.33) (for $p=m$), which in turn imply

$$\psi_{+,j}(z,0) = I_m, \quad \psi_{+,j}^{[1,j]}(z,0) = \widehat{M}_{+,0,j}(z), \quad z\in\mathbb{C}\backslash[0,\infty),\ j=1,2 \tag{2.44}$$

and the unique solvability of the initial value problems in (2.29). We summarize this discussion in the following result proved in [13]:

Theorem 2.3. *Assume Hypothesis* 2.1 *and let* $\alpha_0 = (I_m\ 0)$. *Denote by*

$$U_+(\zeta,\,\cdot\,,\alpha_0) = (u_{+,1}(\zeta,\,\cdot\,,\alpha_0)\ u_{+,2}(\zeta,\,\cdot\,,\alpha_0))^\top, \quad \zeta\in\mathbb{C}\backslash\mathbb{R}, \tag{2.45}$$

the Weyl–Titchmarsh solution corresponding to $\mathcal{D}$, *and by* $M_+^D(\,\cdot\,,\alpha_0)$ *the* $m\times m$ *matrix-valued half-line Weyl–Titchmarsh function corresponding to* $\mathcal{D}$. *Then the* $m\times m$ *matrix-valued Weyl–Titchmarsh solutions associated with* τ_j, *denoted by* $\psi_{+,j}(z,\,\cdot\,)$, $j=1,2$, *are given by* (2.39) *and* (2.40), *and the* $m\times m$ *matrix-valued generalized Dirichlet-type Weyl–Titchmarsh functions* $\widehat{M}_{+,0,j}$ *of* τ_j, $j=1,2$, *are given by* (2.41) *and* (2.42). *In particular,*

$$\widehat{M}_{+,0,1}(z) = \zeta M_+^D(\zeta,\alpha_0) = -z\widehat{M}_{+,0,2}(z)^{-1}, \quad z=\zeta^2,\ \zeta\in\mathbb{C}\backslash\mathbb{R}. \tag{2.46}$$

The subscript "0" in $\widehat{M}_{+,0,j}$, $j=1,2$, indicates that these generalized Weyl–Titchmarsh matrices correspond to a Dirichlet boundary condition at the reference point $x=0$ in the corresponding generalized half-line Schrödinger operators $H_{+,0,j}$, $j=1,2$, in $L^2([0,\infty))^m$ defined by

$$\begin{aligned}
&(H_{+,0,j}u)(x) = (\tau_j u)(x) = -\big(u^{[1,j]}\big)'(x) + (-1)^{j+1}\phi(x)u^{[1,j]}(x) \text{ for a.e. } x>0,\\
&u\in\operatorname{dom}(H_{+,0,j}) = \big\{v\in L^2([0,\infty))^m \,\big|\, v, v^{[1,j]}\in AC([0,R])^m \text{ for all } R>0;\\
&\qquad v(0)=0;\ \big[\big(v^{[1,j]}\big)' + (-1)^j\phi v^{[1,j]}\big]\in L^2([0,\infty))^m\big\}, \quad j=1,2.
\end{aligned} \tag{2.47}$$

(For more general Sturm–Liouville operators in the scalar case $m=1$ we refer to [11] and the references therein.) The corresponding Green's function of $H_{+,0,j}$ is then of the familiar form

$$\begin{aligned}
G_{+,0,j}(z,x,x') &= (H_{+,0,j}-zI)^{-1}(x,x')\\
&= \begin{cases} s_j(z,x)\psi_{+,j}(\overline{z},x')^*, & x\leqslant x',\\ \psi_{+,j}(z,x)s_j(\overline{z},x')^*, & x'\leqslant x, \end{cases}
\end{aligned} \tag{2.48}$$

$$z \in \mathbb{C}\backslash[0,\infty),\ x, x' \in [0,\infty),\ j = 1,2.$$

Of course, (2.39)–(2.46), (2.48) extend as usual to all z in the resolvent set of $H_{+,0,j}$, $j = 1,2$.

We conclude this section by detailing some properties of $\widehat{M}_{+,0,j}$: First, we recall the fundamental identity

$$\operatorname{Im}\big(\widehat{M}_{+,0,j}(z)\big) = \operatorname{Im}(z) \int_0^\infty dx'\, \psi_{+,j}(z,x')^* \psi_{+,j}(z,x'), \quad z \in \mathbb{C}\backslash\mathbb{R},\ j = 1,2, \tag{2.49}$$

implying that $\widehat{M}_{+,0,j}$, $j = 1,2$, are matrix-valued Nevanlinna–Herglotz functions. Moreover, one has the following result.

Lemma 2.4. *Assume Hypothesis* 2.1 *and denote by $\widehat{M}_{+,0,j}$, $j = 1,2$, the generalized Dirichlet-type $m \times m$ matrix-valued Weyl–Titchmarsh functions associated to $H_{+,0,j}$, $j = 1,2$, as defined by* (2.41) *and* (2.42). *Then $\widehat{M}_{+,0,j}$, $j = 1,2$, are $m \times m$ matrix-valued Nevanlinna–Herglotz functions of maximal rank m. In particular* (*for $j = 1,2$*),

$$\operatorname{Im}\big(\widehat{M}_{+,0,j}(z)\big) \geqslant 0, \quad z \in \mathbb{C}_+, \tag{2.50}$$

$$\widehat{M}_{+,0,j}(\overline{z}) = \widehat{M}_{+,0,j}(z)^*, \tag{2.51}$$

$$\operatorname{rank}\big(\widehat{M}_{+,0,j}(z)\big) = m, \tag{2.52}$$

$$\lim_{\varepsilon\downarrow 0} \widehat{M}_{+,0,j}(\lambda + i\varepsilon) \text{ exists for a.e. } \lambda \in \mathbb{R}. \tag{2.53}$$

3. Inverse Spectral Theory for Half-Line Dirac-Type and Schrödinger Operators

Several equivalent forms of self-adjoint Dirac-type systems have been considered in the literature. In particular, the case of self-adjoint Dirac-type systems of the form

$$\frac{d}{dx}\Upsilon(\zeta,x) = i(\zeta\mathfrak{S}_3 + \mathfrak{S}_3\mathcal{V}(x))\Upsilon(\zeta,x) \text{ for a.e. } x > 0, \tag{3.1}$$

where

$$\mathfrak{S}_3 = \begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix}, \qquad \mathcal{V}(x) = \begin{pmatrix} 0 & \mathcal{Q}(x) \\ \mathcal{Q}(x)^* & 0 \end{pmatrix}, \quad x \geqslant 0, \tag{3.2}$$

$\mathcal{Q}$ is an $m \times m$ matrix-valued function defined a.e. on $[0,\infty)$, and $\zeta \in \mathbb{C}$ represents the spectral parameter, was recently studied in [40]. The procedure described in [40] to solve the inverse spectral problem of recovering $\mathcal{Q}$ from the underlying matrix-valued half-line Weyl–Titchmarsh function is based on the method of operator identities [41, 42, 43] (see also the references therein).

In the special case when

$$\mathcal{Q}(x) = -\mathcal{Q}(x)^* \text{ for a.e. } x > 0, \tag{3.3}$$

the system (3.1) is equivalent to the half-line Dirac-type system

$$(\mathcal{D}U)(\zeta,x) = \zeta U(\zeta,x), \quad \mathcal{D} = J\frac{d}{dx} + \begin{pmatrix} 0 & \phi(x) \\ \phi(x) & 0 \end{pmatrix}, \quad x > 0, \tag{3.4}$$

where

$$J = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}, \qquad \phi(x) = \phi(x)^* \text{ for a.e. } x > 0, \tag{3.5}$$

studied in the first part of Section 2.

The explicit connection between systems (3.1) and (3.4) is given by the relations

$$U(\zeta,x)=W\Upsilon(\zeta,x),\quad \phi(x)=-i\mathcal{Q}(x),\quad W:=\frac{1}{\sqrt{2}}\begin{pmatrix}-iI_m & iI_m\\ I_m & I_m\end{pmatrix}. \tag{3.6}$$

Indeed, one easily verifies that

$$-W^*J\,W=i\mathfrak{S}_3,\qquad -W^*\begin{pmatrix}0 & \phi(x)\\ \phi(x) & 0\end{pmatrix}W=\mathcal{V}(x),\quad x>0, \tag{3.7}$$

where W is unitary (i.e., $W^*W=WW^*=I_{2m}$).

In order to apply the results from [40] to the Dirac-type system (3.4), we need some preparations. First, we recall the normalized fundamental $2m\times 2m$ solution $\Psi(\zeta,x,\alpha)$ of (3.4) as introduced in (2.12), (2.13), with α satisfying (2.2)–(2.6).

The $m\times m$ matrix-valued Weyl–Titchmarsh function $M_+^D(\cdot,\alpha)$, of the system (3.4) on $[0,\infty)$ is then introduced by the relation

$$\Psi(\zeta,x,\alpha)\begin{pmatrix}I_m\\ M_+^D(\zeta,\alpha)\end{pmatrix}\in L^2\big([0,\infty)\big)^{2m\times m},\quad \zeta\in\mathbb{C}_+. \tag{3.8}$$

On the other hand, the fundamental solution $\widehat{\Psi}(\zeta,x)$ of the Dirac-type system (3.1) in [40] is normalized at $x=0$ by

$$\widehat{\Psi}(\zeta,0)=I_{2m},\quad \zeta\in\mathbb{C}, \tag{3.9}$$

and the corresponding Weyl–Titchmarsh matrix $\widehat{M}^D$ is introduced in [40, eq. (1.5)] by the relation

$$\widehat{\Psi}(\zeta,x)\begin{pmatrix}I_m\\ \widehat{M}^D(\zeta)\end{pmatrix}\in L^2\big([0,\infty)\big)^{2m\times m},\quad \zeta\in\mathbb{C}_+. \tag{3.10}$$

In view of (3.6), (2.12) and (3.9) one concludes that

$$\Psi(\zeta,x,\alpha)=W\widehat{\Psi}(\zeta,x)W^*\Psi(\zeta,0,\alpha),\quad \zeta\in\mathbb{C},\ x\geqslant 0, \tag{3.11}$$

and one notes that according to (2.2), the initial value $\Psi(\zeta,0,\alpha)$ is unitary. It is immediate that the unitary matrix $W^*\Psi(\zeta,0,\alpha)$ is given by

$$W^*\Psi(\zeta,0,\alpha)=\frac{1}{\sqrt{2}}\begin{pmatrix}\alpha_2^*+i\alpha_1^* & \alpha_1^*-i\alpha_2^*\\ \alpha_2^*-i\alpha_1^* & \alpha_1^*+i\alpha_2^*\end{pmatrix}=\begin{pmatrix}\alpha_1^*-i\alpha_2^* & 0\\ 0 & \alpha_1^*+i\alpha_2^*\end{pmatrix}W^*, \tag{3.12}$$

where, according to (2.2), one has

$$(\alpha_1+i\alpha_2)(\alpha_1^*-i\alpha_2^*)=I_m. \tag{3.13}$$

Taking into account (3.8) and (3.10)–(3.13), one derives the equality

$$\widehat{M}^D(\zeta)=(\alpha_1^*+i\alpha_2^*)\big[M_+^D(\zeta,\alpha)-iI_m\big]\big[M_+^D(\zeta,\alpha)+iI_m\big]^{-1}(\alpha_1+i\alpha_2),\quad \zeta\in\mathbb{C}_+, \tag{3.14}$$

relating the matrix-valued Weyl–Titchmarsh functions for systems (3.1) and (3.4). We note that the Weyl–Titchmarsh matrices for both systems are unique (due to the limit point property of $\mathcal{D}$ at ∞) and that $\widehat{M}^D$ is contractive on $\mathbb{C}_+$.

Since $\phi=-i\mathcal{Q}$ (see (3.6)), using (3.14) we can now reformulate [40, Theorems 1.4 and 4.4] for the case of the half-line Dirac systems at hand. For that purpose, we partition $\widehat{\Psi}(0,x)$ into the $m\times m$ blocks β_1, β_2, γ_1, and γ_2:

$$\widehat{\Psi}(0,x)=\begin{pmatrix}\beta(x)\\ \gamma(x)\end{pmatrix}=\begin{pmatrix}\beta_1(x) & \beta_2(x)\\ \gamma_1(x) & \gamma_2(x)\end{pmatrix},\quad x\geqslant 0, \tag{3.15}$$

and recover ϕ from those blocks. The properties of β and γ, which we give below, are essential for their recovery and follow immediately from (3.1), (3.2), and (3.9):

$$\beta(0) = \begin{pmatrix} I_m & 0 \end{pmatrix}, \quad \gamma(0) = \begin{pmatrix} 0 & I_m \end{pmatrix}; \quad \beta\mathfrak{S}_3\beta^* \equiv I_m, \quad \gamma\mathfrak{S}_3\gamma^* \equiv -I_m, \tag{3.16}$$

$$\beta\mathfrak{S}_3\gamma^* \equiv 0, \quad \beta'\mathfrak{S}_3\beta^* = \gamma'\mathfrak{S}_3\gamma^* \equiv 0, \quad \beta'\mathfrak{S}_3\gamma^* = \phi. \tag{3.17}$$

Next, we introduce the operator of integration, $\mathcal{A}_x \in \mathcal{B}\big(L^2\big([0,x]\big)^m\big)$, $x > 0$, by

$$\big(\mathcal{A}_x f\big)(y) = -i \int_0^y f(t)dt; \quad y \in [0,x], \ f \in L^2\big([0,x]\big)^m, \tag{3.18}$$

acting componentwise on f.

A direct application of [40, Theorems 1.4 and 4.4] then implies the following inverse spectral result for the half-line Dirac operator $D_+(\alpha)$:

Theorem 3.1. *Assume Hypothesis* 2.1 *and consider the half-line Dirac-type operator $D_+(\alpha)$ in* (2.8)*, with associated Weyl–Titchmarsh matrix $M_+^D(\,\cdot\,,\alpha)$. Then $M_+^D(\,\cdot\,,\alpha)$ uniquely determines $\phi(\cdot)$ a.e. on $[0,\infty)$.*

In order to explicitly recover $\phi(\cdot)$ from $M_+^D(\,\cdot\,,\alpha)$, one first recovers the $m\times m$ matrix-valued function $\Lambda(\cdot)$ via equality (3.14) *and the formula*

$$\Lambda(x) = (2\pi i)^{-1} e^{x\eta} \operatorname*{l.i.m.}_{a\to\infty} \int_{-a}^{a} d\xi \, \frac{e^{-ix\xi}}{\xi + i\eta} \widehat{M}^D\left(\frac{\xi + i\eta}{2}\right), \quad x > 0, \tag{3.19}$$

where $\eta > 0$ is arbitrary and l.i.m. denotes the entrywise limit in the norm of $L^2\big([0,\infty)\big)$. Then

$$\Lambda \in H^1_{\rm loc}([0,\infty))^{m\times m}. \tag{3.20}$$

Introducing the bounded operator $\Pi_x \in \mathcal{B}\big(\mathbb{C}^{2m}, L^2\big([0,x]\big)^m\big)$, $x > 0$, via

$$\big(\Pi_x g\big)(\cdot) = \Lambda(\cdot)g_1 + g_2, \quad x > 0, \ g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad g_1, g_2 \in \mathbb{C}^m, \tag{3.21}$$

the following operator identity,

$$\mathcal{A}_x\mathcal{S}_x - \mathcal{S}_x\mathcal{A}_x^* = i\Pi_x\mathfrak{S}_3\Pi_x^*, \quad x > 0, \tag{3.22}$$

leads to the boundedly invertible and strictly positive operator $\mathcal{S}_x \in \mathcal{B}\big(L^2\big([0,x]\big)^m\big)$, $x > 0$, given by

$$\big(\mathcal{S}_x f\big)(y) = f(y) - \frac{1}{2}\int_0^x ds \int_{|y-s|}^{y+s} dt \, \Lambda'\left(\frac{t+y-s}{2}\right)\Lambda'\left(\frac{t+s-y}{2}\right)^* f(s) \tag{3.23}$$

for $y \in [0,x]$ and every $f \in L^2([0,x])^m$. Moreover,

$$\Pi_x^*\mathcal{S}_x^{-1}\Pi_x \in AC_{\rm loc}([0,\infty))^{m\times m}, \tag{3.24}$$

and hence one can define the Hamiltonian H of the corresponding canonical system,

$$H(x) = \gamma(x)^*\gamma(x) = \frac{d}{dx}\big(\Pi_x^*\mathcal{S}_x^{-1}\Pi_x\big) \text{ for a.e. } x > 0. \tag{3.25}$$

Using (3.16) *and* (3.17)*, one uniquely recovers γ and β from H as described in Remark* 3.2 *below. Finally, one obtains ϕ via*

$$\phi(x) = \beta'(x)\mathfrak{S}_3\gamma(x)^* \text{ for a.e. } x > 0. \tag{3.26}$$

Remark 3.2. We describe the recovery of β and γ satisfying (3.16) and (3.17) from H given by (3.25) in greater detail. First, one recovers $\gamma_2^{-1}\gamma_1$ via

$$\gamma_2^{-1}\gamma_1 = [\gamma_2^*\gamma_2]^{-1}\gamma_2^*\gamma_1 = \left(\begin{pmatrix}0 & I_m\end{pmatrix} H \begin{pmatrix}0\\ I_m\end{pmatrix}\right)^{-1} \begin{pmatrix}0 & I_m\end{pmatrix} H \begin{pmatrix}I_m\\ 0\end{pmatrix}. \tag{3.27}$$

Next, one recovers γ_2 from the differential equation and initial condition below,

$$\gamma_2' = \gamma_2\big(\gamma_2^{-1}\gamma_1\big)'\big(\gamma_2^{-1}\gamma_1\big)^*\big(I_m - \gamma_2^{-1}\gamma_1(\gamma_2^{-1}\gamma_1)^*\big)^{-1}, \quad \gamma_2(0) = I_m. \tag{3.28}$$

Given γ_2 and $\gamma_2^{-1}\gamma_1$, one recovers γ_1 and γ. Finally, one recovers β via the relations,

$$\beta = \beta_1\breve{\beta}, \quad \breve{\beta} := \begin{pmatrix}I_m & \gamma_1^*\big(\gamma_2^*\big)^{-1}\end{pmatrix}, \tag{3.29}$$

$$\beta_1' = -\beta_1\big[\breve{\beta}'\mathfrak{S}_3\big(\breve{\beta}\big)^*\big]\big[\breve{\beta}\mathfrak{S}_3\big(\breve{\beta}\big)^*\big]^{-1}, \quad \beta_1(0) = I_m. \tag{3.30}$$

Next, combining (2.46) and (3.14) one also obtains (employing $\alpha_0 = (I_m \quad 0)$)

$$\widehat{M}^D(\zeta) = (-1)^{j+1}\big[\widehat{M}_{+,0,j}(\zeta^2) - i\zeta I_m\big]\big[\widehat{M}_{+,0,j}(\zeta^2) + i\zeta I_m\big]^{-1}, \quad \zeta \in \mathbb{C}_+, \ j = 1,2. \tag{3.31}$$

Thus, given ϕ, one has actually reconstructed the distributional potential coefficients $V_j = \phi^2 + (-1)^j\phi'$ in the generalized half-line Schrödinger operators $H_{+,0,j}$, $j = 1,2$:

Corollary 3.3. *Assume Hypothesis* 2.1 *and consider the generalized half-line Schrödinger operators $H_{+,0,j}$, $j = 1,2$, with associated Dirichlet-type matrix-valued Weyl–Titchmarsh functions $\widehat{M}_{+,0,j}$, $j = 1,2$. Then either one of $\widehat{M}_{+,0,1}$ and $\widehat{M}_{+,0,2}$ uniquely determines $\phi(\cdot)$ a.e. on $[0,\infty)$, and hence also $V_j = \phi^2 + (-1)^j\phi'$, $j = 1,2$.*

In addition, $\phi(\cdot)$ is recovered from $\widehat{M}_{+,0,1}$ (resp., $\widehat{M}_{+,0,2}$) along the lines of (3.19)–(3.26) *upon employing* (3.31) *on the right-hand side of* (3.19).

For inverse spectral problems with distributional potentials in the scalar context $m = 1$ we also refer to [12].

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