Math 4510/5010 **Final Exam** **Name:** ____________________
Spring 2011 **UTCID#:** ____________________

Directions: Read all the questions carefully and answer them completely. No credit will be given for undocumented responses. **Show all your work to get full credit.**

Please note: Do 10 problems from questions 1 to 11. 17 points each.

1. (17 pts) A function $f:(a,b)\to R$ satisfies a Lipschitz condition at $x\in(a,b)$ iff there is $M>0$ and $\epsilon>0$ such that $|x-y|<\epsilon$ and $y\in(a,b)$ imply that $|f(x)-f(y)|\le M|x-y|$.
(a) (5 pts) Give an example of a function that fails to satisfy a Lipschitz condition at a point of continuity.

(b) (12 pts) If f is differentiable at x, prove that f satisfies a Lipschitz condition at x.

2. (17 pts) Define $f(x)=\begin{cases} x+2x^2\sin\dfrac{1}{x}, & x\neq 0,\\ 0, & x=0.\end{cases}$ Prove that f is differentiable everywhere (12 pts). Show that there exists a number a such that $f'(a)>0$, but there does not exists a neighborhood of a in which f is increasing (5 pts).

3. (17 pts) If $0 < p < 1$ and $h > 0$, then show that

$$(1 + h)^p < 1 + ph.$$

4. (17 pts) A set $A \subset [0, 1]$ is dense in $[0, 1]$ iff every open interval that intersects $[0, 1]$ contains a point of A. Suppose $f : [0, 1] \to R$ is integrable and $f(x) = 0$ for all $x \in A$ with A dense in $[0, 1]$. Show that $\int_0^1 f(x)dx = 0$.

5. (17 pts) Prove Theorem 5.3, i.e., show that if $f : [a, b] \to R$ is increasing, then $f \in R(x)$ on $[a, b]$.

6. (17 pts) Use induction to show that $1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$ for any $n \in J$. Can you use this fact to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converges or diverges? Justify your reasoning.

7. (17 pts) Determine those values of p for which $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges.

8. (17 pts) Let $\sum_{n=1}^{\infty} a_n$ be an infinite series and $\{n_k\}_{n=1}^{\infty}$ a subsequence of the sequence of positive integers. Prove that if $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_{n_k}$ converges absolutely. What can be concluded if $\sum_{n=1}^{\infty} a_n$ converges conditionally? Justify your answers.

9. (17 pts) Prove that the sequence $\{k_n\}_{n=1}^{\infty}$, defined by

$$k_n(x) = \frac{x}{1 + nx^2}$$

for all $x \in R$ and $n \in J$, converges uniformly on R.

10. (17 pts) For each $n \in J$ and $x \in [0, 1]$, define $f_n(x) = \dfrac{nx}{1 + n^2x^4}$. Show that $\{f_n\}_{n=1}^{\infty}$ does not converge uniformly on $[0, 1]$ (10 pts) . Is the equality

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx$$

true? Justify your answers (7 pts).

11. (17 pts) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. If, for any $p \in J$, we have

$$\lim_{n\to\infty} (a_{n+1} + a_{n+2} + \ldots + a_{n+p}) = 0,$$

is the series $\sum_{n=1}^{\infty} a_n$ convergent (5 pts)? Justify your answers (10 pts).

Everyone needs do the following Problem 12.

12. (30 pts) For each of the following, determine whether the statement is true and justify your answer.

(a) (6 pts) If $f : D \to R$ is uniformly continuous and differentiable on D, then the derivative f' of f is bounded on D.

(b) (6 pts) If f is uniformly continuous on D, then f is differentiable on D.

(c) (6 pts) Let $f : [0,1] \to R$ be defined by

$$f(x) = \begin{cases} 2, & \text{if } x \text{ is rational,} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$$

Then $f \in R(x)$ on $[0,1]$.

(d) (6 pts) If the power series $\sum_{n=0}^{\infty} a_n x^n$ diverges at $x = x_1$, then it must diverge for all $x > x_1$.

(e) (6 pts) The sequence $f_n(x) = x^n - x^{2n}$ converges uniformly on the interval $[0,1]$.