Math 4510/5010 **Exam 3** **Name:** ____________________
Spring 2011 **UTCID#:** ____________________

Directions: Read all the questions carefully and answer them completely. No credit will be given for undocumented responses. **Show all your work to get full credit**.

1. (10 pts) State carefully the meaning of the following by definition.
(a) (4 pts) An infinite series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(b) (6 pts) A sequence $\{f_n(x)\}_{n=1}^{\infty}$ of functions does not converge uniformly to $f(x)$ on E.

2 (12 pts) For each of the following, determine whether the statement is true and justify your answer.
(a) (6 pts) The sequence $f_n(x) = x^n - x^{n+1}$ converges uniformly on the interval $[0, 1]$.

(b) (6 pts) If $f_n(x)$ converges uniformly to $f(x)$ on E and $g_n(x)$ converges uniformly to $g(x)$ on E, then $f_n(x)g_n(x)$ converges uniformly to $f(x)g(x)$ on E.

3. (20 pts) Finish the following.
(a) (10 pts) Prove that the first mean value theorem for integrals, i.e., prove that if $f : [a, b] \to R$ is continuous and $g : [a, b] \to R$ is integrable on $[a, b]$ with $g(x) \geq 0$ for all $x \in [a, b]$, then there is $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx.$$

(b) (10 pts) Use the first mean value theorem for integrals to prove that

$$\lim_{n\to\infty} \int_0^{\pi/2} \sin^n x dx = 0.$$

4. (10 pts) Assume that $\sum_{n=1}^{\infty} a_n$ converges and $a_n \geq a_{n+1} > 0$ for all $n \in J$. Prove that $\lim_{n\to\infty} na_n = 0$.

5. (10 pts) Assume that $\lim_{n\to\infty} na_n = a \neq 0$. Prove that $\sum_{n=1}^{\infty} a_n$ diverges.

6. (10 pts) Let $f : R \to R$ be uniformly continuous, and for each $n \in J$ and $x \in R$, define

$$f_n(x) = f\left(x + \frac{1}{n}\right).$$

Prove that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on R.

7. (12 pts) For any $n \in J$, define $h_n : [0, 1] \to R$ by

$$h_n(x) = \begin{cases} \frac{1}{x}, & \frac{1}{n} \leq x \leq 1, \\ n^2 x, & 0 \leq x < \frac{1}{n}. \end{cases}$$

Show that $\{h_n\}_{n=1}^{\infty}$ does not converge uniformly on $[0, 1]$, but does converge uniformly on $[\rho, 1]$ for any $0 < \rho < 1$.

8. (16 pts) For any $n \in J$, define

$$f_n(x) = \frac{1}{n} \arctan(x^n).$$

(a) (5 pts) Show that $f_n(x)$ converges uniformly on $(-\infty, \infty)$.

(b) (5 pts) Show that

$$\left[\lim_{n\to\infty} f_n(x)\right]'_{x=1} \neq \lim_{n\infty} f'_n(1).$$

(c) (6 pts) Show that $f'_n(x)$ does not converge uniformly on $(0, 2)$.