

Rotational Behavior of Einsteinian Space

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Abstract

The rotational transformation between the Schwarzschild metric and the Kerr metric is obtained for weak field at large distance. The transformation on the equatorial plane is discussed in detail and the rotational time dilation is derived.

Key Words: general relativity, gravitation, time dilation, Schwarzschild metric, Kerr metric

1. Introduction

The special theory of relativity describes the physics of inertial systems. The dynamics is based on the Lorentz transformation, from which the concepts such as time dilation, length contraction, velocity addition, mass- energy relationship, and the transformation of electromagnetic fields can be derived. It can be said that the Lorentz transformation is the mathematical foundation of the special theory of relativity. The relativistic transformation between rotating frames, however, does not exist. The situation is rather perplexing, considering the fact that rotational transformation can be routinely done in classical mechanics. After all, the principle of general relativity states that "The laws of physics must be of such nature that they apply to systems of reference in any kind of motion"[1]. If we trust that the universe is isotropic, there is no reason for us to consider a system with certain orientation any better than others.

The study of rotating reference systems played an important part in the origin of general relativity theory[2,3]. Einstein's sample of a rotation roundabout experiment suggested that the new theory should be geometrical, and the geometry had to be non-Euclidean. The rotational motion of the planets of the solar system helped to fix the constants in Einstein's field equation. Since then experiments were designed to test the general theory of relativity. Many of these experiments, such as the experiments to test transverse Doppler effect by Hay et al.[4], the Doppler shift experiment in circular orbit by Chempeney and Moon[5], and the circumnavigating cesium clock experiment by Hafeile and Keating[6], involved rotational motion. In the design and analysis of these experiments, some classical relationships of rotation were taken for granted, for any value of angular momentum throughout the whole space, without much explanation. The rotational motion will continue to play an important role in the experimental tests of relativity, for two reasons. First, the particles accelerated by synchrotrons and the equipment sent into space are all in rotational motion; Second, the rotational motion is the only practical way to maintain sustained acceleration of a non inertial system. It is therefore of both theoretical and practical importance to have a thorough understanding of the rotational behavior of the Einsteinian Space.

It is generally believed that the non-inertial coordinate systems, including rotational systems, are treated in the general theory of relativity. Such treatment has to be derived from Einstein's equation and its solutions. The two well-known ones are the Schwarzschild solution for a non-rotating spherical field [7] and the Kerr solution for a rotating spherical field [8,9]. These solutions give metric tensors or the space-time intervals of the corresponding fields. The space-time intervals, however, do not reveal full information of the coordinate transformation. To see this, let us consider

an example with two inertial systems S and S'. The coordinate transformation is given by the Lorentz transformation:

$$\begin{cases} t = \gamma (t' + (v/c^2)x') \\ x = \gamma (vt' + x') \end{cases} \quad (1)$$

The space-time intervals are

$$dS^2 = (c dt)^2 - dx^2 \quad (2)$$

$$dS'^2 = (c dt')^2 - (dx')^2 \quad (3)$$

Eq. (2) does not uniquely lead to Eq. (1) unless we impose

$$dS^2 = dS'^2 \quad (4)$$

and other postulations.

The fact that the space-time intervals do not reveal full information of the coordinate transformation can also be seen from the classical rotational transformation described in the Euler equations:

$$\begin{cases} x = x' \cos \phi' - y' \sin \phi' \\ y = x' \sin \phi' + y' \cos \phi' \end{cases} \quad (5)$$

Again we have $dS^2 = dx^2 + dy^2 = dx'^2 + dy'^2 = dS'^2$, which is identical to Eq. (2) and (3). From Eqs, (1) and (5) we see that the interval can remain invariant under different transformations. It is clear that although the Kerr solution describes the field of a rotating mass, it alone does not give the rotational transformation between the rotating coordinate systems. In this article we will obtain an explicit rotational transformation consistent with the Schwarzschild solution and the Kerr solution.

2. The rotational transformation

Classically, the transformation between coordinate systems in constant rotation is given by:

$$\begin{cases} t = t' \\ \phi = \omega t' + \phi' \end{cases} \quad (6)$$

or,

$$\begin{cases} dt = dt' \\ d\phi = \omega dt' + d\phi' \end{cases} \quad (7)$$

where ω is the angular velocity of relative rotation. The translational counterpart of Eq. (7) is the Galilean transformation

$$\begin{cases} t = t' \\ x = vt' + x' \end{cases} \quad (8)$$

where v is the linear velocity of relative motion. In relativity, Eq. (8) is replaced by the Lorentz transformation, Eq.(1), and we expect the transformation (6) and (7) to be modified in a similar fashion. Namely, we expect the relativistic rotational transformation to have the general form

$$\begin{cases} t = A t' + B \phi \\ \phi = \alpha (\omega t' + \phi') \end{cases} \quad (9)$$

or,

$$\begin{cases} dt = A dt' + B d\phi' \\ d\phi = \alpha (\omega dt' + d\phi') \end{cases} \quad (10)$$

The constants A , B and α have to be so chosen that the transformation is consistent with the Schwarzschild metric and the Kerr metric.

3. Rotational transformation between the Schwarzschild metric and the Kerr metric

Let us consider the gravitational field of a rotating mass with spherical symmetry depicted in Fig. 1. The coordinate system (x,y,z) , or (r,θ,ϕ) , is fixed with the mass M , which is rotating in the coordinate system (x',y',z') , or (r',θ',ϕ') . The gravitational field of the mass M can be measured in either of the two systems. We will use the more convenient spherical coordinate system and the natural unit system ($\hbar = c = 1$).

The gravitational field of a non-rotating isotropic spherical mass M is given by the Schwarzschild solution

$$\begin{aligned} ds^2 &= (1-2GM/r) dt^2 - (1-2GM/r)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \\ &= g_{\mu\nu} dx^\mu dx^\nu \end{aligned} \quad (11)$$

$$\text{with } dx^\mu = (dt, dr, d\theta, d\phi) \quad (12)$$

and

$$[g_{\mu\nu}] = \begin{pmatrix} 1 - \frac{2GM}{r} & 0 & 0 & 0 \\ 0 & 1 + \frac{2GM}{r} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix} \quad (13)$$

where we have used the approximation $(1 - 2GM/r)^{-1} = 1 + 2GM/r$.

In the system (t', r', θ', ϕ') , the mass M and its rest reference frame (t, r, θ, ϕ) are rotating with an angular velocity ω . The gravitational field surrounding the rotating mass is given by the Kerr solution [8,9]:

$$ds'^2 = dt'^2 - (\rho'^2/\Delta')dr'^2 - \rho'^2 d\theta'^2 - (r'^2 + a^2) \sin^2\theta' d\phi'^2 - (2GMr'/\rho'^2)(dt' - a \sin^2\theta' d\phi')^2 \quad (14)$$

$$\text{with} \quad \rho'^2 = r'^2 + a^2 \cos^2\theta' \quad (15)$$

$$\text{and} \quad \Delta' = r'^2 - 2GMr' + a^2 \quad (16)$$

where a is the angular momentum per unit mass:

$$a = I \omega / M = k \omega \quad (17)$$

$$\text{with} \quad k = I / M \quad (18)$$

and I is the moment of inertia. In the c.g.s. system, $a = I \omega / (Mc) = k\omega/c$. For a homogeneous solid sphere, $k = 3R^2/5$.

The mass is assumed to be the rest mass on the grounds that the rotation is slow and no particle of the mass is moving at relativistic velocity. We will consider the field at large distance where $GM \ll r'$ and the Kerr metric (14) reduces to

$$ds'^2 = (1-2GM/r')dt'^2 - (1+2GM/r')dr'^2 - r'^2 d\theta'^2 - r'^2 \sin^2\theta' d\phi'^2 + (4GMa/r') \sin^2\theta' d\phi' dt' \\ = g'_{\lambda\rho} dx'^{\lambda} dx'^{\rho} \quad (19)$$

$$\text{with} \quad dx'^{\lambda} = (dt', dr', d\theta', d\phi') \quad (20)$$

and

$$[g'_{\lambda\rho}] = \begin{pmatrix} 1 - \frac{2GM}{r'} & 0 & 0 & \frac{2GMa}{r'} \sin^2\theta' \\ 0 & 1 + \frac{2GM}{r'} & 0 & 0 \\ 0 & 0 & -r'^2 & 0 \\ \frac{2GMa}{r'} \sin^2\theta' & 0 & 0 & -r'^2 \sin^2\theta' \end{pmatrix} \quad (21)$$

We now seek a rotational coordinate transformation

$$\begin{pmatrix} t \\ r \\ \theta \\ \phi \end{pmatrix} = \mathbf{A} \begin{pmatrix} r' \\ \theta' \\ \phi' \end{pmatrix} \quad (22)$$

$$\text{or,} \quad x^\mu = A_{\mu\lambda} x'^{\lambda} \quad (23)$$

where $A_{\mu\lambda}$ are the elements of the 4x4 matrix \mathbf{A} , in consistency with Eqs. (13) and (21). Substituting Eq. (23) into Eq. (11) gives the space-time interval in terms of dx'^{λ} :

$$ds^2 = g_{\mu\nu} A_{\mu\lambda} A_{\nu\rho} dx'^{\lambda} dx'^{\rho} = ds'^2 \quad (24)$$

Since the space-time interval is unique in a certain system, Eqs. (19) and (24) have to be the same.

The consistency demands:

$$g'_{\lambda\rho} = g_{\mu\nu} A_{\mu\lambda} A_{\nu\rho} = A^T_{\lambda\mu} g_{\mu\nu} A_{\nu\rho} \quad (25)$$

or in matrix form,

$$[g'_{\lambda\rho}] = \mathbf{A}^T [g_{\mu\nu}] \mathbf{A} \quad (26)$$

where \mathbf{A}^T is the transpose of \mathbf{A} . Note that the r and θ components are identical in both the Schwartzchild metric and the Kerr metric, the transformation for r and θ is simply the identity matrix. This is not surprising at all since we are looking for a transformation of spatial rotation, which should keep r and θ unchanged. We then have $r = r'$ and $\theta = \theta'$. The transformation (26) is reduced to a two dimensional transformation between (t, ϕ) and (t', ϕ') . For the reasons stated in the previous section, this transformation should assume the form of Eqs. (9) or (10). Namely, we look for a 2x2 matrix:

$$\mathbf{A} = \begin{pmatrix} p & q \\ \alpha\omega & \alpha \end{pmatrix} \quad (27)$$

which satisfies Eq.(26) with $\mu, \nu = 0, 3$.

To determine the constants p , q and α , we substitute Eqs. (10), (13) and (21) into Eq. (26) to obtain:

$$g_{00} p^2 = g'_{00} - g_{33} \alpha^2 \omega^2 \quad (28)$$

$$g_{00} p q = g'_{03} - g_{33} \alpha^2 \omega \quad (29)$$

$$g_{00} q^2 = g'_{33} - g_{33} \alpha^2 \quad (30)$$

Solving Eqs. (28), (29) and (30) yields

$$\alpha = \sqrt{\frac{g'_{00} g_{33} - g'^2_{03}}{g_{33}(g'_{33} \omega^2 - 2g'_{03} \omega + g'_{00})}} \quad (31)$$

$$p = \frac{g'_{00} - g'_{03} \omega}{\sqrt{g_{00}(g'_{33} \omega^2 - 2g'_{03} \omega + g'_{00})}} \quad (32)$$

$$q = \frac{g'_{03} - g'_{33} \omega}{\sqrt{g_{00}(g'_{33} \omega^2 - 2g'_{03} \omega + g'_{00})}} \quad (33)$$

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{\alpha}{\Delta} & \frac{-q}{\Delta} \\ -\frac{\alpha\omega}{\Delta} & \frac{p}{\Delta} \end{pmatrix} \quad (34)$$

where $\Delta = \alpha (p - q \omega)$, and

$$\frac{\alpha}{\Delta} = \sqrt{\frac{g_{00}}{g'_{33} \omega^2 + 2g'_{03} \omega + g'_{00}}} \quad (35)$$

$$- \frac{\alpha \omega}{\Delta} = - \omega \sqrt{\frac{g_{00}}{g'_{33} \omega^2 + 2g'_{03} \omega + g'_{00}}} \quad (36)$$

$$- \frac{q}{\Delta} = \frac{\sqrt{g_{33}} (g'_{00} \omega - g'_{03})}{\sqrt{(g'_{33} \omega^2 + 2g'_{03} \omega + g'_{00}) (g'_{00} g'_{33} - g'_{03}{}^2)}} \quad (37)$$

$$\frac{p}{\Delta} = \frac{\sqrt{g_{33}} (g'_{00} - g'_{03} \omega)}{\sqrt{(g'_{33} \omega^2 + 2g'_{03} \omega + g'_{00}) (g'_{00} g'_{33} - g'_{03}{}^2)}} \quad (38)$$

To obtain α , p and q in terms of ω and the coordinates, we substitute into the above expressions the matrix elements of Eqs.(13) and (21). Keeping in mind that $r = r'$, $\theta = \theta'$, we have

$$\alpha = \frac{1}{\sqrt{1 - r^2 \omega^2 \sin^2 \theta}} \quad (39)$$

$$p = \frac{1}{\sqrt{1 - r^2 \omega^2 \sin^2 \theta - 2GM/r}} \quad (40)$$

$$q = \frac{r^2 \omega \sin^2 \theta}{\sqrt{1 - r^2 \omega^2 \sin^2 \theta}} \quad (41)$$

In obtaining Eqs. (39) - (41), we have made the approximations assuming $GM \ll r$, and $(r \omega) \ll r^2$. Note that $(r \omega)$ is not an infinitesimal quantity. In the c.g.s. system, $(r \omega)$ should be replaced by $(r\omega/c) = v/c$. This quantity is close to unity when the linear velocity of the point at large distance is close to the speed of light.

If we consider the case

$$1 - r^2 \omega^2 \sin^2 \theta \gg 2GM/r \quad (42)$$

then Eq.(40) can be approximated to be

$$p = \frac{1}{\sqrt{1 - r^2 \omega^2 \sin^2 \theta}} = \alpha \quad (43)$$

We obtain the transformation matrix

$$\mathbf{A} = \alpha \begin{pmatrix} 1 & r^2 \omega \sin^2 \theta \\ \omega & 1 \end{pmatrix} \quad (44)$$

The complete rotational transformation is

$$\begin{cases} t = \alpha (t' + r^2 \omega \sin^2 \theta \phi') \\ \phi = \alpha (\omega t' + \phi) \\ r = r' \\ \theta = \theta' \end{cases} \quad (45)$$

In the c.g.s. system, the first equation in Eq.(45) should read

$$t = \alpha (t' + (r/c)^2 \omega \sin^2 \theta \phi') \quad (46)$$

and α should read

$$\alpha = \frac{1}{\sqrt{1 - r^2 (\omega/c)^2 \sin^2 \theta}} \quad (47)$$

The inverse transformation is

$$\mathbf{A}^{-1} = \alpha \begin{pmatrix} 1 & -r^2 \omega \sin^2 \theta \\ -\omega & 1 \end{pmatrix} \quad (48)$$

Eq.(48) shows that the inverse transformation can be obtained simply by replacing ω with $-\omega$ in the transformation matrix (44).

4. Discussion

A. Existence of rotational transformation

In order for the transformation to be physically meaningful, α has to be real. We must have

$$r \omega < c \quad (49)$$

This condition is not surprising at all since $(r \omega)$ is the local linear velocity. The condition (49) says that no rotational transformation can be made consistent with the Schwartzchild and Kerr solutions to Einstein's equation outside the spherical surface $r_\omega = c/\omega$. If the gravitational field is strong, the more stringent condition (42) instead of (49) should be imposed.

B. Non uniformity of transformation

Within the domain of rotational transformation $r < r_\omega$, the transformation matrix \mathbf{A} is dependent on the coordinates r and θ , i.e., there does not exist an uniform rotational transformation between (t, ϕ) and (t', ϕ') for all the values of r and θ . On any cylindrical surface coaxial with the Z-axis with certain radius $(r \sin \theta)$, however, the rotational transformation is uniform.

C. Rotation of the Z- axis ($\sin \theta = 0$)

On the rotational axis, $\sin \theta = 0$. The transformation reduces to a trivial one:

$$\begin{cases} t = t' \\ \phi = \omega t' + \phi' \end{cases} \quad (50)$$

with $\alpha = 1$. This is identical to the classical rotational transformation due to the fact that the linear velocity of any point on the Z- axis is zero.

D. Rotation of the equatorial plane ($\sin\theta = 1$)

On the equatorial plane, $\sin\theta = 1$. The transformation (46) becomes

$$\begin{cases} t = \alpha (t' + (r/c)^2 \omega \phi') \\ \phi = \alpha (\omega t' + \phi') \end{cases} \quad (51)$$

with

$$\alpha = \frac{1}{\sqrt{1 - r^2 \omega^2 / c^2}} = \frac{1}{\sqrt{1 - v^2 / c^2}} \quad (52)$$

Since the product $(r \omega)$ is the local linear velocity v , the constant α is the familiar γ factor in the special theory of relativity. Actually, the transformation is locally identical to the Lorentz transformation if $(r \phi)$ is replaced with the linear arc distance x .

E. The rotational time dilation

Let us look at the differential form of the transformation (51):

$$\begin{cases} dt = \alpha (dt' + (r/c)^2 \omega d\phi') \\ d\phi = \alpha (\omega dt' + d\phi') \end{cases} \quad (53)$$

Suppose there is a clock fixed in the (t', r', θ', ϕ') system, $d\phi' = 0$. From Eq.(52) we have

$$dt = \alpha dt' = \frac{dt'}{\sqrt{1 - \left(\frac{r \omega}{c}\right)^2}} \quad (54)$$

which shows that the time dilation due to rotational motion is locally identical to that of special relativity.

F. The angle contraction

Now suppose the observer in the (t', r', θ', ϕ') system measures a certain angular displacement $d\phi'$. This angular displacement is measured by the observer in the (t, r, θ, ϕ) system to be $d\phi$, but it has to be measured simultaneously, namely, dt must equal zero. We therefore have

$$d\phi = \frac{d\phi'}{\alpha} = \sqrt{1 - \left(\frac{r \omega}{c}\right)^2} d\phi' \quad (55)$$

The angle contraction is locally consistent with the Lorentz length contraction.

5. Conclusion

We have obtained a relativistic rotational transformation between the Schwarzschild metric and the Kerr metric for weak fields within the domain $r \omega < c$. Since the Schwarzschild metric and the Kerr metric are the solutions to Einstein's field equation for non-rotating and rotating masses, the rotational transformation thus obtained can be considered a direct consequence of Einstein's field equation. The transformation is local in nature and no global uniform rotational transformation is possible. The two interesting results derived from this transformation are the rotational time dilation and angle contraction that are locally consistent with the time dilation and length contraction of special relativity of a co-moving system with a linear velocity $v = r \omega$.

References

1. A Einstein, "The Foundation of the General Theory of Relativity" in *The Principle of Relativity*, Methuen, London (1923), Dover reprint.
2. L. Marder, *Time and The Space-Traveller*, University of Pennsylvania Press, Philadelphia, 1970.
3. H. Arzelies, *Relativistic Kinematics*, Pergamon, Oxford, 1966.
4. H.J. Hay, J.P. Schiffer, T.E. Cranshaw and P.A. Egelstaff, *Measurement of the Red Shift in an Accelerated System using the Mossbauer Effect in Fe*, *Phys. Rev. Lett.*, **4**, 165 (1960).
5. D.C. Champeney, and P.B. Moon, *Absence of Doppler Shift for Gamma Ray Source and Detector on Same Circular Orbit*, *Proc. Phys. Soc.*, **77**, 350 (1961).
6. J.C. Hafele and R.E. Keating, *Relativistic Time for Terrestrial Circumnavigations*, *Amer. J. Phys.* **40**, 81, (1972).
7. Schwarzschild, *Berl. Ber.*, 1916, p189.
8. R.P. Kerr, *Phys. Rev. Lett.* **11**, 237, (1963).
9. R.H. Boyer and R.W. Lindquist, *J. Math. Phys.* **8**, 265 (1967).

Caption to the Figure

Figure 1. The rotational transformation of coordinates. The un-primed coordinate system is fixed with the spherical mass M that is rotating in the primed system with angular velocity ω . The clocks of the two systems are synchronized when $\phi = \phi'$.

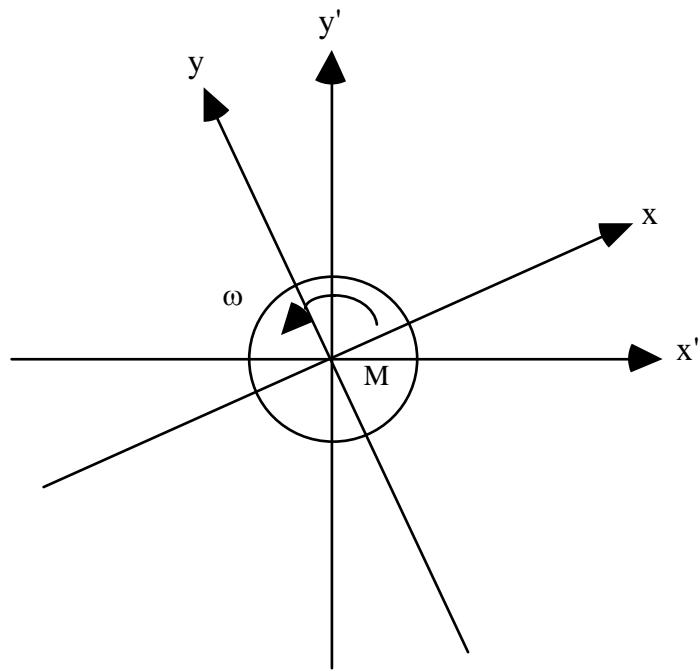


Figure 1