2008 ITQ ALGEBRA/STATISTICS

CURRICULUM MATERIALS

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1 TI-84+ and CBL2 Setup

TI-84+ Set-up

(when you get your calculator)

1. Open the TI-84+ box.
2. Insert the batteries into the TI-84+.

CBL2 Set-up

(when we go to the lab)

1. Open the CBL2 box.
2. Insert the batteries into the CBL2.
3. Take the cover off of the TI-84+.
4. Insert the calculator into the cradle, top end first, with the bottom of the calculator next to the cradle. Then snap the calculator down.
5. Insert the CBL2, head first, into the bottom of the cradle, with the top of the CBL2 next to the cradle. Snap the CBL2 into the cradle.
6. Put one end of the 15 inch cable into the port of the calculator, and the other end into the port of the CBL2.

Installing DATAMATE

1. On the TI-84+ press
2. 2nd [LINK]
3. ► RECEIVE ENTER .
4. Press **TRANSFER** on the CBL2. CBL2 will beep once at the beginning of the transfer, then twice following the successful completion of the transfer. The program or App is transferred and appears in the calculator's program list or application list.
5. When transfer is complete, press 2nd [QUIT] on the calculator.

2 Basic Calculator Notes

MODE menu

NORMAL SCI ENG	Numeric notation.
FLOAT 0123456789	Number of decimal places.
RADIAN DEGREE	Unit of angle measure.
FUNC PAR POL SEQ	Type of graphing.
CONNECTED DOT	Whether to connect graph points.
SEQUENTIAL SIMUL	Whether to plot simultaneously.
REAL a+bi re^θi	Real, rectangular complex, or polar complex.
FULL HORIZ G-T	Full screen, two split-screen modes.

2nd [FORMAT] menu

RectGC PolarGC	Sets cursor coordinates.
CoordOn CoordOff	Sets coordinates display on or off.
GridOff GridOn	Sets grid off or on.
AxesOn AxesOff	Sets axes on or off.
LabelOff LabelOn	Sets axes label off or on.
ExprOn ExprOff	Sets expression display on or off.

WINDOW menu

Xmin	minimum x-value.
Xmax	maximum x-value.
Xscl	distance between thick marks on x-axis.
Ymin	minimum y-value.
Ymax	maximum y-value.
Yscl	distance between thick marks on y-axis.
Xres	sets pixel resolution. If Xres is 1, the function is evaluated at every pixel on the x-axis.

To select (highlight) a setting, press ▼ ▲ ► ◄ to move the cursor to the setting you want.

ZOOM menu

1:ZBox Draws a box to define the viewing window.

1. Move the zoom cursor to any spot you want to define as a corner of the box,
2. ENTER
3. Use ◀ ▶ ▲ ▼ to define the box.
4. ENTER

2:ZoomIn Magnifies the graph around the cursor.

1. Move the zoom cursor to the point that is to be the center of the new viewing window.
2. ENTER

3:ZoomOut Views more of a graph around the cursor. Same procedure as in **2:ZoomIn**.

6:ZStandard Sets **Xmin**=−10, **Xmax**=10, **Ymin**=−10, **Ymax**=10.

9:ZoomStat Redefines the viewing windows so that all statistical data points are displayed.

Entering functions

To enter a function, $y_1 = x^2 + x$. Y= X,T,Θ,n x^2 + X,T,Θ,n

To enter a function, $y_2 = 2x - 2$. Y= ▼ 2 X,T,Θ,n − 2

To select or deselect a function.

1. Y=
2. Use ▼ ▲ to move the cursor to the function you want to select/deselect.
3. ◀ ENTER

Graphing a function

To graph the selected functions. GRAPH

To move the cursor on the Cartesian plane. ◀ ▶ ▲ ▶

To move the cursor along a line. TRACE ◀ ▶

To move the cursor to any valid x-value on the graph in **TRACE** mode. Enter the value, press ENTER.

To move the cursor from one function to another in **TRACE** mode. Press ▲ ▼ .

Calculating a function

To calculate the y-value for a given x.

1. 2nd [CALC]
2. **1:value**
3. Enter a real value for x between **Xmin** and **Xmax**.
4. ENTER . The correspondent y-coordinate on the first selected function is displayed.
5. In case, use ▲ ▼ to move the cursor from function to function.

Table of Values for a function.

1. 2nd [TBLSET]
2. Enter the starting value for x, e.g. $x = -2$.
3. Enter Δx, e.g. $\Delta x = 2$.
4. 2nd [TABLE]

To find the x-intercepts of a function.

1. 2nd [CALC]
2. **2:zero**
3. **Left Bound?** is displayed in the bottom-left corner. Use ▲ ▼ to move the cursor to the desired function.
4. Use ◄ ► (or enter a value) to select a point to the left of the x-intercept. Press ENTER .
5. **Right Bound?** is displayed. Use ► (or enter a value) to select a point to the right of the x-intercept. Press ENTER .
6. **Guess?** is displayed. Use ◄ (or enter a value) to select a point near the zero of the function. Press ENTER .
7. The cursor is on the solution and the coordinates are displayed.

To find a local minimum (resp. maximum) of a function.

1. [2nd] [CALC]
2. **3:minimum** (resp. **4:maximum**)
3. **Left Bound?** is displayed in the bottom-left corner. Use [▲] [▼] to move the cursor to the desired function.
4. Use [◄] [►] (or enter a value) to select a point to the left of the point of local minimum (resp. maximum). Press [ENTER].
5. **Right Bound?** is displayed. Use [►] (or enter a value) to select a point to the right of the point of local minimum (resp. maximum). Press [ENTER].
6. **Guess?** is displayed. Use [◄] (or enter a value) to select a point near the point of local minimum (resp. maximum). Press [ENTER].
7. The cursor is on the point of local minimum (resp. maximum) and the coordinates are displayed.

To find the point of intersection of two functions.

1. [2nd] [CALC]
2. **5:intersect**
3. **First Curve?** is displayed in the bottom-left corner. Use [▲] [▼] to move the cursor to the first function. Press [ENTER].
4. **Second Curve?** is displayed in the bottom-left corner. Use [▲] [▼] to move the cursor to the second function. Press [ENTER]
5. **Guess?** is displayed. Use [◄] [►] (or enter a value) to select a point near the intersection of the two functions. Press [ENTER].
6. The cursor is on the intersection and the coordinates are displayed.

Lists

To enter a list.

1. [STAT]
2. **1:Edit**
3. Use [◄] [►] to select the desired list.
4. After entering each data value, press [ENTER].

To edit a list.

1. [STAT]
2. **1:Edit**
3. Use [◄] [►] to select the list to be edited.
4. Use [▼] [▲] to select the data value to be edited.
5. Edit the data value. Then press [ENTER].

To insert/delete a data value from a list.

1. [STAT]
2. **1:Edit**
3. Use [◄] [►] to select the list to be edited.
4. Use [▼] [▲] to select the position where to delete/insert a data value.
5. Press [DEL] to delete the data value.
6. Press [2nd] [INS] to insert a new data value. Insert the data value, then press [ENTER].

To clear a list.

1. [STAT]
2. **4:ClrList**
3. [2nd] [L1] to clear L_1.
4. [2nd] [L1] [,] [2nd] [L2] to clear both L_1 and L_2.

To clear all the lists:

1. [2nd] [MEM]
2. **4:ClrAllLists**
3. [ENTER]

To find the minimum (resp. maximum) of a list.

1. [MATH] [►] NUM
2. **6:min(** (resp. **7:max(**)
3. Insert the list, e.g. [2nd] [L1] for L_1.
4. [)] [ENTER]

To find mean, variance, etc. of a list.

1. [STAT] [►] CALC
2. **1:1-Var Stats**
3. Insert the list, e.g. [2nd] [L1] for L_1.
4. [ENTER]

To perform Linear, Power, Exponential Regression between two lists, say L_1 and L_2, and store the regression line in Y_1.

1. [STAT] [►] CALC
2. **4:LinReg(a+bx)** for Linear Regression, or
3. **A:PwrReg** for Power Regression, or
4. **0:ExpReg** for Exponential Regression.
5. [2nd] [L1] [,] [2nd] [L2] [,]
6. [VARS] [►] Y-VARS
7. **1:Function**
8. **1:Y1**
9. [ENTER]

Storing a statistical parameter

Example. Store the sample variance Sx in **S**.

1. [VARS]
2. **5:Statistics. . .**
3. **3:Sx**
4. [STO►] [ALPHA] **S**

Example. Store the correlation coefficient r in **R**.

1. [VARS]
2. **5:Statistics. . .**
3. [►] [►] EQ
4. **7:r**
5. [STO►] [ALPHA] **R**

Scatter Plots and Regression Lines

To obtain a scatter plot.

1. [2nd] [STAT PLOT]
2. **1:Plot1**
3. **On**
4. Select the first plot type (scatter plot)
5. Xlist: [2nd] [L1]
6. Ylist: [2nd] [L2]
7. Mark: +
8. [ZOOM]
9. **9:ZoomStat**

Normal Distribution

To compute $P(z_1 < Z < z_2)$

1. [2nd] [DISTR]
2. **2:normalcdf** (z_1, z_2)
3. [ENTER]

To compute $P(x_1 < X < x_2)$ where $X = N(\mu, \sigma)$

1. [2nd] [DISTR]
2. **2:normalcdf** (x_1, x_2, μ, σ)
3. [ENTER]

To compute z_1 so that $P(Z \leqslant z_1) = A$

1. [2nd] [DISTR]
2. **3:invNorm** (A)
3. [ENTER]

To compute x_1 so that $P(X \leqslant x_1) = A$ where $X = N(\mu, \sigma)$

1. [2nd] [DISTR]
2. **3:invNorm** (A, μ, σ)
3. [ENTER]

3 Correlation, Scatter Plots, Linear Regression

A *correlation* is a relationship between two variables. The *independent variable* x is (usually) the variable that can be controlled or manipulated while the *dependent variable* y is the one that cannot be. The determination of the x and y variables is not always clear-cut. The values of the independent and dependent variables can be used to form ordered pairs (x, y). A *scatter plot* is the graph of the ordered pairs (x, y) in which the independent variable x is plotted on the horizontal axis and the dependent variable y is plotted on the vertical axis. The scatter plot is a visual way to describe the relationship between the variables. That is, by inspecting the scatter plot one can determine whether a linear (straight line) correlation exists between the two variables (or an exponential correlation, a quadratic correlation, etc.).

Example. Construct a scatter plot for the sample data below by hand and check your work by constructing it on the calculator.

x	1	2	4	5
y	4	24	8	32

The *(population) correlation coefficient* ρ is a measure of the strength and the direction (positive or negative) of a linear relationship between two variables. The symbol r denotes the *sample correlation coefficient* and is given by

$$r = \frac{n\left(\sum xy\right) - \left(\sum x\right)\left(\sum y\right)}{\sqrt{n\left(\sum x^2\right) - \left(\sum x\right)^2}\sqrt{n\left(\sum y^2\right) - \left(\sum y\right)^2}}$$

To calculate r:

a. Calculate $\sum x$, the sum of the x-values;
b. Calculate $\sum y$, the sum of the y-values;
c. Multiply each x-value by its corresponding y-value and calculate the sum $\sum xy$.
d. Square each x-value and calculate the sum $\sum x^2$, the sum of the squares of the x-values.
e. Square each y-value and calculate the sum $\sum y^2$, the sum of the squares of the y-values.
f. Use (a) - (e) to calculate r.

Example. Calculate r for the sample data above.

Properties of r

1. $-1 \leqslant r \leqslant 1$.
2. If r is close to 1 (resp. -1), there is a significant positive (resp. negative) linear correlation between x and y, i.e., there is strong evidence that the relationship is linear and the line of the best fit has a positive (resp. negative) slope. If r is close to 0, there is no linear relationship between x and y.
3. The value of r does not change if all values of either variable are converted to a different scale.

4. The value of r remains unchanged if we swap the choices of x and y.
5. r only measures the strength of a linear relationship between x and y.
6. The value of r^2 is the proportion of the variation in y that is due to its linear relationship with x. The rest, $1 - r^2$, is due to random variation of the data and other factors not included in the study.

Rule of Thumb

Perfect Correlation	$\Leftrightarrow$	$\lvert r\rvert = 1$
Good Correlation	$\Leftrightarrow$	$0.7 \leqslant \lvert r\rvert < 1$
Fair Correlation	$\Leftrightarrow$	$0.4 \leqslant \lvert r\rvert < 0.7$
Poor Correlation	$\Leftrightarrow$	$0 < \lvert r\rvert < 0.4$
No Correlation	$\Leftrightarrow$	$r = 0$

Example. Determine the percentage of the variation in y that is due to its linear relationship with x.

Given a set of paired sample data (x_1, y_1), (x_2, y_2), ..., (x_n, y_n), the *linear regression equation* is given by

$$\hat{y} = mx + b \tag{1}$$

in which

$$m = \frac{n\left(\sum xy\right) - \left(\sum x\right)\left(\sum y\right)}{n\left(\sum x^2\right) - \left(\sum x\right)^2} \qquad \text{and} \qquad b = \frac{\left(\sum y\right)\left(\sum x^2\right) - \left(\sum x\right)\left(\sum xy\right)}{n\left(\sum x^2\right) - \left(\sum x\right)^2}.$$

The centroid $(\overline{x}, \overline{y})$ in which $\overline{x}$ is the average of the x-values and $\overline{y}$ is the average of the y-values, must lie on the regression line and can be used together with m to calculate b.

The linear regression equation describes the *linear* relationship between x and y and its graph is called the *regression line* (or the *least-squares line* or the *line of best fit*) for the data.

Example. Calculate and graph the regression line for the sample data.

We can use the regression line to predict a value of y corresponding to a given value of x. The farther the x-value is from $\overline{x}$, the less trust we should place in the predicted value. In fact, the regression line should not be used to predict values of y for any values of x that lie outside the limits of the data values of x, i.e., you cannot extrapolate. See the graphs in Fig. 12.12.

Example. Use the regression equation to predict the value of y corresponding to $x = 3$ and $x = 6$.

For a given sample data point (x, y), the residual e is the difference $y - \hat{y}$ between y, the observed y-value, and $\hat{y}$, the value of y that is predicted for x using the regression equation.

Example. Compute the residuals for each data point. What is the sum of the residuals?

The regression equation $\hat{y} = mx + b$ is the linear equation that minimizes the sum of the squares of the residuals, i.e., minimizes $e_1^2 + \cdots + e_n^2 = (y_1 - \hat{y}_1)^2 + \cdots + (y_n - \hat{y}_n)^2$, in which (x_i, y_i) is the ith data point and $\hat{y}_i = mx_i + b$, $i = 1, \ldots, n$.

Example. Compute the sum of the squares of the residuals for the regression line of the example. What do the residuals signify graphically?

Example. Now use the **LinReg(ax+b)** on the TI-84+ to determine the linear regression line, r and r^2. Does this agree with your work?

LINEAR REGRESSION ON THE TI-84+

To clear all the lists:

1. 2nd [MEM]
2. **4:ClrAllLists**
3. ENTER

To clear L_1 and L_2:

1. STAT
2. **4:ClrList**
3. 2nd [L1] , 2nd [L2]

To obtain a scatter plot.

1. 2nd [STAT PLOT]
2. **1:Plot1**
3. **On**
4. Select the first plot type (scatter plot)
5. Xlist: 2nd [L1]
6. Ylist: 2nd [L2]
7. Mark: +
8. ZOOM
9. **9:ZoomStat**

To obtain the regression line.

1. 2nd [CATALOG]
2. **Diagnostic On** (so that you compute r and r^2 also).
3. STAT
4. ▶ CALC
5. **4:LinReg(ax+b)**
6. 2nd [L1] , 2nd [L2] ,
7. VARS
8. ▶ Y-VARS
9. **1:Function**
10. **1:Y1**
11. ENTER

To obtain the graph of the scatter plot together with the regression line.

1. 2nd [STAT PLOT]
2. **1:Plot1**
3. **On**
4. Select the first plot type (scatter plot)
5. Xlist: 2nd [L1]
6. Ylist: 2nd [L2]
7. Mark: +
8. ZOOM
9. **9:ZoomStat**

Class Example. Consider the sample data below.

x	1	3	5	7
y	3	4	6	7

a. Construct a scatter plot for the data.
b. Compute the linear correlation coefficient r.
c. Determine the percentage of the variation in y that is due to its linear relationship with x.
d. Compute and graph the regression line. ANS: $y = 0.7x + 2.2$
e. If possible, use the regression equation to predict the value of y corresponding to $x = 4$.
f. Compute the residuals for each data point.
g. Compute the sum of the squares of the residuals.
h. Check your work by using the linear regression routine on your TI84+.

Class Example. We are interested in using data to predict a student's GPA at the end of the freshman year based on his or her high school GPA.

Student	High School GPA x	Freshmen GPA y
1	2.00	1.60
2	2.25	2.00
3	2.60	1.80
4	2.65	2.80
5	2.80	2.10
6	3.10	2.00
7	2.90	2.65
8	3.25	2.25
9	3.30	2.60
10	3.60	3.00
11	3.25	3.10

a. Detaermine the percent of variation in y that is due to its linear relationship with x.

b. Determine the regression line for the data.

c. Graph the scatterplot and the regression line.

d. If possible, predict the freshman GPA of a student whose high school GPA is 3.00 (2.00).

Remark: You must turn stat plots off before trying to graph ordinary functions, else you will get a dimension error.

To turn off stat plots:

1. 2nd [STAT PLOT]
2. **1:Plot1**
3. **Off** (repeat using arrows for any other stat plots)
4. ENTER

Correlation and Causation The fact that two variables are strongly correlated does not in itself imply a cause-and-effect relationship between the variables. To decide, one should consider the following questions:

1. Is there a direct cause-and-effect relationship between the variables?
2. Is there a reverse cause-and-effect relationship between the variables?
3. Is it possible that the relationship between the variables can be caused by a third variable or perhaps a combination of several other variables?
4. Is it possible that the relationship between two variables may be a coincidence?

4 Linear Cost/Revenue/Profit Models

Let x denote the number of units of a product manufactured or sold. The *total production cost function* $C(x)$ gives the total cost of manufacturing x units of the product.

Suppose a firm has a fixed *set-up cost* of F dollars, and a *production cost* of c dollars per unit, then the total production cost function is given by

$$C(x) = cx + F.$$

Example. Compudisk, a computer disk manufacturer, sells each disk it produces (so that there is never any inventory). Thus, if x is the number produced, x also equals the number sold. There is a (set-up) cost of \$25 and each disk costs \$5 to produce.

a. The total production cost for producing x disks is given by the model $y = C(x) = \dots$
b. Make a rough graph of this equation.
c. How does the set-up cost relate to the graph of the equation?
d. How does unit cost relate to the graph of the equation?
e. What is the production cost if 20 disks are produced?
f. How many disks were produced if the total cost is \$175?
g. If Compudisks budget limits it to a total production cost of \$500, what is the maximum number of disks it can produce?

The *revenue function* $R(x)$ is the total revenue realized from the sale of x units of the product. If the selling price per unit (*unit price*) is p, the the revenue function is given by

$$R(x) = px.$$

Example. Consider the previous example, and suppose Compudisk sells each disk for \$10 each.

h. The revenue function is ...
i. Calculate the revenue if the company produces and sells 15 disks.
j. How many disks were sold if the total revenue is \$450?

The *profit* equals the revenue minus the cost, so that the *profit function* is given by

$$P(x) = R(x) - C(x) = (p - c)x - F.$$

Example. With regard to the previous examples,

k. Determine the profit $y = P(x)$.
l. Determine the profit from 10 disks.
m. How many disks were produced and sold if the profit is \$500?

The *break-even point* is the level of production where the revenue equals the total production cost, that is, where the profit is equal to zero. To find the break-even point, we must solve the equation $P(x) = 0$.

Example. With regard to the previos examples,

n. Graph the cost and revenue equations on the same set of axes.
o. What does the break-even point mean geometrically (graphically) in terms of the cost and revenue equations?
p. Determine the break-even point algebraically.
q. Give the intervals for x when there is a (positive) profit and the intervals where there is a loss (negative profit).

Homework. A company producing computers has total production costs given by $y = C(x) = 50x + 40,000$ dollars and total revenue given by $y = R(x) = 850x$ dollars where x is the number produced and sold.

a. What does 50 represent?
b. What does $40,000$ represent?
c. What does 850 represent?
d. Give the profit function.
e. Use the profit function to determine the production level at which the company breaks even.
f. Give a table of values for the profit for production levels of 10, 20, 30, 40, 50.

Homework. A manufacturer has a monthly fixed cost of \$40,000 and a production cost of \$8 for each unit produced. The product sells for \$12/unit.

a. What is the cost function?
b. What is the revenue function?
c. What is the profit function?
d. Compute the profit (loss) corresponding to production levels of 8000 and $12,000$ units.
e. Determine the break-even point of production.
f. Determine the intervals where the production level yields a profit or a loss.

Homework. Auto-Time, a manufacturer of 24-hour variable timers, has a monthly fixed cost of \$48,000 and a production cost of \$8 for each timer manufactured. The timers sell for \$14 each.

a. What is the cost function?
b. What is the revenue function?
c. What is the profit function?

d. Compute the profit (loss) corresponding to production levels of 4000, 6000, and 10,000 timers, respectively.

e. Determine the break-even level of production.

f. Graph the cost and revenue functions and give the intervals where the production level yields a profit or a loss.

Definition. A function that is defined by more than one rule is called a *piece-wise function*.

An example of piecewise function is

$$f(x) = \begin{cases} 2x - 1 & 0 \leqslant x \leqslant 3 \\ x + 2 & 3 < x \leqslant 7. \end{cases}$$

In Cost/Revenue/Profit Models, piece-wise functions are used when the cost per unit (or the selling price per unit) varies depending on the amount x of units produced/sold.

Example. You decide to buy a salad for lunch. The cost is \$3.00 for the first 4 ounces or less, plus 50 cents an ounce for anything above 4 ounces.

a. Give the function $C(x)$ representing the cost of x ounces of salad where $x > 0$.

b. Graph $C(x)$ by hand.

Example. You park downtown where the daily rates are \$2 for the first hour or fraction thereof, plus \$1.20 per hour for anything above 1 hour up to a maximum daily cost of \$8. (you have to be out after 24 hours.)

a. Give the function $C(x)$ which represents the daily cost of x hours of parking.

b. Graph $C(x)$.

Homework. A steel company charges \$5 per pound for the first 2000 pounds ordered, \$4 per pound for everything over 2000 pounds but not more than 5000 pounds and \$3 for everything over 5000 pounds.

a. Give the function $C(x)$ representing the charge for x pounds of steel ordered.

b. Graph $C(x)$.

5 Average Rate of Change

Let $y = f(x)$. The *average rate of change* of f on the interval $[a, b]$ is $\dfrac{f(b) - f(a)}{b - a}$.

Example. Let $f(x) = x^2 - 2x - 15$.

a. Find the average rate of change of f on the interval $[-4, -1]$.
b. Find the average rate of change of f on the interval $[2, 6]$.

Graphically, what does the average rate of change on $[a, b]$ represent?

Homework. Find the average rate of change of $f(x) = x^2 - x$ on

a. $[1, 3]$;
b. $[-5, -1]$;
c. $[2, 6]$.

If $h(t)$ describes the position of an object as a function of the time t, the average rate of change of h on the time interval $[t_0, t_1]$ is called the *average velocity on the interval* $[t_0, t_1]$.

Example. Let $h(t) = -16t^2 + 64t + 128$.

a. Find the average velocity between $t = 0$ and $t = 2$.
b. Find the average velocity between $t = 2$ and $t = 4$.

As the width of the time interval $[t_0, t_1]$ decreases to zero, the average velocity will approach the so-called *instantaneous velocity*.

Example. In the previous example,

a. Find the average velocity between $t = 2.5$ and $t = 3.5$.
b. Find the average velocity between $t = 2.9$ and $t = 3.1$.
c. Find the average velocity between $t = 2.99$ and $t = 3.01$.

6 Quadratic Cost/Revenue/Profit Models

Typically, as the unit price p increases, the demand x decreases, due to the fact that fewer people are willing to buy at a higher price. Thus, the unit price for a product does not remain constant and is related to the demand (the number sold).

Example. Suppose $p = 120 - 6x$ or $x = 20 - \frac{p}{6}$. Then, each \$6 increase in price brings about a 1-unit decrease in demand.

Example. The research department in a small company that manufactures VCRs has established that the unit cost of each VCR is \$40, the fixed cost is \$1100, and the demand equation is $p = 200 - 5x$ dollars where p is the unit price and x is the number sold, $0 \leqslant x \leqslant 40$.

a. Determine the revenue function, the cost function, and the profit function.

b. Using your calculator, graph the cost and revenue functions on the same set of axes.

c. Use the cost and revenue functions to determine the break-even points algebraically and check your answer on your calculator.

d. Using parts (b) and (c), determine the production levels where you sustain a loss and where you make a profit.

e. Determine the output that produces the maximum revenue, find the maximum revenue, and find the optimal price in regard to revenue (the price at which you attain maximum revenue).

f. Graph the profit function on the same set of axes. What do the break-even production levels represent in terms of this graph?

g. Determine the output that produces maximum profit, find the maximum profit, and find the optimal price in regard to profit (the price at which you obtain maximum profit).

h. Determine the average rate of change of revenue with respect to the production level x, and the average rate of change of profit with respect to the production level x as the production is increased from 5 units to 10 units.

Homework. A company produces large screen TVs and it has been determined that the unit cost is \$500, the fixed cost is \$14,000, and the demand equation is $p = 1400 - 10x$ where p is in dollars and x is the demand, $0 \leqslant x \leqslant 140$.

a. Determine the revenue function, the cost function, and the profit function.

b. Using your calculator, graph the cost and revenue functions on the same set of axes.

c. Use the cost and revenue functions to determine the break-even points algebraically and check your answer on your calculator.

d. Using parts (b) and (c), determine the production levels where you sustain a loss and where you make a profit.

e. Determine the output that produces the maximum revenue, find the maximum revenue, and find the optimal price in regard to revenue (the price at which you attain maximum revenue).

f. Graph the profit function on the same set of axes. What do the break-even production levels represent in terms of this graph?

g. Determine the output that produces maximum profit, find the maximum profit, and find the optimal price in regard to profit (the price at which you obtain maximum profit).

h. Determine the average rate of change of revenue with respect to the production level x, and the average rate of change of profit with respect to the production level x as the production is increased from 10 units to 20 units.

Homework. A company produces mountain bicycles which have a fixed cost of \$36,000 and a unit cost of \$600. The demand equation is $p = 2400 - 20x$ where p is in dollars and x is the number sold, $0 \leqslant x \leqslant 120$.

a. Determine the revenue function, the cost function, and the profit function.

b. Using your calculator, graph the cost and revenue functions on the same set of axes.

c. Use the cost and revenue functions to determine the break-even points algebraically and check your answer on your calculator.

d. Using parts (b) and (c), determine the production levels where you sustain a loss and where you make a profit.

e. Determine the output that produces the maximum revenue, find the maximum revenue, and find the optimal price in regard to revenue (the price at which you attain maximum revenue).

f. Graph the profit function on the same set of axes. What do the break-even production levels represent in terms of this graph?

g. Determine the output that produces maximum profit, find the maximum profit, and find the optimal price in regard to profit (the price at which you obtain maximum profit).

h. Determine the average rate of change of revenue with respect to the production level x, and the average rate of change of profit with respect to the production level x, as the production is increased from 20 units to 30 units.

7 Basic Concepts of Probability

Larson and Farber: Sec. 3.1-3.4

Exploring Mathematics with the Probability Simulation Application: Activity 1

SAMPLE SPACES AND PROBABILITY

Definitions:

- *Probability* is the chance of an event occurring.
- A *probability experiment* is a chance process that leads to well-defined results.
- An *outcome* is the result of a single trial of a probability experiment.
- The *sample space* of a probability experiment is the set of all possible outcomes.
- A *tree diagram* is a schematic which is used to determine the sample space of a probability experiment.
  Draw tree diagrams for the examples below.
- An *event* is any collection of outcomes or simply a subset of the sample space. A *simple event* is one consisting of a single outcome. A *compound event* is any event consisting of 2 or more outcomes.

Examples:

1. Flip a coin.
2. Roll a die.
3. Draw a card from a standard deck.
4. Roll a pair of dice.

Notation:

- P denotes probability.
- A, B, C, ... denote events.
- $P(A)$ = probability of event A.

Equally likely events are events that have the same probability of occurring.

1. CLASSICAL PROBABILITY

In classical probability we make the assumption that a procedure has n possible outcomes, each of them equally likely. If this assumption does not hold, the probabilities determined by the classical interpretation of probability will not be accurate. The probability of an event E under the classical interpretation of probability is computed by taking the ratio of the number n_E of outcomes favorable to E and the total number of outcomes n. That is,

$$P(E) = \frac{n_E}{n}.$$

Roundoff Rule: When expressing the value of a probability, either give the exact fraction or round off to 3 significant digits (ignoring leading zeros).

Examples of Classical Probability:

1. A jar contains 3 red balls, 5 blue balls, 7 yellow balls, and 9 black balls. You draw a ball at random. What is the probability you get
   a. a red ball.
   b. a non-blue ball.
   c. a red ball or a yellow ball.
   d. Homework: a yellow ball.
   e. Homework: a non-red ball.
   f. Homework: a blue ball or a red ball.
2. Card Problems: a standard deck of cards has 52 cards, divided into 4 suits: Spades and clubs (both black) and hearts and diamonds (both red). Thus, each of the four suits has 13 cards which consist of aces, face cards (kings, queens, and jacks) and spot cards (2's, 3's, . . . , 9's, and 10's). You draw a card from a standard deck; what is the probability it is
   a. a face card?
   b. an ace?
   c. a spot card?
   d. a non-face card?
   e. Homework: a spot card between 5 and 9, inclusive?
   f. Homework: a non-ace?
   g. Homework: a non-spot card?
3. Dice problems: a single die is a cube and has 6 sides numbered 1 to 6. Assuming the die is fair, each side is equally likely to turn up. You roll a die. What is the probability you get
   a. a 3?
   b. an even number?
   c. Homework: a 5?
   d. Homework: an odd number?
4. You roll a pair of dice. This can be considered a multi-stage event and a tree diagram can be used to determine the sample space; then you can list them in a table. You roll a pair of dice. What is the probability that you get a sum
   a. of 7?
   b. of 10?
   c. less than 6?
   d. that is even?
   e. Homework: of 3?
   f. Homework: of 6?
   g. Homework: greater than 7?
   h. Homework: that is odd?

2. EMPIRICAL PROBABILITY
the Relative Frequency Approach to Probability:

In empirical probability we repeat the experiment a large number of times, observe the frequency of an event, and use this frequency to determine the probability. If a procedure is repeated a large number of times and event E occurs $p\%$ of the time, then $p\%$ should be a good approximation to the actual probability of event E.

Symbolically, if a procedure is conducted n different times and if event E occurs on f of these trials, then the probability $P(E)$ of event E is approximated by

$$P(E) \approx \frac{f}{n}.$$

We say approximately because we think of the actual probability $P(E)$ as being the relative frequency of the occurrence of event E over a very large number of observations or repetitions of phenomenon. The fact that we can check probabilities that have a relative frequency interpretation (by simulating many repetitions of the procedure) makes this interpretation very appealing.

Law of Large Numbers: As a procedure is repeated again and again, the empirical (relative frequency) probability of an event E approaches the actual (classical) probability.

3. SUBJECTIVE PROBABILITIES:

The probability of an event is found by simply guessing or by estimating its value based upon knowledge of the relevant circumstances.

- Probability Rule 1: The probability of any event E is a number (either a fraction or decimal) between and including 0 and 1. This is denoted by $0 \leqslant P(E) \leqslant 1$.
- Probability Rule 2: If an event E cannot occur (i.e., the event contains no members in the sample space), its probability is 0.
- Probability Rule 3: If an event E is certain to happen, then the probability of E is 1.
- Probability Rule 4: The sum of the probabilities of the outcomes in the sample space is 1.

The *complement of an event* A, denoted by $\overline{A}$, is the set of all outcomes in which A does not occur.

Class Examples:

1. If a family has three children, find the probability that all the children are girls.
2. If the probability that a person lives in an industrialized country of the world is 1/5, find the probability that a person does not live in an industrialized country. You may use the fact that $P(A) = 1 - P(\overline{A})$.
3. Consider a company that selects employees for random drug tests. The company uses a computer to randomly select an employee number that ranges from 1 to 6296. Find the probability of selecting a number that is not divisible by 1000.
4. Hospital records indicated that maternity patients stayed in the hospital for the number of days shown in the distribution.

Number of days stayed	Frequency
3	15
4	23
5	56
6	19
7	5

Find the probability that

a. a patient stayed at most 4 days;
b. a patient stayed at most 5 days.

8 Exponential functions and Logarithms

Definition. $f(x) = b^x$, where $b > 0$, $b \neq 1$ is called an *exponential function with base b and exponent x*. Its domain is the set of real numbers.

Laws of exponents. Let b, c be positive numbers and x, y be real numbers.

1. $b^x b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$
4. $(bc)^x = b^x c^x$
5. $\left(\dfrac{b}{c}\right)^x = \dfrac{b^x}{c^x}$

Definition. Let $b > 0$, $b \neq 1$. The *logarithm to the base b of x*, denoted by $\log_b x$, is the power to which b must be raised to produce x. Therefore

$$y = \log_b x \qquad \Leftrightarrow \qquad b^y = x.$$

Thus, we see that a logarithm is an exponent (this is the basis for converting log expressions to exponential expressions and vice versa). Also, this implies that logs are defined for positive values of x only.

$\log_{10} x$ is called the *common logarithm* of x, and is denoted by $\log x$.

$\log_e x$ is called the *natural logarithm* of x, and is denoted by $\ln x$.

Laws of Logarithms. Let b, r, s be positive numbers, with $b \neq 1$,

1. $\log_b rs = \log_b r + \log_b s$
2. $\log_b \dfrac{r}{s} = \log_b r \log_b s$
3. $\log_b r^s = s \log_b r$
4. $\log_b 1 = 0$
5. $\log_b b = 1$
6. $\log_b b^s = s$

Examples. Solve for x.

a. $\log_3 x = 4$
b. $\log 4 = x$
c. $\log_x 8 = 3$
d. $2e^{x+2} = 5$
e. $5 \ln x + 3 = 0$
f. $e^{0.4x} = 8$
g. $2e^{-0.2x} - 4 = 6$

9 Exponential Growth/Decay

In the physical and natural sciences it is often observed the rate of growth (rate of decay) is proportional to the number or amount y present at any time t. That is,

$$\text{Growth Equation:} \quad \frac{dy}{dt} = ky, \quad y(0) = y_0$$
$$\text{Decay Equation:} \quad \frac{dy}{dt} = -ky, \quad y(0) = y_0$$

where $k > 0$ and y_0 is the initial amount present. Solving, we obtain the (exponential) growth equation

$$y = y_0 e^{kt}$$

or the (exponential) decay equation

$$y = y_0 e^{-kt}.$$

Radioactive substances decay exponentially according to the decay equation $y = A(t) = A_0 e^{-kt}$ where A_0 is the initial amount present and $k > 0$ is the decay constant. The half-life of a radioactive substance is the time required for the given amount to be reduced by one-half.

The amount of Carbon-14 (C-14), a radioactive isotope of carbon, in a living organism is constant. But, when it dies it stops absorbing new quantities of C-14 and the amount of C-14 in the remains diminishes due to the natural decay of the radioactive substance. So the approximate age of a fossil can be determined by measuring the amount of C-14 it contains at present.

Example. C-14 has a half-life of 5770 years.

a. What is its decay constant?
b. If pottery from an archeological dig has one-fifth the amount of C-14 that it originally contained, approximately how old is the pottery?
c. If a skull is $15,000$ years old, what percentage of its original C-14 remains?
d. If there was 50g of C-14 present after 8000 years, how much was present initially?

Example. The half-life of radium is 1600 years. Assume there was 200 milligrams present originally in an organism.

a. Determine its decay constant.
b. When will there be 40 milligrams left?
c. How much will be left after 800 years?

Homework. Logs recovered from an archaeological dig contain 30% of the C-14 they originally contained. How long ago did the trees die?

Homework. The skeletal remains of the Pittsburgh Man were discovered in Pennsylvania. It was determined that the remains had lost 82% of the C-14 they originally contained. How old were the bones?

Homework. Phosphorus 32 has a half-life of 14.2 days.

a. If 500g of this substance was present in a material initially, what is the amount present after t
b. What is the amount present after a week?
c. How long will it be before the material has lost 70% of its Phosphorus 32?

Homework. The radioactive element polonium has a decay constant of 0.00495 where time t is measured in days.

a. Determine its half-life.
b. If there were 280g present after 20 days, how much is present initially?
c. How much will be present after 100 days?

Homework. Strontium 90 is a radioactive isotope of strontium and its decay constant is -0.0257, where the time t is measured in years.

a. Determine the half-life of Strontium 90.
b. What percentage of the original amount of Strontium 90 will be left after 10 years?
c. How long will it take for 80% of the Strontium 90 to decay?

Homework. As altitude increases, pressure decreases. Specifically, the atmospheric pressure y in millibars (mb) at a given altitude of x meters is given by $y = p(x) = Ce^{-kx}$, where $k > 0$.

a. The atmospheric pressure at 0 meters is 1013mb and at $10,000$ meters is 265mb. Find the atmospheric pressure $p(x)$ at an altitude of x meters.
b. Find the atmospheric pressure at $20,000$ meters.
c. If the atmospheric pressure is 55mb, what is the altitude?

10 Exponential Regression

Suppose that we wish to find the best exponential fit (rather than the best linear fit)

$$\hat{y} = Ce^{kx}, \text{ where } C = y(0) = y_0 > 0. \tag{2}$$

for a set of paired sample data $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in which $y_i > 0$. Taking natural logs, the equation (2) can be transformed into the linear equation

$$\hat{Y} = kx + b \tag{3}$$

in which $\hat{Y} = \ln \hat{y}$ and $b = \ln C$. Thus, the paired data has an exponential fit (2) if and only if the transformed paired data $(x_i, \ln y_i)$, $i = 1, \ldots, n$, has a linear fit (3). The equation (2) will then be the exponential fit that minimizes the sum of the squares of the residuals $r = \ln y - \ln \hat{y}$. The correlation coefficient r is computed from the transformed data. r^2 is the percentage of variation in y that is due to its exponential relationship with x.

Example. For the data

x	-1	2	5
y	e^2	e^4	e^8

find the best exponential fit $\hat{y} = Ce^{kx}$ (on your calculator) by first finding the best linear fit for the transformed data. Then determine the percentage of variation in y that is due to its exponential relationship with x. Also, find the best exponential fit and graph the scatter plot along with the best exponential fit using **0:ExpReg** on your calculator. Compare the two r^2 values.

Example. Repeat for

x	-1	1	3	4
y	5	4	2	1

Class Example. Repeat for

x	-3	-1	0	2
y	e^4	e^2	e	e^{-2}

Class Example. Repeat for

x	-4	-1	1	3
y	1	2	5	10

Remark. If we seek a best fit of the form $\hat{y} = ap^x$, note that this is equivalent to an exponential fit since $p^x = (e^{\ln p})^x$.

11 Geometry with Determinants

AREAS AND DISTANCES WITH DETERMINANTS

The area A of the triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is given by

$$A = \frac{1}{2}\left|\det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}\right|.$$

This formula can be used to answer the following questions:

- How would you write a formula for the area of the parallelogram with (x_1, y_1), (x_2, y_2), and (x_3, y_3) as three of its four vertices?
- How would you check the collinearity of the three points (x_1, y_1), (x_2, y_2), and (x_3, y_3)?
- How would you find the area of a quadrilateral?
- How would you find the (perpendicular) distance from a point to straight line?
- How would you find the distance between two parallel straight lines?

Examples.

1. Find the area of the triangle with the given vertices.

   a. $(-1, -1)$, $(6, 6)$, and $(7, 5)$;
   b. $(3, -4)$, $(5, 10)$, and $(-9, 12)$.

2. Find the area of the parallelogram with the three given points as three of its four vertices.

   a. $(-1, 0)$, $(6, 5)$, and $(-2, 4)$;
   b. $(2, -3)$, $(5, -1)$, and $(-3, 10)$.

3. For the given four points A, B, C, and D, find the area of the quadrilateral $ABCD$.

   a. $A(3, 1)$, $B(5, 1)$, $C(4, 5)$, and $D(1, 6)$;
   b. $A(-2, 1)$, $B(4, -1)$, $C(-3, -5)$, and $D(-6, 0)$.

4. Find the distance from the given point to the given straight line.

   a. $(-1, 2)$; $3x - 4y = 24$;
   b. $(2, -5)$; $12y = 5x - 2$.

5. Find the distance between the given two parallel straight lines.

   a. $3x - 4y = 23$ and $3x - 4y = 3$;
   b. $2y = 5x - 3$ and $4y = 10x + 5$.

12 Curve Fitting with Determinants [1]

1. Write the equation $Ax+By+C=0 \Leftrightarrow x(A)+y(B)+1(C)=0$ of the unique straight line passing through the two points $P(x_1,y_1)$ and $Q(x_2,y_2)$.

$$\det\begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0.$$

2. Write the (determinantal) equation $Ay+Bx^2+Cx+D=0 \Leftrightarrow$ ________________ of the vertical parabola that fits the three points $P(x_1,y_1)$, $Q(x_2,y_2)$, and $R(x_3,y_3)$.

$$\det\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} = 0.$$

3. Write the (determinantal) equation $Ax^2+Ay^2+Bx+Cy+D=0 \Leftrightarrow$ ________________ of the circle that contains the three points $P(x_1,y_1)$, $Q(x_2,y_2)$, and $R(x_3,y_3)$.

$$\det\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} = 0.$$

Examples.

1. Find the equation of the straight line that passes through the two given points.
   a. $(1,1)$ and $(3,7)$;
   b. $(-1,11)$ and $(2,-10)$.
2. Find the equation of the vertical parabola that passes through the three given points.
   a. $(0,3)$, $(1,1)$, and $(2,-5)$;
   b. $(-1,1)$, $(1,5)$, and $(2,16)$.
3. Find the equation of the circle that passes through the three given points.
   a. $(-1,-1)$, $(6,6)$, and $(7,5)$;
   b. $(3,-4)$, $(5,10)$, and $(-9,12)$.

[1] Most of the problems in this section are taken (with minor changes, if any) from: Elementary Linear Algebra, C. H. Edwards, Jr. and David E. Penny.

13 Free-Falling Objects

An object is thrown upward from an initial height h_0 with an initial velocity of v_0 feet/sec. (We assume the positive direction is away from the Earth and that the acceleration due to gravity is $a(t) = 32$ft/sec/sec.) Then, for this free-falling body, the position equation (equation of motion) and the velocity equation are

- Equation of Motion: $h(t) = -16t^2 + v_0t + h_0, \quad t \geqslant 0;$
- Velocity Equation: $v(t) = -32t + v_0, \quad t \geqslant 0.$

Key Questions:

1. What is the maximum height?
2. What is the terminal velocity (velocity upon impact)?
3. What is the average velocity over a given time interval?
4. How long does it take for the object to hit the ground?

Example. A ball is thrown upward from the top of a 160-ft high building with an initial velocity of 48 feet per second.

a. Give the velocity equation.
b. Give the position equation.
c. Find the apex.
d. Find the velocity when it strikes the ground.
e. Find the average velocity in the first 2 seconds of its downward flight.
f. Find the average velocity in the last 3 seconds of flight.
g. How long does it take for the ball to hit the ground?

Homework. A ball is thrown upward from the top of a 960-ft high building with an initial velocity of 112 feet per second.

a. Give the velocity equation.
b. Give the position equation.
c. Find the apex.
d. Find the velocity when it strikes the ground.
e. Find the average velocity in the first 2 seconds of flight.
f. Find the average velocity in the last 3 seconds of flight.
g. How long does it take for the ball to hit the ground?

Homework. A ball is thrown upward from the top of a 1280-ft high building with an initial velocity of 32 feet per second.

a. Give the velocity equation.
b. Give the position equation.
c. Find the apex.
d. Find the velocity when it strikes the ground.
e. Find the average velocity in the first 2 seconds of its downward flight.
f. Find the average velocity in the last 3 seconds of flight.
g. How long does it take for the ball to hit the ground?

14 Power Regression

Suppose that we wish to find the best power fit

$$\hat{y} = ax^c, \qquad a > 0 \tag{4}$$

for a set of paired sample data (x_1, y_1), (x_2, y_2), …, (x_n, y_n) in which $x_i, y_i > 0$, $i = 1, \ldots, n$. Taking natural logarithms, the equation (4) can be transformed into the linear equation

$$\hat{Y} = cX + b \tag{5}$$

in which $\hat{Y} = \ln \hat{y}$, $X = \ln x$, and $b = \ln a$. Thus, the paired data has a power fit (4) if and only if the transformed paired data $(\ln x_i, \ln y_i)$, $i = 1, \ldots, n$, has a linear fit (5). The correlation coefficient r is computed from the transformed data.

Example. Use linear regression to find the best power fit $\hat{y} = ax^c$ for the data

x	2	4	5
y	50	750	1800

Check your answer by using **A:PwrReg** to find the best power fit.

Example. Repeat for

x	1	3	4	7
y	4	0.5	0.25	0.08

Class Example. Repeat for

x	0.1	2	5	7
y	450	1	0.2	0.1

15 The Addition Rule

Two events A and B are *mutually exclusive*, (or *disjoint*) if (when the procedure is performed a single time) the occurrence of one of the events precludes the possibility of the occurrence of the other. That is, they cannot occur simultaneously.

Addition Rule 1: (The Addition Rule for Mutually Exclusive Events)
If two events A and B are mutually exclusive, then the probability that either event will occur is

$$P(A \text{ or } B) = P(A \text{ occurs or } B \text{ occurs}) = P(A) + P(B).$$

The Addition Rule generalizes for any finite number of pairwise mutually exclusive events.

Examples:

1. You draw a card from a standard deck.
   a. What is the probability it is a king or a spot card?
   b. Homework: a club or a red card? Ans. 3/4.
2. A jar contains 4 red balls, 3 blue balls, 5 yellow balls, and 2 black balls. You draw a ball at random.
   a. What is the probability you get a red ball or a yellow ball.
   b. Homework: a blue ball or a red ball? Ans. 1/2.
3. You roll a pair of dice.
   a. What is the probability that you get a total of 7 or a total $geqslant 10$?
   b. Homework: a total of 9 or a total $\leqslant 6$? Ans. 7/18.

Addition Rule 2: (The General Addition Rule)
If A and B are two events, then the probability that A or B occurs is

$$P(A \,\text{or}\, B) = P(A \text{ occurs or } B \text{ occurs or both occur}) = P(A) + P(B) - P(A \,\text{and}\, B)$$

where $P(A \,\text{and}\, B)$ denotes the probability that A and B occur simultaneously.

Observe that the second rule reduces to the first when A and B are mutually exclusive.

Examples:

1. You draw a card from a standard deck. What is the probability that
   a. it is a face card or a spot card?
   b. it is red or a face card?

   Homework:
   c. it is black or a non-face card? Ans. 23/26.
   d. it is an ace or a spot card? Ans. 10/13.
2. You roll a pair of dice. What is the probability you get a total

a. of 5 or 9?
b. of 7 or an even total?
c. less than 5 or an even total?

Homework:
d. of 3 or 4? Ans. 5/36.
e. of 4 or an odd total? Ans. 7/12.
f. less than 10 or an odd total? Ans. 8/9.
g. less than 9 or an even total? Ans. 5/6.

3. a. Given $P(A) = 2/5$, $P(B) = 1/2$, and $P(A \,\text{or}\, B) = 4/5$, $P(A \,\text{and}\, B) = ?$
b. Given $P(A) = 3/5$, $P(B) = 2/5$, and $P(A \,\text{and}\, B) = 1/5$, $P(A \,\text{or}\, B) = ?$

Homework:
c. Given $P(A) = 1/3$, $P(B) = 1/2$, and $P(A \,\text{or}\, B) = 2/3$, $P(A \,\text{and}\, B) = ?$ Ans. 1/6
d. Given $P(A) = 2/3, P(B) = 2/5$, and $P(A \,\text{and}\, B) = 1/4$, $P(A \,\text{or}\, B) = ?$ Ans. 49/60

Application of the (Simple) Addition Rule: The Probability of $\overline{A}$:

Recall that $\overline{A}$ is the event that A does not occur.

- $P(A \,\text{or}\, \overline{A}) = 1$ (why?)
- $P(A) + P(\overline{A}) = 1$ (why?)

So $P(\overline{A}) = 1 - P(A)$ or, equivalently, $P(A) = 1 - P(\overline{A})$.

Example: If $P(A) = 3/11$, what is $P(\overline{A})$? If $P(\overline{A}) = 7/16$, what is $P(A)$?

Class Example:

1. You draw a card from a standard deck. What is the probability

   a. it is a non-face card?
   b. Homework: it is a non-spot card?

2. You roll a pair of dice. What is the probability

   a. you do not get doubles?
   b. you do not get a total of 4?
   c. Homework: you do not get a total $\geqslant 11$? Ans. 11/12.
   d. Homework: you do not get a total divisible by 4? Ans. 3/4.

16 The Multiplication Rule; Conditional Probability

Two events A and B are *independent* if the occurrence of one does not affect the probability of the occurrence of the other. Otherwise, we say A and B are *dependent*. An example of two independent events is the roll of a pair of dice or successive rolls of a die.

Multiplication Rule 1: (The Multiplication Rule For Independent Events)

If A and B are independent events, then

$$P(A \,\text{and}\, B) = P(A)P(B).$$

In general, the probability of any sequence of independent events is simply the product of their individual probabilities.

Example: You roll a pair of dice and draw a card from a standard deck. What is the probability you get

a. a total of 7 and a face card?
b. a total < 5 and an ace?
c. Homework: a total of 10 and a spot card? Ans. 3/52.
d. Homework: an even total bigger than 2 and a heart? Ans. 17/144.

Multiplication Rule 2: (The General Multiplication Rule) (for dependent events)

If A and B are (possibly dependent) events, then

$$P(A \,\text{and}\, B) = P(A)P(B \mid A)$$

where $P(B \mid A)$ represents the probability of B occurring given that A has occurred, read "the probability of B given A."

If we draw items *with* replacement, we are dealing with independent events. On the other hand, if we draw items *without* replacement, we are dealing with dependent events.

Example: Draw two cards from a standard deck with replacement. What is the probability you get

a. 2 kings?
b. no face cards?
c. Homework: 2 spot cards? Ans. 0.479.
d. Homework: no spot cards? Ans. 0.0947.
e. Homework: Repeat (c.) and (d.) drawing without replacement. Ans. c. 0.475; d. 0.905.
f. Homework: 2 red cards? Ans. 0.25.
g. Homework: 2 hearts? Ans. 0.0625.
h. Repeat (f.) and (g.) drawing without replacement. Ans. f. 0.245 g. 1/17.

Example: You draw 3 cards without replacement.

a. Homework: What is the probability that all 3 are hearts? Ans. 0.0129.
b. Repeat with replacement. Ans. 0.0156.

More on Complements: The probability of "at least one" "At least one" is equivalent to "one or more." Thus, the complement of "at least one" is "none." Therefore, to find the probability of at least one, we find the probability of none and subtract this probability from 1. That is,

$$P(\text{at least one}) = 1 - P(\text{none}).$$

Class Example:

1. A jar contains 5 red balls, 3 blue balls, 4 yellow balls, and 3 black balls. Draw 2 balls from the jar with replacement. What is the probability you draw
   a. at least one black ball.
   b. at least one red ball.
   c. Homework: at least one yellow ball. Ans. 0.462.
   d. Homework: at least one non-blue ball. Ans. 0.960.
   e. Repeat without replacement. Ans. 0.476; 0.971.
2. Draw 2 cards from a standard deck with replacement. Find the probability you obtain
   a. at least one ace.
   b. at least one face card.
   c. Homework: at least one spot card. Ans. 0.9053.
   d. Repeat without replacement. Ans. 0.9095.

CONDITIONAL PROBABILITY

If we solve the general multiplication formula for $P(B \mid A)$ by dividing by $P(A)$, we obtain

The Formula for the Conditional Probability of B given A

$$P(B \mid A) = \frac{P(A \,\text{and}\, B)}{P(A)}.$$

Examples:

1. A jar contains 6 red balls, 2 blue balls, 3 yellow balls, and 5 black balls. You draw a ball at random. What is the probability you get
   a. a red ball given that you did not draw a blue ball.
   b. a blue ball given that you did not draw a red ball.
   c. a yellow ball given that you did not draw a blue ball nor a black ball.
   d. Homework: a yellow ball given that you drew a non-blue ball. Ans. 0.214.

e. Homework: a red ball given that you did not draw a black ball. Ans. 0.545.
f. Homework: a blue ball or a red ball given that you did not draw a yellow ball. Ans. 0.615.

2. Draw a card from a standard deck. Find the probability you draw

   a. a red spot card $geqslant 8$ given that you drew a spot card.
   b. a jack given that you drew a non-ace.
   c. Homework: a king given that you drew a face card. Ans. 0.333.
   d. Homework: an ace given that you did not draw a face card. Ans. 0.100.

Venn Diagram for Conditional Probability: What does the conditional probability formula mean in terms of the Venn diagram?

Answer: Since we are assuming that event A has occurred, we are restricting our sample space to the outcomes comprising event A. Thus, the outcomes favorable to event B in this revised sample space are the outcomes comprising event $A \cap B$, i.e., the outcomes that are favorable to both A and B.

Condition for Independence: A and B are independent if and only if $P(A \text{ and } B) = P(A)P(B)$ if and only if $P(B \mid A) = P(B)$.

Examples:

1. A recent survey asked 100 people if they thought women in the armed forces should be permitted to participate in combat. The results of the survey are shown.

Gender	Yes	No	Total
Male	32	18	50
Female	8	42	50
Total	40	60	100

   Find the probability that

   a. a respondent answered yes;
   b. a respondent answered yes, given that the respondent was a female;
   c. the respondent was a male;
   d. the respondent was a male, given that the respondent answered no.
   e. Are the events of answering yes and being female independent or dependent?

2. The following table shows the results of a study in which researchers examined a childs IQ and the presence of a specific gene in the child.

	Gene present	Gene not present	Total
High IQ	33	19	
Normal IQ	39	11	
Total			

Find the probability that

a. a child does not have the gene;
b. a child does not have the gene, given that the child has a normal IQ;
c. a child has a high IQ;
d. a child has a high IQ, given that the child has the gene.
e. Are the events of presence of the gene and the child having a high IQ independent or dependent?

3. An urn contains 3 red balls, 2 blue balls, and 5 white balls. A ball is selected and its color is noted. Then a second ball is selected (without replacing the first selection) and its color is noted. Find the probability of

   a. selecting 2 blue balls;
   b. selecting a blue ball and then a red ball.

4. Three cards are drawn from a deck without replacement. Find the probability that

   a. all are clubs;
   b. all are red cards.

5. A coin is tossed 5 times. Find the probability of getting at least one tail.

6. A multiple-choice quiz has three questions, each with five answer choices. Only one of the choices is correct. You have no idea what the answer is to any question and have to guess each answer. Find the probability of answering

   a. all three questions correctly;
   b. none of the questions correctly;
   c. at least one of the questions correctly.

7. The probability that a person in the United States has type O+ blood is 83%. Three unrelated people in the United States are selected at random. Find the probability that

   a. all three have type O+ blood;
   b. none of the three has type O+ blood;
   c. at least one of the three has type O+ blood.

8. In a statistics class there are 18 juniors and 10 seniors; 6 of the seniors are females and 12 of the juniors are males. If a student is selected at random, find the probability of selecting

   a. a junior or a female;
   b. a senior or a female;
   c. a junior or a senior.

9. Three cable channels (6, 8, and 10) have quiz shows, comedies, and dramas. The number of each is shown here.

Type of Show	Channel 6	Channel 8	Channel 10
Quiz show	5	2	1
Comedy	3	2	8
Dramas	4	4	2

If a show is selected at random, find the probability that

a. the show is a quiz show or it is shown on channel 8;
b. the show is shown on channel 10 or it is a drama;
c. the show is a drama or a comedy.

10. If $P(A) = 0.35$, $P(B) = 0.35$, $P(C) = 0.30$, $P(A \text{ and } B) = 0.15$, $P(A \text{ and } C) = 0.25$, $P(B \text{ and } C) = 0.19$, and $P(A \text{ and } B \text{ and } C) = 0.03$. Find $P(A \text{ or } B \text{ or } C)$.
11. Urn 1 contains 5 red balls and 3 black balls. Urn 2 contains 3 red balls and 1 black ball. Urn 3 contains 4 red balls and 2 black balls. If an urn is selected at random and a ball is drawn, find the probability it will be red.
12. Three people are selected at random. Assume 365 days in a year. Find the probability that

    a. all three share the same birthday;
    b. none of the three share the same birthday.

17 Measures of Center and Variability; Empirical Rule

MEASURES OF CENTER

Previously, we looked at frequency tables and graphs that illustrated the (frequency) distribution of data. Here, we are interested in finding values that represent the center of a data set.

A *measure of center* is a value at the center or middle of a data set. The primary measures of center are the (arithmetic) mean, the median, the mode, and the midrange.

Mean. The average or *arithmetic mean* or, simply, the *mean* $\overline{x}$ of a data set is obtained by summing the values and dividing by the total number of values. It is the most commonly used measure of center. Its advantage is that it takes each value into account. Geometrically, the mean is the *balance point* of the data, i.e., the sum of the deviations from the mean is zero.

$$\text{sample mean} \quad = \quad \overline{x} \quad = \quad \frac{\sum x}{n}.$$

Example: 3, 7, 1, 11, 8; 3, 1, 7, 1, 2, 10.

Sample statistics are usually denoted with English letters and population parameters with Greek letters. The *population mean* is denoted by μ.

$$\mu = \frac{\sum x}{N}$$

where N is the size of the population.

Median. The *median* $\tilde{x}$ of a data set if the middle value when the data is arranged in increasing or decreasing order. It is the middle value if n is odd and the average of the middle values if n is even. Because the mean is greatly affected by extremely large (or small) observations and the median is not, the median is preferred in locating the center of skewed distributions. Skewed distributions often arise in economic or sociological data. For large sets of measurements, the median is often a better measure of center than the mean.

Mode. The *mode* M of a data set is the most frequent value. There may be more than one mode, but a set in which each value occurs once is said to have no mode. The mode is not usually used with numerical data, but is the only measure of center when dealing with nominal data. The mode is very useful in business planning to identify those products that are in greatest demand. When the data is described with a relative frequency histogram, the mode is the midpoint of the class interval having the largest relative frequency.

Midrange. The *midrange* MR is the average of the largest and smallest values and is thus a measure of center, but is seldom used since it only uses 2 values from the data set and is very sensitive to extremes, even more so than the mean.

(The *range* R is the highest value less the smallest value and is a measure of variability).

MEASURES OF VARIATION

1. Variation refers to the spread of the data.
2. Values relatively close together have lower measures of variation, while those spread farther apart have higher variation.

Range. The *range* $R =$ (largest value) $-$ (smallest value).

Thus, the range depends upon only two values and is very susceptible to extremely large or small values. So we prefer a measure of variation that includes all data values. What about the sum of the deviations from the mean, i.e., $\sum(x - \overline{x})$?

Population Standard Deviation (Variance). The *population variance* is denoted σ^2 while its square root, the *population standard deviation*, is denoted by σ.

$$\sigma^2 = \frac{\sum(x-\mu)^2}{N}$$

Sample Standard Deviation (Variance). The *sample variance* is denoted s^2 while its square root, the *sample standard deviation*, is denoted by s. (On your calculator, σx denotes the population standard deviation and Sx denotes the sample standard deviation.)

$$s^2 = \frac{\sum(x-\overline{x})^2}{n-1} = \frac{n(\sum x^2) - (\sum x)^2}{n(n-1)} \text{(Short-cut Formula)}$$

Justification: Why do we divide by $n-1$ rather than n? The sample variance s^2 is an unbiased estimator of the population variance σ^2, i.e., the values of s^2 target the value of σ^2, rather than systematically overestimating or underestimating. If we divide (instead) by n, we get a biased (under-shooting each time) estimate of σ^2, the population variance.

Example: Calculate the range, sample variance (two ways) and sample standard deviation for the data sets

a. 3, 7, 1, 11, 8;
b. 3, 1, 7, 1, 2, 10.

Then calculate the population variance and population standard deviation for these two data sets. p. 85-87, Larson/Farber

RANGE RULE OF THUMB

This rule is based on the principle that, for a bell-shaped distribution, 95% of the sample values lie within 2 standard deviations from the mean. That is, 95% lie between $\overline{x} - 2s$ and $\overline{x} + 2s$. Thus, since the range equals the maximum minus the minimum,

$$\text{range} = \max - \min \approx (\overline{x} + 2s) - (\overline{x} - 2s) = 4s$$

so that

$$s \approx \frac{\text{range}}{4}$$

(This is called the range rule of thumb.)

The range rule of thumb can be used to check the plausibility of your calculated value for s.

Example: Use R to estimate s in the first example on page 43 of the notes. Then calculate $\overline{x}$ and s on your calculator.

Class Example: Use R to estimate s in the second example on page 43 of the notes. Then calculate $\overline{x}$ and s on your calculator.

Using μ and σ ($\overline{x}$ and s) in inferential statistics. $\overline{x} - 2s$ and $\overline{x} + 2s$ are called the minimum "usual" value and the maximum "usual" value, respectively. The closed interval with these values as endpoints is called the *interval of usual observations.*

Example: Weights of men have a mean of 160 lb and a standard deviation of 8 lb. Use the range rule of thumb to find the minimum and maximum "usual" weights and the interval of usual observations. Based on these assumptions, would it be unusual for a man to weigh 180 lbs?

Class Example: Chilean women have a mean height of 65 inches and a standard deviation of 2.5 inches. Find the minimum and maximum usual heights and the interval of usual observations. Would it be unusual to find a Chilean woman with a height of 58 inches?

EMPIRICAL RULE

(valid for data with a bell-shaped or normal frequency distribution)

1. Approximately 68% of all values fall within 1 standard deviation of the mean.
2. Approximately 95% of all values fall within 2 standard deviations of the mean.
3. Approximately 99.7% of all values fall within 3 standard deviations of the mean.

This holds true for both the normally distributed population and samples taken from the population. These percentages can also be interpreted as the probability that a randomly selected observation falls in the interval. For example, $P(\mu - \sigma) < x < \mu + \sigma) = 0.68$. It is advantageous to draw a sketch and label it.

Example: Heights of men have a bell-shaped distribution with a mean of 70 inches and a standard deviation of 3 inches. Then,

a. 68% of the heights are between __________ and __________ inches.

b. 95% are between __________ and __________ inches.

c. 99.7% are between __________ and __________ inches.

Class Example: The life of a brand of light bulbs has a bell-shaped distribution with a mean of 150 hours and a standard deviation of 6 hours.

a. 68% of the bulbs last between ____________ and ____________ hours.

b. 95% are between ____________ and ____________ hours.

c. 99.7% are between ____________ and ____________ hours.

Example: Weights of men have a bell-shaped distribution with a mean of 160 lb and a standard deviation of 6 lb. Then,

a. ____________% of the weights are between 148 and 172 lb.

b. ____________% are between 142 and 178 lb.

c. ____________% are between 154 and 166 lb.

Class Example: The weight of a loaf of bread has a bell-shaped distribution with a mean of 16 oz and a standard deviation of 2 oz.

a. ____________% of the loaves weigh between 10 and 22 oz.

b. ____________% of the loaves weigh between 14 and 18 oz.

c. ____________% of the loaves weigh between 12 and 20 oz.

Examples

1. Water buffalo have weights that have a bell-shaped distribution with a mean of 1240 lb and a standard deviation of 60 lb. Would it be unusual to encounter a water buffalo weighing at least 1350 lb?
2. Times for the 100 meters for the top male athletes are bell-shaped with a mean of 9.92 seconds and a standard deviation of 0.08 seconds. Would it be unusual for one of these athletes to run 100 meters in 9.84 seconds or less?
3. Great white sharks have jaw widths that are normally distributed with a mean of 15.7 inches and a standard deviation of 2.8 inches. Sketch a normal curve showing the jaw widths at one, two, and three standard deviations from the mean. Use the sketch to find the percentage of sharks having jaw widths of

   a. between 12.9 and 21.3 inches.

b. less than 10.1 inches.
c. greater than 18.5 inches.
d. between 15.7 and 18.5 inches.
e. less than 12.9 inches.
f. greater than 7.3 inches.

4. A college entrance exam has two parts: mathematical and verbal. The scores for each part are normally distributed with a mean of 500 and a standard deviation of 100. Find the percentage of scores on the verbal part that are

   a. less than 400.
   b. more than 700.
   c. between 300 and 600.
   d. more than 300.
   e. less than 800.
   f. between 600 and 700.

18 Counting Principles

Larson and Farber: Sec. 3.4

Statistics Handbook for the TI-83: Activity 6, Topic 22

The Fundamental Counting Principle

Fundamental Counting Rule: In a sequence of n events in which the first one has k_1 possibilities and the second event has k_2 and the third has k_3, and so forth, the total number of possibilities of the sequence will be

$$k_1 \cdot k_2 \cdot k_3 \cdot \cdots \cdot k_n.$$

Permutations of n DISTINCT objects taken r at a time.

$${}_nP_r = \frac{n!}{(n-r)!}$$

Distinguishable permutations of n objects. (n_1 of type 1, n_2 of type 2, ...)

$$P(n_1, \ldots, n_k; n) = \frac{n!}{n_1! n_2! \cdots n_k!}, \qquad \text{where } n_1 + n_2 + \cdots + n_k = n.$$

Combinations of n objects taken r at a time.

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

Examples:

1. a. How many different ID cards can be made if there are 6 digits on a card and no digit can be used more than once?
   b. How many different ID cards can be made if there are 6 digits on a card?
2. From a pool of 14 candidates, the offices of president, vice-president, secretary, and treasurer will be filled. How many ways the offices can be filled?
3. How many different ways can 9 trophies be arranged on a shelf?
4. How many ways can a person select 8 videotapes from 10 tapes?
5. a. In how many ways can the letters A, B, C, D, E, F, and G be arranged for a sevenletter security code?
   b. In how many ways can 4 letters A and 3 letters B be arranged to for a sevenletter security code?
6. How many ways can a jury of 6 women and 6 men be selected from 10 women and 12 men?
7. Problem 40, p. 159 from Larson and Farber.

19 Continuous Probability Distributions: Uniform, Normal

Continuous Random Variables. A *continuous random variable* X is one that can assume any value within some interval or intervals.

A function $f(x)$ is a probability density function (probability distribution) for the random variable X if

a. f is defined for all real numbers x,
b. $f(x) \geqslant 0$ for all x, and
c. the area under the graph of f between two points a and b is $P(a < X < b)$, the probability that X assumes a value between a and b.

Note that:

1. Because there is no area over a point, say $X = a$, it follows that $P(X = a)$, the probability associated with a particular value of X, is zero. Thus,

$$P(a < X < b) = P(a \leqslant X < b) = P(a < X \leqslant b) = P(a \leqslant X \leqslant b).$$

2. Because areas over intervals represent probabilities, it follows that the total area under a probability distribution, the probability assigned to the set of all possible values of X is 1.

The Uniform Distribution. Perhaps the simplest of all probability distributions is the *uniform probability distribution*, in which the continuous random variable X has equally likely outcomes over its range of possible values. It is useful to study uniform distribution first since the calculations of probabilities involves only the areas of rectangles in this case. Suppose that the uniform random variable X can assume values only in the interval $a \leqslant x \leqslant b$. Then, since each outcome is equally likely, the distribution will have a rectangular shape. This forces $f(x)$ to have the following form:

$$y = f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \leqslant x \leqslant b \\ 0 & \text{otherwise} \end{cases} .$$

Then, $P(c < X < d) =$ the area of the rectangle over the interval $[c, d]$.

Example: Monthly rainfall X ranges from 3 to 23 inches with a uniform distribution.

a. Give the density function for X.
b. $P(7 < X \leqslant 9) =$
c. $P(12 \leqslant X \leqslant 19) =$
d. $P(X < 14) =$
e. $P(X > 16) =$

Class Examples:

1. In Santa Domingo, daily temperature X ranges from 65°F to 115°F with a uniform distribution.

    a. Give the density function for X.
    b. $P(69 < X \leqslant 75) =$ 0.12
    c. $P(78 \leqslant X \leqslant 92) =$ 0.28
    d. $P(X < 86) =$ 0.42
    e. $P(X > 76) =$ 0.78

2. The length of time X waiting for a cab has a uniform distribution with times ranging from 2 minutes to 32 minutes.

    a. Give the density function for X.
    b. $P(7 < X < 16) =$ 0.3
    c. $P(13 \leqslant X \leqslant 19) =$ 0.2
    d. $P(X < 14) =$ 0.4
    e. $P(X > 17) =$ 0.5

The Normal Distribution. The most important continuous distribution is the *normal distribution.* Its probability density function for a given mean μ and standard deviation σ is

$$y = f(x) = \frac{e^{-[(x-\mu)/\sigma]^2}/2}{\sigma\sqrt{2\pi}}.$$

In this case, the random variable X is $N(\mu, \sigma)$, i.e., normally distributed with mean μ and standard deviation σ.

Computing areas over intervals for a normal distribution is a difficult task. Consequently, we will use tables for the computed areas in the case where $\mu = 0$ and $\sigma = 1$ (called the *standard normal distribution*); a random variable with a standard normal distribution is typically denoted by the symbol Z. In this case,

$$y = f(z) = \frac{1}{\sqrt{2\pi}} e^{-(1/2)z^2}.$$

We will learn how to convert all normal random variables to the standard normal (and then use the standard normal tables).

Properties of a Normal Distribution.

1. The mean, median, and mode are equal.
2. The normal curve is bell-shaped and is symmetric about the mean.
3. The total area under the normal curve is equal to one.
4. The normal curve approaches, but never touches, the x-axis as it extends farther and farther away from the mean.
5. The tables in Larson/Farber give the area under the standard normal curve to the left of z, in which z is rounded to two decimal places. That is, they give $P(Z \leqslant z)$ in which Z is $N(0, 1)$. This is called the cumulative distribution for Z.

Probabilities Corresponding to z-scores. Assume that the reading Z on thermometers is normally distributed with a mean of 0.00°C and a standard deviation of 1.00°C. A thermometer is randomly selected and its reading z is recorded. In each case, draw a sketch and find the probabilities.

Examples:

a. $P(Z < 1.54°) =$
b. $P(Z < -2.75°) =$
c. $P(-0.97° < Z < 1.24°) =$
d. $P(0.55° < Z < 2.16°) =$
e. $P(-1.57° < Z < -1.11°) =$
f. $P(Z > 1.23°) =$
g. $P(Z > -0.47°) =$

Examples: p. 225-228, Larson/Farber

Finding z scores corresponding to probabilities: We have been finding the probability that corresponds to a given z score; now we will do the reverse. That is, given a probability, we will find the corresponding z score.

Key fact: The numbers in the body of the table correspond to areas or probabilities while the numbers on the boundary correspond to z-scores.

Procedure:

1. Find the number in the BODY of the table (NOT the boundary) that is closest to the given probability.
2. Take the z-score (on the boundary of the table, right?) that corresponds to this number. If the probability is equally close to two numbers, average the two corresponding z scores. It is important to draw a graph with the relevant labels as before. You can then use common sense to see if your results are reasonable.

Class Examples: (from Triola's Elementary Statistics)

h. Find P_{90}, the 90th percentile. This is the temperature reading separating the bottom 90% from the top 10%. Answer: 1.28.
i. Find P_{20}, the 20th percentile. Answer: -0.84
j. Find Q_1, the temperature reading that is the first quartile. Answer: -0.67
k. Find D_3, the temperature reading that is the third decile. Answer: -0.52
l. If 5% of the thermometers are rejected because they have readings that are too low, but all other thermometers are acceptable, find the reading that separates the rejected thermometers from the others. Answer: -1.645
m. If 12% of the thermometers are rejected because they have readings that are too high, but all other thermometers are acceptable, find the reading that separates the rejected thermometers from the others. Answer: 1.18

n. A troubleshooter wants to examine thermometers that give readings in the bottom 3%. What reading separates the bottom 3% from the others? Answer: -1.88

o. If 2.5% of the thermometers are rejected because they have readings that are too high and another 2.5% are rejected because they have readings that are too low, find the two readings that are cutoff values separating the rejected thermometers from the others. Answer: $-1.96, 1.96$

Finding Probabilities From x-values. Recall that the z-score of a measurement x is given by

$$z = \frac{x - \mu}{\sigma} \qquad \left(z = \frac{x - \overline{x}}{s}\right). \tag{6}$$

Property of Normal Distributions. If X is a normal random variable with mean μ and standard deviation σ, then the random variable Z defined by the formula

$$Z = \frac{X - \mu}{\sigma}$$

has a standard normal distribution. In particular, if $z_1 = \frac{x_1 - \mu}{\sigma}$ and $z_2 = \frac{x_2 - \mu}{\sigma}$, then

$$P(x_1 < X < x_2) = P(z_1 < Z < z_2).$$

Procedure for Finding Probabilities Corresponding to x-values.

1. Sketch the normal distribution with its mean μ. Then shade the area corresponding to the probability you seek.
2. Convert the boundaries of the shaded area from x-values to z-values. Show these z-values under the corresponding x-values in your sketch.
3. Use the tables to find the areas corresponding to the z-values.

Class Examples: (from Triola's Elementary Statistics)

1. Assume that women's weights are normally distributed with mean $\mu = 143$ lb and standard deviation $\sigma = 29$ lb (based on data from the National Health Survey). Assume that a woman is randomly selected and her weight X is recorded. Draw a graph and find the indicated probabilities.

   a. $P(143 \text{ lb} < X < 172 \text{ lb})$ Answer: 0.3413
   b. $P(150 \text{ lb} < X < 180 \text{ lb})$ Answer: 0.3049
   c. $P(X > 150 \text{ lb})$ Answer: 0.4052
   d. $P(X > 130 \text{ lb})$
   e. $P(X < 186.5 \text{ lb})$ Answer 0.9332
   f. $P(X < 140 \text{ lb})$

2. Assume the heights of women are normally distributed with a mean $\mu = 63.6$ inches and a standard deviation σ of 2.5 inches.

a. The Beanstalk Club, a social organization for tall people, has a requirement that women must be at least 70 in. (or 5 ft 10 in.) tall. What percentage of women meets that requirement: Answer: 0.52%
b. In order to fit into a Russian Soyuz spacecraft, an astronaut must have a height between 64.5 in. and 72 in. What percentage of women meets that requirement? Answer: 35.93%

3. The lengths of pregnancies are normally distributed with a mean of 268 days and a standard deviation of 15 days. If we stipulate that a baby is premature if born at least three weeks early, what percentage of babies are born prematurely? Answer: 8.08%
4. p. 235, #13-30, Larson/Farber

Finding x-values corresponding to probabilities. We can solve the z-score formula for x and obtain

$$x = \mu + z\sigma. \tag{7}$$

So, given a probability, first find the z-score corresponding to that probability, then find the x-score that corresponds to the z-score via (7).

Class Examples:

1. Assume that women's weights are normally distributed with mean $\mu = 143$ lb and standard deviation $\sigma = 29$ lb (based on data from the National Health Survey). Assume that a woman is randomly selected and her weight X is recorded. Draw a graph and find the indicated weights. Determine the women's weight X that
   a. corresponds to the 35th percentile P_{35}
   b. corresponds to the 22nd percentile P_{22}
   c. corresponds to the 80th percentile P_{80}
   d. corresponds to the 63rd percentile P_{63}

   e. 42% of the women's weights are above ________ lb

   f. 28% of the women's weights are above ________ lb

   g. 14% of the women's weights are above ________ lb

   h. 68% of the women's weights are above ________ lb

   i. 92% of the women's weights are above ________ lb
2. Assume the heights of women are normally distributed with a mean $\mu = 63.6$ inches and a standard deviation σ of 2.5 inches. To be eligible for the U.S. Army, a woman's height must be between 58 in. and 80 in. If that requirement is changed so that only the shortest 1% and tallest 1% are excluded, find the minimum and maximum acceptable heights. Answer: 57.8 in., 69.4 in.
3. The lengths of pregnancies are normally distributed with a mean of 268 days and a standard deviation of 15 days. If we stipulate that a baby is premature if the length of pregnancy is in the lowest 4%, find the length of pregnancy that separates premature babies from those who are not premature. Answer: 242 days

4. IQ scores are normally distributed with a mean of 100 and a standard deviation of 15. If we redefine the category of "intellectually very superior" to be scores in the top 2%, what does the minimum score become? Answer: 131

Probabilities corresponding to z-scores using the TI84+:

To compute $P(z_1 < Z < z_2)$

1. [2nd] [DISTR]
2. **2:normalcdf** (z_1, z_2)
3. [ENTER]

Examples:

a. $P(-1.25 < Z < 2.13) =$ ________;

b. $P(Z < -1.45) =$ ________;

c. $P(Z > 1.37) =$ ________;

Class Examples:

a. $P(1.02 < Z < 2.43) =$ ________;

b. $P(Z > -1.54) =$ ________;

c. $P(Z < 2.06) =$ ________.

Probabilities corresponding to x-values using the TI84+:

To compute $P(x_1 < X < x_2)$ where $X = N(\mu, \sigma)$

1. [2nd] [DISTR]
2. **2:normalcdf** (x_1, x_2, μ, σ)
3. [ENTER]

Example: $X = N(10, 2)$; $P(-7.50 < X < 14.38) =$ ________

Class Examples: $X = N(7.50, 1.20)$; $P(6.00 < X < 9.60) =$ ________

z-scores corresponding to probabilities using the TI84+:

To compute z_1 so that $P(Z \leqslant z_1) = A$

1. [2nd] [DISTR]
2. **3:invNorm** (A)
3. [ENTER]

Examples: $P(Z < z_1) = 0.26$; $P(Z > z_2) = 0.32$.

Class Examples: $P(Z \leqslant z_1) = 0.65$; $P(Z \geqslant z_2) = 0.30$.

x-values corresponding to probabilities using the TI84+:

To compute x_1 so that $P(X \leqslant x_1) = A$ where $X = N(\mu, \sigma)$

1. [2nd] [DISTR]
2. **3:invNorm** (A, μ, σ)
3. [ENTER]

Examples: $X = N(10, 2)$; $P(X \leqslant x_1) = 0.10$; $P(X \geqslant x_2) = 0.25$.

Class Examples: $X = N(15, 5)$; $P(X \leqslant x_1) = 0.72$; $P(X \geqslant x_2) = 0.77$.