UTC Mathematics Department
Student Problem Competition
Past Student Problems

Problem 1

Calculate the sum of the infinite series $\sum_{k=1}^{\infty} \sin\left(\frac{2x}{3^k}\right) \sin\left(\frac{x}{3^k}\right)$.

Winner: Jeremy Higdon

Problem 2

Find all positive integers m, n, and x such that $m > 1$, $n > 1$ and $x^n = \sum_{k=1}^{m} k!$.

Winner: Jeremy Higdon

Problem 3

How many real solutions does the equation $\sqrt[7]{x} - \sqrt[5]{x} = \sqrt[3]{x} - \sqrt{x}$ have?

Problem 4

Solve the system of simultaneous equations for real numbers x and y.

$$x^4 - 6x^2y^2 + y^4 = 1$$
$$4x^3y - 4xy^3 = 1$$

Problem 5

Between what two integers does the sum S lie, where S is the sum

$$\sum_{n=1}^{10^9} n^{-2/3} = 1 + \frac{1}{\sqrt[3]{2^2}} + \frac{1}{\sqrt[3]{3^2}} + ... + \frac{1}{\sqrt[3]{(10^9)^2}}.$$

Problem 6

If n is a positive integer, how many real solutions are there, as a function of n, to $e^x = x^n$?

Problem 7
If $A = (0, -10)$ and $B = (2, 0)$, find the point(s) C on the parabola $y = x^2$ which minimize(s) the area of triangle ABC.

Winner: Jonathan McGlumphy

Problem 8
For each real value of a_1, determine if the sequence $\{a_n\}$ defined by $3a_{n+1} = 2 + a_n^3$ converges or diverges.

Problem 9
Show that the equation $3^x + 4^x = 5^x$ has exactly one real root.

Problem 10
Suppose that the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is R and $0 < R < \infty$. Evaluate the radius of convergence of $\sum_{n=1}^{\infty} \frac{n^n}{n!} a_n x^n$.

Problem 11
Consider the series

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - ...,$$

obtained by rearranging the terms of the harmonic series in such a way that each positive term is followed by two negative terms. Find the sum of this series.

Problem 12
Ten (not necessarily distinct) positive integers have the property that if all but one are added the possible results are 82, 83, 84, 85, 87, 89, 90, 91, and 92. What are the integers?

Winner: Adam Crowley

Problem 13
Define $\{a_n\}$ recursively by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + \sqrt{a_n}}$ for $n \geq 1$. Prove the sequence converges and find an algebraic form of the limit.

Problem 14
Let $f(x)$ be continuous on $[a, b]$ and let M be the maximum value of $|f(x)|$.
Prove $M = \lim_{n\to\infty} \left[\int_a^b |f(x)|^n \, dx\right]^{1/n}$.

Problem 15
Derive the antiderivative of $\dfrac{1}{x^4+1}$ and use to get a closed form value of $\displaystyle\int_0^\infty \frac{1}{x^4+1}\, dx$.

Problem 16
Find the value of the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.
Winner: Eric Williams

Problem 17
Find all positive integers n so that $f(n) = \dfrac{12n^3 - 5n^2 - 251n + 389}{6n^2 - 37n + 45}$ is an integer.

Problem 18
Find a formula for S_n the sum of the greatest odd divisors of the first 2^n positive integers. For example, $S_2 = 1 + 1 + 3 + 1 = 6$.

Problem 19
Let $S = \{n_1, n_2, ...\}$ denote the collection of those positive integers that do not involve the digit 0 in their decimal representation. For example, $7 \in s$ but $101 \notin S$. Show that $\sum_{k=1}^{\infty} \frac{1}{n_k}$ converges and has a sum less than 90.

Problem 20
Prove that, for any natural number n, the expression
$2903^n - 803^n - 464^n + 261^n$ is divisible by 1897.
Winner: Kossivi Adjoh

Problem 21
Let a_1 be an arbitrary real number. Define the sequence $\{a_n\}$ by $3a_{n+1} = 2 + a_n^3$. For each real number a_1, find the limit of $\{a_n\}$, if it exists.

Winner: Kossivi Adjoh

Problem 22
Between what two consecutive integers does the sum S lie, where

$$S = \sum_{n=1}^{10^9} n^{-2/3} \quad ?$$

Winners: Kossivi Adjoh and Tony McDaniel

Problem 23
Let $D_n = \ln n - 2(\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1})$ for $n \geq 2$.
Show that $\{D_n\}$ is an increasing sequence and find $\lim_{n\to\infty} D_n$.

Kossivi Adjoh gave a partial solution.

Problem 24
Prove that $\sqrt[3]{2} - \sqrt{3}$ is irrational.

Winners: Kossivi Adjoh and Jeremy Higdon

Problem 25
Find all integers solutions to

$$2(x^5 + y^5 + 1) = 5xy(x^2 + y^2 + 1).$$

Problem 26
Find the area of the region that is in the interior of the four circles of radius one with centers at each of the four vertices of the unit square.

5 winners shared the prize: Milton White, Kyle Lange, Zach Patterson, Andrew Hranek, and Kossivi Adjoh

Problem 27
Show that for any positive real numbers a, b, c,

$$abc \leq \frac{(a+b+c)^3 - (a^3+b^3+c^3)}{24} \leq \frac{(a+b+c)^3}{27}.$$

Winners: Rocky Gao and Sam Chill

Problem 28
For a positive integer greater than 2, define $H(N)$ by

$$H(N) = \frac{1}{N} + \frac{1}{N+3} + \frac{1}{N+6} + ... + \frac{1}{N+3(N-1)}.$$

Prove that H(N) is convergent and determine its limit.
Winner: Kossivi Adjoh

Problem 29
Determine all pairs (k,n) of positive integers such that

$$1 + 2 + ... + k = (k+1) + (k+2) + ... + n.$$

Kossivi Adjoh gave a partial solution.

Problem 30
Derive the indefinite integral of $f(x) = 1/(x^4+1)$ and use it to find

$$\int_0^\infty \frac{1}{x^4+1}\, dx.$$

Winners: Kossivi Adjoh and Daniel Long

Problem 31
Find $\displaystyle\sum_{n=1}^{\infty} \arctan\left(\frac{2}{n^2}\right)$ in closed form.